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LOW-ORDER DPG METHOD

Carsten Carstensen, Dietmar Gallistl, Friederike Hellwig* and Lucy Weggler

Department of Mathematics, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany, { cc,gallistl,hellwigf,wegglerl }@math.hu-berlin.de, http://www.math.hu-berlin.de/∼cc

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Considered is a conventional second-order boundary value problem subject to the discontinuous Petrov Galerkin (dPG) method where best-approximation properties of the numerical solution $\xi_h \in X_h \subset X$ are guaranteed in the sense that

$$||x - \xi_h||_X \leq C \operatorname{dist}(x, X_h)\,.$$

Apart from this commonly known fact, there are two further properties of the dPG methods, namely,

- The dPG methods qualify for a wider range of solution spaces since the functional analytical setting requires only that the solution space X and the test space Y are reflexive Banach spaces.
- The dPG methods are very interesting in the context of adaptive schemes as they come with a built-in a posteriori error estimator

  $$||x - \xi_h||_X \leq C\, ||b(x - \xi_h, \bullet)||_{Y^*}\,.$$

  However, this error estimate suffers from the fact that it is computationally not accessible as it contains the norm of a dual space Y^*.

This presentation addresses the numerical analysis of the dPG methods in an alternative approach and starts with a stability analysis for the dPG methods directly on the discrete level. New insight into the relations between the finite-dimensional spaces and the least-squares character of the dPG method is gained by a discussion of the discrete inf-sup condition

$$0 < \beta_h := \inf_{x_h \in S(X_h)} \sup_{y_h \in S(Y_h)} b(x_h, y_h)\,,$$

with $S(\mathcal{X}) := \{x \in \mathcal{X} \mid ||x||_{\mathcal{X}} = 1\}$, as it will lead to the fundamental observation that a stable dPG method requires the boundedness of the modified discrete inf-sup condition which reads

$$\gamma_h := ||\Pi_h||^{-1}\beta_h \leq \inf_{x_h \in S(X_h)} \sup_{y_h \in S(M_h)} b(x_h, y_h)\,.$$

Here, Π_h is a projector onto a subspace $M_h \subset Y_h$ that guarantees non-degeneracy of the bilinear form $b : X_h \times Y_h \to \mathbb{R}$ on X_h.

Provided $\gamma_h > 0$ we further prove that the error of the numerical solution can be estimated as follows

$$\beta_h||x - \xi_h||_X \leq ||\Pi||\,||F - b(\xi_h, \bullet)||_{Y_h^*} + ||F \circ (1 - \Pi)||_{Y^*}\,.$$

In fact, this estimate holds not only for the numerical solution, but for any $\xi_h \in X_h$. Thus, the dual norm from above does not appear in the principal term anymore. A posteriori error control is addressed in the same framework which motivates adaptive mesh-refining strategies.

For a model problem in the L^2-Hilbert space setting it is proved that the low-order dPG method is stable. A principal result from this analysis is that in order to design a stable dPG scheme it is not necessary to increase the polynomial order. Numerical experiments confirm the theoretical considerations.

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