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HOMOGENIZATION OF THE ONE-DIMENSIONAL WAVE EQUATION WITH PERIODIC COEFFICIENTS

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This work is devoted to the periodic homogenization of the wave equation in a one-dimensional open bounded domain where the time-independent coefficients are ε - periodic with small period $\varepsilon > 0$. Corrector results for the low frequency waves have been published in [2]. These works were not taking into account fast time oscillations, so the models reflect only a part of the physical solution. In [1], an homogenized model has been developed to cover the time and space oscillations occurring both at low and high frequencies. Unfortunately, the boundary conditions of the homogenized model were not found. Therefore, establishing the boundary conditions of the homogenized model is critical and is the goal of the present work.

For a bounded open set $\Omega = (0, \alpha)$ and a finite time interval $I = [0, T) \subset \mathbb{R}^+$, we consider the wave equation with Dirichlet boundary conditions,

$$\left\{ \begin{array}{l} \rho\left(\frac{x}{\varepsilon}\right)\partial_{tt}^2 u^\varepsilon(t,x) - \partial_x\left(a\left(\frac{x}{\varepsilon}\right)\partial_x u^\varepsilon(t,x)\right) = f^\varepsilon(t,x) \text{ in } I \times \Omega, \\ u^\varepsilon(t=0,x) = u_0^\varepsilon(x) \text{ and } \partial_t u^\varepsilon(t=0,x) = v_0^\varepsilon(x) \text{ in } \Omega, \\ u^\varepsilon(t,0) = u^\varepsilon(t,\alpha) = 0 \text{ in } I, \end{array} \right. \tag{1}$$

where $\varepsilon > 0$ denotes a small parameter intended to go to zero. The two functions $a^\varepsilon = a\left(\frac{x}{\varepsilon}\right)$ and $\rho^\varepsilon = \rho\left(\frac{x}{\varepsilon}\right)$ are Lipschitzian, positive and periodic with respect to a lattice of reference cell $\varepsilon Y \subset \mathbb{R}$.

The wave equation is written under the form of a system with unknown the vector of first-order derivatives $U^\varepsilon := \left(\sqrt{a^\varepsilon}\partial_x u^\varepsilon, \sqrt{\rho^\varepsilon}\partial_t u^\varepsilon\right)$. For $k \in \mathbb{R}$, the modulated two-scale transform W_k^ε is applied to the solution U^ε as in [1]. For $n \in \mathbb{N}^*$ and $k \in \mathbb{R}$, the n^{th} eigenvalue λ_n^k of the Bloch wave problem with $k-$quasi-periodic boundary conditions satisfies $\lambda_n^k = \lambda_n^{-k}$, in addition λ_n^k may be double for $k \in \mathbb{Z}/2$, so the corresponding waves are oscillating with the same frequency. The homogenized model is thus derived for pairs of the fibers $\{-k,k\}$ if $k \neq 0$ and for the fiber $\{0\}$ otherwise for which the boundary conditions are derived.

For any fixed $K \in \mathbb{N}^*$, using the definition of the set L_K^* of fibers introduced in [1] and $Y_K = KY$, the weak limit $U(t,\tau,x,y)$ of $\sum_{k\in L_K^*} W_k^\varepsilon U^\varepsilon$ in $L^2(I\times\Lambda\times\Omega\times Y_K)^2$ can be decomposed as

$$U(t,\tau,x,y) = U_H(t,x,y) + \sum_{k\in L_K^*}\sum_{n\in\mathbb{Z}^*} U_n^k(t,x)\, e^{2i\pi s_n\tau} e_n^k(y). \tag{2}$$

The term U_H is the low frequency part. The other terms represent the high frequency waves where e_n^k is a Bloch mode and U_n^k is a solution of a system of macroscopic equations which boundary conditions constitute one of the main contributions of this work. We deduce an approximation of the physical solution,

$$U^\varepsilon(t,x) \approx U_H\left(t,x,\frac{x}{\varepsilon}\right) + \sum_{k\in L_K^*}\sum_{n\in\mathbb{Z}^*} U_n^k(t,x)\, e^{is_n\sqrt{\lambda_{|n|}^k}t/\varepsilon} e_n^k\left(\frac{x}{\varepsilon}\right). \tag{3}$$

The figures below represent numerical results.

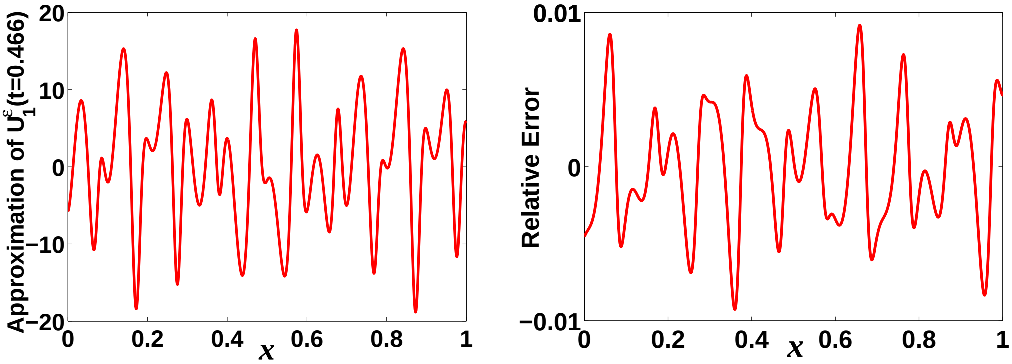

Figure 1: *Left: Space distribution of two-scale approximation of the first component U_1^ε of U^ε at t=0.466. Right: Relative error between the physical solution U_1^ε and its approximation in $L^2(\Omega)$ - norm is 7e-3.*

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