11th World Congress on Computational Mechanics (WCCM XI)
5th European Conference on Computational Mechanics (ECCM V)
6th European Conference on Computational Fluid Dynamics (ECFD VI)
July 20–25, 2014, Barcelona, Spain

# ENERGY BOUNDS FOR HOMOGENIZATION OF STOKES' EQUATIONS USING PERIODIC BOUNDARY CONDITIONS

**Carl Sandström[1], Fredrik Larsson[2]**

[1] Chalmers University of Technology, Gothenburg, Sweden, carl.sandstrom@chalmers.se
[2] Chalmers University of Technology, Gothenburg, Sweden, fredrik.larsson@chalmers.se

**Key words:** *Multiscale modelling, porous media, periodicity, bounds*

We consider a two-scale problem where the subscale is represented as a Stokes flow on a strongly heterogeneous domain consisting of a fluid within an open pore system. Furthermore, we adopt the concept of Variationally Consistent Homogenization and produce a macroscale representation in the form of a Darcy flow where the seepage and its tangent are computed on the subscale. In the linear case, the tangent represents the permeability tensor whereas in the non-linear case it serves as the consistent tangent in Newton iterations.

In order to capture the effective properties of the subscale, the Stokes flow is solved on a Representative Volume Element (RVE) which should be large enough to represent the true subscale yet small enough to be as computationally efficient as possible. Moreover, the RVE problem must be formulated using the proper prolongation conditions as to satisfy the macrohomogeneity condition, ensuring energy equilibrium production on the two scales. It has been shown in [1], that a proper choice of boundary conditions for this problem is that of periodicity.

The classical approach in Finite Element Analysis is to enforce periodic boundary conditions by replacing two degrees of freedom on either side of a domain with one. Although computationally effective, this approach calls for a mesh which has identical discretization on either opposite sides. It has been shown in [2] and [3] that the dependence on mesh periodicity can be void by imposing so called weakly periodic boundary conditions on the subscale RVE. In [2], periodicity is achieved by using global base functions on the boundary for the unknown while in [3] periodicity is imposed by using unknown loads on the boundary. The RVE problem is stated by adding periodicity constraints to a minimization problem pertinent to a Stokes flow. For the subsequent minimization, we consider the RVE $\Omega_\Box$ with pore volume $\Omega_\Box^{\mathrm{F}}$ and introduce the mean potential as

$$\pi_\Box(p^{\mathrm{M}}, \boldsymbol{v}, p^{\mathrm{S}}, \boldsymbol{\beta}, \gamma) = \frac{1}{|\ \Omega_\Box\ |}\left(\int_{\Omega_\Box^{\mathrm{F}}} \Phi(\boldsymbol{v} \otimes \boldsymbol{\nabla})\mathrm{d}V - \int_{\Omega_\Box^{\mathrm{F}}} p^{\mathrm{S}}(\boldsymbol{\nabla} \cdot \boldsymbol{v})\mathrm{d}V + \int_{\Omega_\Box^{\mathrm{F}}} \boldsymbol{\nabla} p^{\mathrm{M}} \cdot \boldsymbol{v}\mathrm{d}V - \int_{\Gamma_\Box^{\mathrm{F}+}} [\![\boldsymbol{v}]\!] \cdot \boldsymbol{\beta} + [\![p^{\mathrm{S}}]\!]\gamma \mathrm{d}S\right)$$

where $p^{\mathrm{M}}$ is the smooth macroscale pressure, $p^{\mathrm{S}}$ is the fluctuating subscale pressure, $\boldsymbol{v}$ is the fluid velocity and $\boldsymbol{\beta}$ and $\gamma$ are Lagrange multipliers due to the weakly periodic boundary conditions defined on the image side of the boundary $\Gamma_\Box^{\mathrm{F}}$. The stationary point for the RVE problem is then given as

$$\psi_\Box(p^{\mathrm{M}}) = \inf_{\boldsymbol{v}\in\mathcal{V}_\Box} \sup_{\substack{p^{\mathrm{S}}\in\mathcal{P}_\Box^{\mathrm{S}} \\ \boldsymbol{\beta}\in\mathcal{B}_\Box}} \inf_{\gamma\in\mathcal{G}_\Box} \pi_\Box(\boldsymbol{v}, p^{\mathrm{M}}, p^{\mathrm{S}}, \boldsymbol{\beta}, \gamma)$$

where $\psi_\Box$ is the energy for an arbitrary macroscale pressure gradient, $\mathcal{V}_\Box$, $\mathcal{P}_\Box^{\mathrm{S}}$, $\mathcal{B}_\Box$ and $\mathcal{G}_\Box$ are function spaces of sufficient regularity and no requirement on peridicity. By introducing Solution Based Shape Functions, we can produce a discretized subspace of $\mathcal{V}_\Box$, containing only strongly periodic functions on the RVE, thus removing the dependence on the pertinent Lagrange multiplier $\boldsymbol{\beta}$ and giving an upper bound for $\psi_\Box$. Likewise, by confining $\mathcal{P}_\Box^{\mathrm{S}}$ to only strongly periodic functions on the RVE, we produce a lower bound on $\psi_\Box$ and void the dependence on $\gamma$. The upper and lower bounds for the strong periodicity are indeed computed on a non-periodic mesh.

As a numerical example, we present a flow around connected solid spheres in a BCC structure where upper and lower bounds are computed along with the weakly periodic solution.

## REFERENCES


[1] C. Sandström, F. Larsson, *Variationally consistent homogenization of Stokes flow in porous media*, Int. J. Multiscale Comput. Engrg. 11 (2) (2013) 117138.

[2] V.-D. Nguyen, G. Becker, L. Noels. *Multiscale computational homogenization methods with a gradient enhanced scheme based on the discontinuous Galerkin formulation*. Computer Methods in Applied Mechanics and Engineering, , Volume 260, 15 June 2013, Pages 63-77.

[3] F. Larsson, K. Runesson, S. Saroukhani, R. Vafadari, *Computeation homogenization based on a weak form of micro-periodicity for RVE-problems* Computer Methods in Applied Mechanics Engineering 200 (1-4) (2011)