11th World Congress on Computational Mechanics (WCCM XI)
5th European Conference on Computational Mechanics (ECCM V)
6th European Conference on Computational Fluid Dynamics (ECFD VI)
July 20–25, 2014, Barcelona, Spain

GLOBAL FORMAT FOR NONLINEAR ENERGY-MOMENTUM CONSERVING TIME INTEGRATION

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Key words: *Time integration, Energy conservation, Nonlinear dynamics.*

Momentum-based time integration methods for dynamic systems have received extensive research attention over the last two decades, starting with the introduction of a particular format for representation of the internal forces, [1], and its extension to general potenrials by a secant representation [2, 3]. In spite of very attractive properties, e.g. in connection with constrained problems in multibody dynamics, these algorithms have remained largely within the academic community. A possible reason is that the standard format makes use of a special representation of the internal force, using the mean states of stress and strain gradient. This ties the method to a special format of the underlying equations, and requires special computation at the element level of the combined mean state properties. The present paper proposes a simple alternative representation of the internal force in terms the arithmetic mean value plus a correction in terms of the increment of the tangent stiffness matrix. The tangent stiffness matrix is typically part of the iteration process, and thus the present formulation is in global form and does not impose any need for special computations, nor any assumption about the form of the energy potential.

Let the equations of motion for a system with displacement vector $\mathbf{u}(t)$ be given by

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{g}(\mathbf{u}) \;=\; \mathbf{f}(t), \tag{1}$$

where $\mathbf{M}$ is the mass matrix, while $\mathbf{g}(\mathbf{u})$ and $\mathbf{f}(t)$ is the internal and external force vectors, respectively. The linear viscous damping matrix $\mathbf{C}$ is mainly included to emphasize the structure of the equations. A consistent algorithmic damping may be introduced as indicated in [4]. After introduction of the velocity $\mathbf{v} = \dot{\mathbf{u}}$ as an independent variable, integration of the corresponding state-space equations gives

$$\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u} \\ \Delta\mathbf{v} \end{bmatrix} + \begin{bmatrix} \int \mathbf{g}(\mathbf{u})\, dt \\ -\int \mathbf{M}\mathbf{v}\, dt \end{bmatrix} = \begin{bmatrix} \int \mathbf{f}\, dt \\ \mathbf{0} \end{bmatrix}. \tag{2}$$

A second order accurate algorithm is obtained by representing the velocity and force integrals by their arithmetic mean values $\bar{\mathbf{v}}$ and $\bar{\mathbf{f}}$, while the internal force integral is represented by its equivalent value $\mathbf{g}_*$. This gives the following discretized form of the equations,

$$\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u} \\ \Delta\mathbf{v} \end{bmatrix} + h \begin{bmatrix} \mathbf{g}_* \\ -\mathbf{M}\bar{\mathbf{v}} \end{bmatrix} = h \begin{bmatrix} \bar{\mathbf{f}} \\ \mathbf{0} \end{bmatrix}, \tag{3}$$

where h is the time increment. The energy balance equation is obtained by pre-multiplication with $[\mathbf{u}^T, \mathbf{v}^T]$. It has the form

$$\big[\, \tfrac{1}{2}\mathbf{v}^T\mathbf{M}\,\mathbf{v} + G(\mathbf{u}) \,\big]_n^{n+1} = \Delta\mathbf{u}^T\bar{\mathbf{f}} - \frac{1}{h}\Delta\mathbf{u}^T\mathbf{C}\,\Delta\mathbf{u}\,, \tag{4}$$

provided that the equivalent internal force satisfies the energy increment relation $\Delta\mathbf{u}^T\mathbf{g}_* = \Delta G(\mathbf{u})$. For an energy potential $G(\mathbf{u})$ in the form of a quartic polynomial in the displacement components $\mathbf{u}$ the equivalent internal force $\mathbf{g}_*$ can be expressed as

$$\mathbf{g}_{\rm q} = \tfrac{1}{2}\big[\, \mathbf{g}_{n+1} + \mathbf{g}_n \,\big] - \tfrac{1}{12}\Delta\mathbf{K}\,\Delta\mathbf{u}\,. \tag{5}$$

This result gives fourth-order accurate conservation of the energy for general energy potential, while full energy conservation is obtained by introducing the secant representation

$$\mathbf{g}_* = \mathbf{g}_{\rm q} + \frac{(G_{n+1} - G_n) - \Delta\mathbf{u}^T\mathbf{g}_{\rm q}}{\Delta\mathbf{u}^T\Delta\mathbf{u}}\Delta\mathbf{u}\,. \tag{6}$$

It is verified by direct substitution that this secant representation satisfies the energy increment condition. The last term vanishes for any fourth-degree energy potential.

The discrete equations (3) are easily arranged in a form where the unknown velocity $\mathbf{v}_{n+1}$ is eliminated, leaving a nonlinear equation in the displacement $\mathbf{u}_{n+1}$, followed by a simple substitution for $\mathbf{v}_{n+1}$. In the present format all quantities in the discretized equations of motion are given in global form, reducing the computational effort to that of a similar central difference form, while retaining full energy conservation.

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