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# Numerical analysis of pressure drop in steady stenotic flows by using Lorentz's reciprocal theorem

**ChangJin JI[1,*], Kazuyasu Sugiyama[2], Shigeho Noda[2], Ying He[1] and Ryutaro Himeno[2]**

[1] Department of Modern Mechanics, University of Science and Technology of China, China
[2] Advanced Center for Computing and Communication, RIKEN, Japan
[*] E-mail: jcj@mail.ustc.edu.cn

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The study of stenotic flow is very helpful in understanding the occurrence and development of many cardiovascular and cerebrovascular diseases, and thus is helpful in the diagnosis and treatment of these diseases. Our focus is on the issue how to systemically model the relation between the pressure drop and the flow rate in terms of the stenosis severity and the convective momentum transport.

Numerical simulations of simplified stenotic flows [Fig. 1] are performed by using a three-dimensional voxel-based simulator[1], and the results based on Lorentz's reciprocal theorem[2] and lubrication theory[3] are analyzed. The effect of stenosis severities with a range of [10%, 80%] and Reynolds numbers with a range of [1-1000], which includes most of the arteries and arterioles in human body, on the pressure drop is considered. We derive an identity accounting for pressure drop mechanism, and demonstrate its validity for a computation of a steady flow.

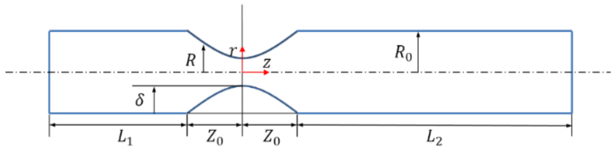


Fig.1 Simplified geometric model for stenosis vessel

**Pressure drop in stentotic flows**

If we call the solutions $\boldsymbol{u}$, $p$ as the NS (Navier-Stokes) solutions and $\boldsymbol{u}^*$, $p^*$ as the Stokes solutions, the Lorentz's reciprocal theorem[4] can be represented as

$$(\nabla \boldsymbol{u}) : \boldsymbol{\sigma}^* = (\nabla \boldsymbol{u}^*) : \boldsymbol{\sigma}. \tag{1}$$

By jointly deducing the Eq.(1) and the continuity and momentum equations of two flows, we arrive at an identical relation

$$\underbrace{\overbrace{\Delta P^* \mathrm{Q}}^{(W_1)} + \overbrace{\frac{2}{Re}\left(\iint n_i u_j D_{ij}{}^*\, dS\right)_{\partial\Omega_{in}\,\&\,\partial\Omega_{out}}}^{(W_2)}}_{(W_L)} =$$

$$\underbrace{\overbrace{\Delta P Q^*}^{(W_3)} + \overbrace{\int \left(u_i u_j D_{ij}{}^*\right) dV}^{(W_4)} + \overbrace{\frac{2}{Re}\left(\iint n_i u_j{}^* D_{ij}\, dS\right)_{\partial\Omega_{in}\,\&\,\partial\Omega_{out}}}^{(W_5)} - \overbrace{\left(\iint n_i \left(u_i u_j u_j{}^*\right) dS\right)_{\partial\Omega_{in}\,\&\,\partial\Omega_{out}}}^{(W_6)}}_{W_R}. \tag{2}$$

Note that in the Stokes flow, the pressure drop $\Delta P^*$ is linear with respect to the flow rate $Q^*$. When $\Delta P^*$is controlled to be $Q^* = Q$, the difference $\Delta P - \Delta P^*$ can be determined from the spatial distribution of the velocity $\boldsymbol{u}$, $\boldsymbol{u}^*$ as indicated in Eq. (2).

The computational results showed that t the values of $W_2$ and $W_5$, which are caused by the nonsymmetrical distribution of velocity at the two ends, are always much less than other terms, which is because that $D_{ij}^*$ are equal at both sides due to the symmetry of Stokes flow and are almost 0 in the axis direction. While the difference of velocities is much less than $Re$, thus $W_2$ can be ignored. The $W_6$ appears to be small at low Reynolds number, but increases significantly with the increase of Reynolds number when $Re$ is larger than 200. It is seen that $W_6$ is almost independent of the stenosis severity. Hence, when we set $Q$=$Q$*, Eq. (2) can be simplified as

$$\Delta P = \Delta P^* + \underbrace{\frac{1}{Q}\int\left(-u_i u_j D_{ij}^*\right) dV}_{\Delta P_V} + \underbrace{\frac{1}{Q}\left(\iint n_i\left(u_i u_j u_j^*\right) dS\right)_{\partial\Omega_{in}\ \&\ \partial\Omega_{out}}}_{\Delta P_S}, \tag{3}$$

which means the pressure drop of NS flow through a stenosis tube can be decomposed into three parts. The first part is the pressure drop of Stokes flow through a stenosis tube and is mainly caused by viscous effect. The second part named as $\Delta P_V$ is the volume integral of $-u_i u_j D_{ij}^*$ within the whole fluid field and the third part named as $\Delta P_S$ is the integral of $u_i u_j u_j^*$ at two ends of tube. The last two terms are both caused by the uneven distributions of velocity on the both sides of stenosis region due to the presence of convective term in the Navier-Stokes Equations. The Eq. (3) offers a new method to analyze the pressure drop in stenotic flows, and the effects of these terms will be analyzed in details.

**Stokes flow in stenosis tubes**

The pressure drop of Stokes flow through a moderate stenosis can be approximately represented as

$$\Delta P^* = \frac{1}{Re}\frac{8Q}{\pi}\int_0^L R(z)^{-4}\, dz. \tag{4}$$

The variation of pressure drop obtained from the lubrication theory and Simulation results show a good agreement at mild and moderate stenosis, however there appears relatively obvious difference between them at higher stenosis severity.

Therefore, by adding a factor expressed as a function of stenosis severity $St$, the lubrication equation (12) can be modified as

$$\Delta P^* = \frac{1}{1-0.1St^2}\frac{1}{Re}\frac{8Q}{\pi}\int_0^L R(z)^{-4}\, dz. \tag{5}$$

The modification reduces the error of pressure drop produced by lubrication theory from 6% to below 0.1%.

## REFERENCES

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