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RECONSTRUCTION OF ELASTIC MODULI FROM NOISY FULL-FIELD MEASUREMENTS

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The recent spread of non-invasive experimental techniques is associated with a variety of novel imaging modalities providing internal full-field measurements of the continuum deformations of (i) biological tissues using, e.g., ultrasound, magnetic resonance imaging or speckle interferometry, and (ii) materials by X-ray, neutron diffraction tomography or digital image correlation. The breakthrough that is the availability of such internal data has led to a paradigm shift in medical imaging and solid mechanics in that the constitutive parameters entering the partial differential equations modeling the behavior of elastic bodies can now be reconstructed given the knowledge of solutions of that PDE. The approach adopted in this study lies at the crossroads of the methodologies developed in the fields of experimental solid mechanics and hybrid inverse problems. It is similar in spirit to the so-called adjoint-weighted variational formulations in elasticity [1]. However, in the present study, advantage is taken of an explicit formulation of a gradient system for the sought moduli, which enables mathematical characterizations of the inversion stability and reconstruction uniqueness. This latter approach stems from the method introduced and discussed in [2] for scalar diffusion equations and which is therefore extended here to the tensorial framework of elasticity.

The inverse problem considered aims at the quantitative reconstruction of the spatially-dependent constitutive moduli, namely the two eigenvalues (α, β) of the elasticity tensor for linear and isotropic bodies. Assuming the availability of two measured displacement fields $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ within the domain of interest, a system of PDE's, whose solutions are the unknown parameters, is constructed by algebraic manipulations. Then, based on an investigated invertibility condition, these equations are explicitly recast into the following simple gradient system which provides the basis for the proposed inversion procedure

$$\begin{bmatrix} \boldsymbol{\nabla}\alpha \\ \boldsymbol{\nabla}\beta \end{bmatrix} + \mathbf{M}(\boldsymbol{u}_1, \boldsymbol{u}_2) \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \boldsymbol{f}(\boldsymbol{u}_1, \boldsymbol{u}_2) \tag{1}$$

Assuming that the data satisfy a number of compatibility conditions, which amount to characterizing an admissible noise, then this gradient system can be integrated using a conventional ordinary differential equation-based approach. This methodology yields a local uniqueness and stability result. Next, it is shown that this approach is no longer applicable when the gradient system features non-admissible noisy data. Assuming the only availability of such polluted measurements, an alternative inversion procedure is implemented based on a weak formulation of the normal equation associated with the system (1). The existence and uniqueness of a corresponding weak solution is finally shown.
The terms $\mathbf{M}$ and $\boldsymbol{f}$ in (1) feature the strain and hessian tensors of measured displacement fields, some de-regularizing terms, that are clearly strongly detrimental to the inversion in the presence of noise. Therefore, it is crucial to characterize the stability of the reconstruction procedure for noisy configurations. Driven by this idea, we propose and analyze a numerical differentiation scheme associated with a prior regularization step based on a L^2-projection of the measurements on coarse, yet high-order, finite element spaces. It is shown how an optimally stable reconstruction of the constitutive moduli can be achieved by an appropriate design of the regularizing operator. A set of numerical results is finally included to assess and highlight the features of the proposed approach.

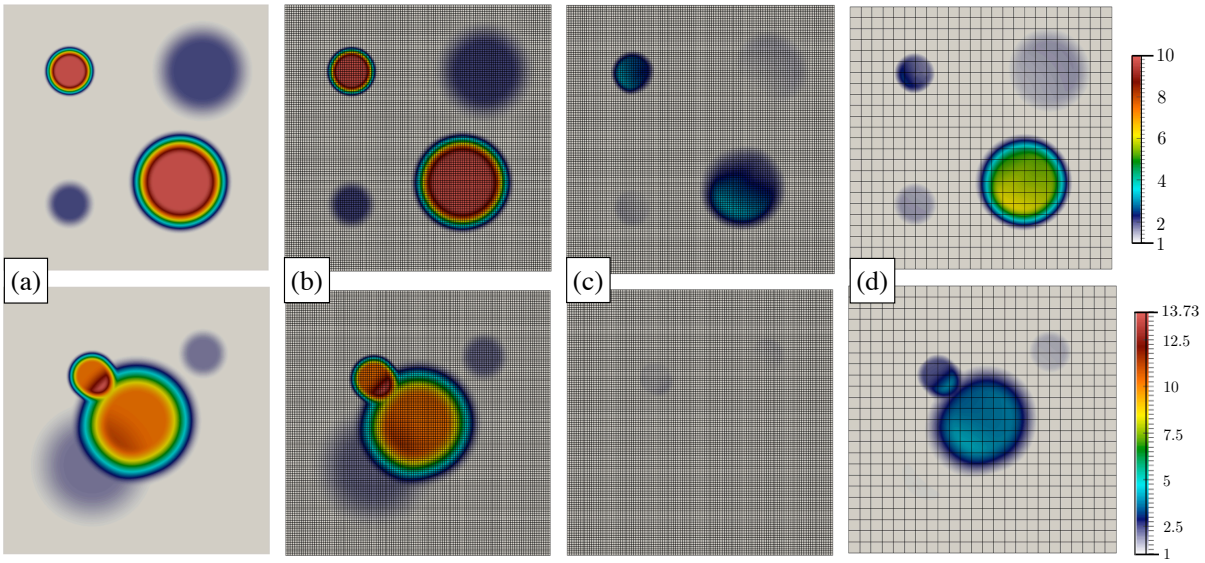

Figure 1: (a) Exact values of $\alpha(\boldsymbol{x})$ *(top)* and $\beta(\boldsymbol{x})$ *(bottom)* in 2D; Corresponding reconstructions with (b) no noise nor regularization, (c) with noise but no regularization, (d) with noise and regularization.

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