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ON THE SCALABILITY OF BDDC-BASED FAST PARALLEL ITERATIVE SOLVERS FOR THE DISCRETE STOKES PROBLEM WITH CONTINUOUS PRESSURES

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The stabilized finite element (FE) discretization of the stationary Stokes equations leads to a *symmetric indefinite* linear system with the following saddle-point structure:

$$\begin{pmatrix} \mathbf{A} & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ g \end{pmatrix}, \tag{1}$$

where $\mathbf{A}$ is a vector Laplacian matrix, C is a stabilization matrix ($C = 0$ for inf-sup stable elements), and $\mathbf{u}$ and p are the velocity and pressure field unknowns of the problem. We can consider that both $\mathbf{A}$ and C are *symmetric and positive definite*. In this work we are interested in effective preconditioners for this problem when continuous FE pressure spaces are used.

We comparatively evaluate two different approaches for the fast parallel iterative solution of (1) on current multicore-based distributed-memory machines. Both approaches ground on the Balancing Domain Decomposition by Constraints (BDDC) preconditioning approach [1], in particular as implemented in [2], for both the extraction of a high degree of parallelism and preconditioning optimality, although in a fairly different way.

Approach #1: BDDC-based monolithic preconditioning. This approach applies the BDDC preconditioning approach to the monolithic discrete Stokes problem (1). For elliptic operators, such as the Laplacian or linear elasticity, it is well known mathematically that these methods are quasi-optimal (algorithmically scalable). In particular, under weak scaling scenarios, they have the very desirable property of keeping the number of PCG iterations (asymptotically) constant with fixed local problem size and increasing number of subdomains. However, for indefinite problems, such as the Stokes problem, there is no mathematical theory that supports such assertion for continuous pressure spaces. Despite this fact, it is possible to mathematically proof that the discrete problem

underlying the BDDC preconditioner is well-posed, both for inf-sup stable and stabilized FEs with continuous pressure spaces [3], so that the method is at least applicable to the monolithic Stokes problem. Indeed, in [4], promising results were obtained when applying this approach to the FE mixed discretization of the Stokes problem using Taylor-Hood FEs. The experimental study is extended to the usage of approximate variants of the BDDC preconditioner where the local saddle-point Neumann/Dirichlet problems and/or the coarse-grid problem are solved only approximately by means of block preconditioned Krylov subspace iterative methods.

Approach #2: BDDC-based block preconditioning. It is well-known [5] that it is possible to derive optimal preconditioners using a block diagonal preconditioner with the inverse of $\mathbf{A}$ and that of the pressure Schur complement, $S = -C - B\mathbf{A}^{-1}B^T$, on the diagonal blocks; the preconditioned matrix has only three distinct eigenvalues [5]. Preconditioners in block LU factored form can be considered as well. Although these preconditioners cannot be used in practice in this form (due to the high computational/memory demands involved in the direct solution of linear systems with $\mathbf{A}$ and S), h-independent (optimal) convergence rates may still be achieved, replacing the preconditioner block-diagonal, i.e., replacing $\mathrm{diag}(\mathbf{A}, S)$, by a spectrally equivalent matrix $\mathrm{diag}(\tilde{\mathbf{A}}, \tilde{S})$. On one hand, $\mathbf{A}$ is a vector-Laplacian-type matrix that can be approximated by its corresponding (optimal and scalable) BDDC preconditioner (in exact or approximate form), whereas S is spectrally equivalent to a diagonal matrix (that can be efficiently inverted in parallel).

The contribution will comprehensively evaluate the weak scalability of both approaches on current multicore-based distributed-memory machines, in terms of both number of iterations and computational time. The main purpose of this study is to answer how far can these methods go in the number of cores and the scale of the problem in practical scenarios to still be within reasonable ranges of parallel efficiency, and how does both approaches (and their variants) compare to each other.

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