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PROGRESSIVE CONSTRUCTION OF REDUCED TENSOR SPACES FOR HIGH-DIMENSIONAL APPROXIMATION

L. Giraldi[1], A. Nouy[2]

LUNAM Université, GeM, UMR CNRS 6183,
Ecole Centrale de Nantes, Université de Nantes, France,
[1]Loic.Giraldi@ec-nantes.fr
[2]Anthony.Nouy@ec-nantes.fr

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We are interested in finding the solution of equations in tensor spaces that arise from the discretization of stochastic or parametric equations. We consider in particular problems that can be formulated as the minimization of a certain distance to the solution:

$$\min_{v\in\mathcal{V}} \mathcal{E}(v,u),$$

where $\mathcal{V} = \bigotimes_{\mu=1}^{d} \mathcal{V}^\mu$ is the tensor product of finite dimensional space $\mathcal{V}^\mu$. For example, $\mathcal{E}(u,v)$ can be some norm of the (preconditioned) residual of the initial equation. When the dimension d is large, this minimization problem is not tractable since the dimension of the tensor space $\mathcal{V}$ grows exponentially with d.

In order to reduce the complexity, several methods have been proposed that exploit possible low-rank structures of the solution (see recent surveys [1, 3] and monograph [4]). They consist in seeking a structured approximation of the solution in low-dimensional low-rank subsets $\mathcal{M} \subset \mathcal{V}$. Of particular interest are the subspace-based low-rank subsets, such as (Hierarchical or Tree-based) Tucker tensors with bounded rank, which are parametrized by reduced spaces. For example, the set $\mathcal{T}_{\mathbf{r}}(\mathcal{V})$ of Tucker tensors with multilinear rank bounded by $\mathbf{r} = (r_1, \dots, r_d)$ is defined by

$$\mathcal{T}_{\mathbf{r}} = \left\{ v \in \bigotimes_{\mu=1}^{d} \mathcal{U}^\mu;\ \mathcal{U}^\mu \subset \mathcal{V}^\mu \text{ such that } \dim(\mathcal{U}^\mu) = r_\mu \right\}.$$

As a consequence, finding the best approximation of u in the set of Tucker tensors $\mathcal{T}_{\mathbf{r}}(\mathcal{V})$ is equivalent to finding the reduced subspaces $(\mathcal{U}^\mu)_{\mu=1}^{d}$ that are solutions to the problem

$$\min_{\mathcal{U}^1} \dots \min_{\mathcal{U}^d} \min_{v\in\bigotimes_{\mu=1}^{d}\mathcal{U}^\mu} \mathcal{E}(v,u) \quad \text{such that} \quad \dim(\mathcal{U}^\mu) = r_\mu.$$

The generalization to hierarchical Tucker tensors is

$$\min_{\mathcal{U}^1} \ldots \min_{\mathcal{U}^d} \min_{v \in \mathcal{M}} \mathcal{E}(v,u) \quad \text{such that} \quad \dim(\mathcal{U}^\mu) = r_\mu \text{ and } \mathcal{M} \subset \bigotimes_{\mu=1}^{d} \mathcal{U}^\mu,$$

where $\mathcal{M}$ is some low-rank hierarchical Tucker subset with moderate dimension.

We here propose an efficient, although suboptimal, approximation of the optimal low-rank approximation. It consists in the progressive construction of an increasing sequence of tensor spaces $\mathcal{U}_m = \bigotimes_{\mu=1}^{d} \mathcal{U}_m^\mu$, $m \geq 1$, defined by

$$\min_{\mathcal{U}_m^1} \ldots \min_{\mathcal{U}_m^d} \min_{v \in \mathcal{U}_m} \mathcal{E}(v,u) \quad \text{such that} \quad \mathcal{U}_{m-1}^\mu \subset \mathcal{U}_m^\mu, \tag{1}$$

with $\mathcal{U}_0 = 0$. Note that when $d = 2$ and $\mathcal{E}(v,u) = \|u - v\|^2$, with $\|\cdot\|$ the canonical induced norm, then the proposed progressive construction yields the classical Singular Value Decomposition of $u \in \mathcal{V}^1 \otimes \mathcal{V}^2$, where $\mathcal{U}_m^1$ (resp. $\mathcal{U}_m^2$) is the $m-$dimensional space spanned by the dominant left (resp. right) singular vectors of u.

In [2], we proposed a suboptimal construction of the sequence of spaces, consisting in computing at step m a rank-one correction $\bigotimes_{\mu=1}^{d} v_m^\mu$ of the approximation in $\mathcal{U}_{m-1}$ and then in defining the next space $\mathcal{U}_m^\mu$ as $\mathcal{U}_{m-1}^\mu + \text{span}\{v_m^\mu\}$. In this suboptimal construction, all subspaces are enriched at the same time.

Here, we rely on the definition (1) of the increasing sequence of reduced spaces and we propose a strategy for the selection of spaces $\mathcal{U}_m^\mu$ to be enriched at each iteration m. This strategy is able to capture a possible anisotropy of the solution.

The performance of the algorithms will be illustrated on numerical examples.

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