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GALERKIN TIME DISCRETIZATION AND MIXED FINITE ELEMENT METHODS

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Numerical simulations of time dependent flow and transport processes in heterogeneous porous media are desirable in several fields of natural sciences and in a large number of branches of technology, for instance, in environmental engineering and hydrology, in oil and gas exploration and recovery or in material sciences and pharmaceutical technology. The accurate numerical approximation of such flow and transport phenomena continues to be a challenging task. The applicability and value of the mixed finite element method (MFEM) and its hybrid variant (MHFEM) have been demonstrated for a wide range of problems. While the discretization in space involves a significant set of complexities, the temporal approximation of transient flow and transport in porous media has received relatively little interest and have most often been limited to traditional non-adaptive low order methods. Rigorous studies of higher order time discretizations are still missing. The Galerkin method is a known approach to solve time dependent problems [2, 3, 4]. So far, this variational approach has been used rarely in practice despite of its significant advantages like a uniform space-time approach for theoretical analyses, the natural construction of higher order methods, the applicability of duality based a posteriori error estimation techniques [1] with automatic mesh adaptation and the provision of a framework suitable for variational multiscale methods. One reason for this might be the higher complexity of the resulting algebraic block matrix systems; cf. (1).

In this contribution we present and analyze variational space-time approximations of a prototype convection-diffusion-reaction model. For the discretization in space mixed finite element methods of Raviart-Thomas type are used. The temporal variable is discretized by at least A-stable continuous and discontinuous Galerkin methods. Stability and error analyses of the schemes as well as implementational issues are addressed. The numerical performance properties are illustrated by test problems of practical interest; cf. Fig. 1.

More precisely, for a nonstationary diffusion problem written in mixed form, we consider

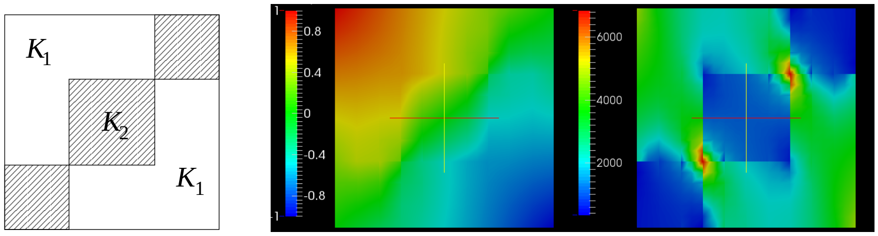

Figure 1: Layered anisotropic medium and computed concentration profile and flux magnitude (*from left to right*) for pure diffusion problem.

and study the following continuous variant of our variational time discretization schemes: *Find* $u_\tau \in \mathcal{X}^r(W)$ *and* $\boldsymbol{q}_\tau \in \mathcal{X}^r(\boldsymbol{V})$ *such that* $u(0) = u_0$ *and*

$$\boxed{\int_0^T \langle \partial_t u_\tau + \nabla \cdot \boldsymbol{q}_\tau, w_\tau \rangle \,\mathrm{d}t = \int_0^T \langle f, w_\tau \rangle \,\mathrm{d}t\,, \qquad \int_0^T \big\{ \langle \boldsymbol{D}^{-1} \boldsymbol{q}_\tau, \boldsymbol{v}_\tau \rangle - \langle u_\tau, \nabla \cdot \boldsymbol{v}_\tau \rangle \big\} \,\mathrm{d}t = 0}$$

for all $w_\tau \in \mathcal{Y}^{r-1}(W)$, $\boldsymbol{v}_\tau \in \mathcal{Y}^{r-1}(\boldsymbol{V})$.

Here, $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2(\Omega)}$. $\mathcal{X}^r(W)$, $\mathcal{Y}^{r-1}(\boldsymbol{V})$ denote spaces of piecewiese polynomial functions in time, $\mathcal{X}^r(X) := \{ u \in C(\bar{I}; X) \;\; \big| \;\; u|_{\bar{I}_n} \in \mathbb{P}_r(\bar{I}_n; X)\,,\ \forall n \}$ and $\mathcal{Y}^r(X) := \{ w \in L^2(I; X) \mid w|_{I_n} \in \mathbb{P}_r(I_n; X)\,,\ \forall n \}$, where $\mathbb{P}_r(J; X) := \big\{ u(t) = \sum_{j=0}^r \xi_n^j \, t^j\,,\ \xi_n^j \in X\,,\ \forall j \big\}$. A discontinuous counterpart of this semidiscretization in time is also studied. By an appropriate choice of test basis functions we recast the variational problem as a time marching scheme. Then, we apply the Gaussian quadrature rule to the integration in time and solve the resulting variational problem in a finite dimensional LBB-stable pair of mixed finite element spaces $W_h \subset W$ and $\boldsymbol{V}_h \subset \boldsymbol{V}$. For instance, in the case $r = 2$ this yields a block matrix system of the following structure that has to be solved for each of the time intervals $I_n = (t_{n-1}, t_n]$, $n = 1, \ldots, N$:

$$\boxed{\begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} & -\boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A} & \boldsymbol{0} & -\boldsymbol{B} \\ \frac{\tau_n}{2} \boldsymbol{B}^\top & \boldsymbol{0} & \hat{\alpha}_{1,1} \boldsymbol{G} & \hat{\alpha}_{1,2} \boldsymbol{G} \\ \boldsymbol{0} & \frac{\tau_n}{2} \boldsymbol{B}^\top & \hat{\alpha}_{2,1} \boldsymbol{G} & \hat{\alpha}_{2,2} \boldsymbol{G} \end{pmatrix} \begin{pmatrix} \boldsymbol{Q}_h^1 \\ \boldsymbol{Q}_h^2 \\ \boldsymbol{U}_h^1 \\ \boldsymbol{U}_h^2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \tilde{\boldsymbol{F}}^1 \\ \tilde{\boldsymbol{F}}^2 \end{pmatrix}} \tag{1}$$

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