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# A NITSCHE FINITE ELEMENT METHOD FOR DYNAMIC CONTACT

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We present a new approximation of elastodynamic frictionless contact problems based both on the finite element method and on an adaptation of Nitsche's method which was initially designed for Dirichlet's condition [4]. The corresponding space semi-discretized weak form reads:

$$\begin{cases} \text{Find a displacement } \mathbf{u}^h : [0,T] \to \mathbf{V}^h \text{ such that for } t \in [0,T] : \\ \langle \rho \ddot{\mathbf{u}}^h(t), \mathbf{v}^h \rangle + A_{\Theta\gamma_h}(\mathbf{u}^h(t), \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma_h} [P_{\gamma_h}(\mathbf{u}^h(t))]_+ P_{\Theta\gamma_h}(\mathbf{v}^h)\, d\Gamma \quad = \quad L(t)(\mathbf{v}^h), \\ \hfill \forall\, \mathbf{v}^h \in \mathbf{V}^h, \\ \mathbf{u}^h(0,\cdot) = \mathbf{u}_0^h, \qquad \dot{\mathbf{u}}^h(0,\cdot) = \dot{\mathbf{u}}_0^h. \end{cases} \tag{1}$$

In the above formulation $\mathbf{V}^h$ is a finite element space built from standard Lagrange finite elements, piecewise linear or quadratic, $A_{\Theta\gamma_h}(\mathbf{u}^h, \mathbf{v}^h) := \int_\Omega \boldsymbol{\sigma}(\mathbf{u}^h) : \boldsymbol{\varepsilon}(\mathbf{v}^h)\, d\Omega - \displaystyle\int_{\Gamma_C} \Theta\gamma_h\, \sigma_n(\mathbf{u}^h)\sigma_n(\mathbf{v}^h)\, d\Gamma$ and $P_{\Theta\gamma_h}(\mathbf{v}^h) := v_n^h - \Theta\gamma_h\, \sigma_n(\mathbf{v}^h)$. The linear form $L(\cdot)$ stands for prescribed body and boundary forces. The domain of the elastic body is denoted by $\Omega$ and the contact boundary by $\Gamma_C$ ; $T$ is the final time of simulation ; $\rho$ is the density of the elastic material ; the notation $v_n^h$ stands for the normal component on $\Gamma_C$ of $\mathbf{v}^h$, and $\sigma_n(\mathbf{v}^h)$ is the normal stress on $\Gamma_C$ ; $\mathbf{u}_0^h$ (resp. $\dot{\mathbf{u}}_0^h$) is an approximation of the initial displacement $\mathbf{u}_0$ (resp. the initial velocity $\dot{\mathbf{u}}_0$). The notation $[\cdot]_+$ stands for the positive part of a scalar quantity, and $\langle \cdot, \cdot \rangle$ stands for the $L^2(\Omega)$ inner product. The parameter $\gamma_h$

is a positive piecewise constant function on the contact interface $\Gamma_C$ ($\gamma_h(x) = \gamma_0 h_T(x)$, where $\gamma_0$ is a positive constant and $h_T(x)$ is the size of the mesh element $T$). Note the additional numerical parameter $\Theta$ which can be freely chosen in $\mathbb{R}$. As in Nitsche's method for (static) unilateral contact, values of interest for $\Theta$ are $-1, 0, 1$ [1].

A main interesting characteristic is that this approximation produces well-posed space semi-discretizations, for any value of $\gamma_0 > 0$, contrary to standard finite element discretizations. We study associated energy conservation properties and manage to prove that :

$$\frac{d}{dt} E^h_\Theta(t) = (\Theta - 1) \int_{\Gamma_C} \frac{1}{\gamma_h} \left[ P_{\gamma_h}(\mathbf{u}^h(t)) \right]_+ \dot{u}^h_n(t) \, d\Gamma$$

where $E^h_\Theta(t) := E^h(t) - \Theta R^h(t)$ is an augmented energy associated to the semi-discrete solution $\mathbf{u}^h(t)$. The mechanical energy is given by $E^h(t) := \frac{1}{2} \left[ \rho \|\dot{\mathbf{u}}^h(t)\|^2_{0,\Omega} + \frac{1}{2} a(\mathbf{u}^h(t), \mathbf{u}^h(t)) \right]$ and $R^h(t) := \frac{1}{2} \left[ \|\gamma_h{}^{\frac{1}{2}} \sigma_n(\mathbf{u}^h(t))\|^2_{0,\Gamma_C} - \|\gamma_h{}^{-\frac{1}{2}} [P_{\gamma_h}(\mathbf{u}^h(t))]_+\|^2_{0,\Gamma_C} \right]$ is an extra term, which represents, roughly speaking, the non-fulfillment of the contact conditions at the semi-discrete level. Note in particular that the symmetric variant ($\Theta = 1$) conserves the discrete augmented energy $E^h_\Theta$.

Various time-discretizations for (1) are then considered, with the families of $\theta$-schemes and Newmark schemes. We also introduce a new hybrid scheme which is second-order accurate and numerically stable without any restriction on the time-step in the case $\Theta = 1$. This new scheme is inspired from [2, 3], and introduces much less dissipation than unconditionally stable variants of $\theta$-schemes and Newmark schemes. We study theoretically the well-posedness of each discrete scheme as well as its energy conservation properties. We finally achieve the corresponding numerical experiments.

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