

# A FINITE ELEMENT EXTERIOR CALCULUS FRAMEWORK FOR THE ROTATING SHALLOW WATER EQUATIONS

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We describe discretisations of the shallow water equations on the sphere using the framework of finite element exterior calculus. The formulation can be viewed as an extension of the classical staggered C-grid energy-entropy conserving [1] and energy-conserving/entropy-dissipating [2] schemes which were defined on latitude-longitude grids. This work is motivated by the need to use pseudo-uniform grids on the sphere (such as an icosahedral grid or a cube grid) in order to achieve good scaling on massively parallel computers, and forms part of the multi-institutional UK “Gung Ho” project which aims to design a next generation dynamical core for the Met Office Unified Model climate and weather prediction system. The rotating shallow water equations are a single layer model that is used to benchmark the horizontal component of numerical schemes for weather prediction models.

We show, within the finite element exterior calculus framework [3], that it is possible to build numerical schemes with horizontal velocity and layer depth that have a conserved diagnostic potential vorticity field, by making use of the geometric properties of the scheme. The schemes also conserve energy and entropy, which arise naturally as conserved quantities out of a Poisson bracket formulation. We show that it is possible to modify the discretisation, motivated by physical considerations, so that entropy is dissipated, either by using the Anticipated Potential Vorticity Method as used in [2], or by inducing stabilised advection schemes for potential vorticity such as SUPG [4] or higher-order Taylor-Galerkin schemes. We illustrate our results with convergence tests and numerical experiments obtained from a FEniCS implementation on the sphere.

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