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TIME-AVERAGED SHALLOW WATER EQUATIONS BY ASYMPTOTIC ANALYSIS

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As it is well known, the equations governing the behavior of a fluid are the Navier-Stokes equations. Due to their strong nonlinearity, high frequency oscillations are produced when the Reynolds number is increased, and the flow becomes unstable and turbulent. When we approximate Naiver-Stokes equations using a shallow water model, if the flow is turbulent, a very small time step must be chosen.

The objective of this paper is to derive, from Navier-Stokes equations, a new bidimensional shallow water model able to reduce these oscillations and then able to achieve good results with larger time steps. Filtering has given good results when working with turbulent Navier-Stokes equations (see [3]) and asymptotic analysis has been applied successfully to obtain and justify shallow water models (see [1], [2]). With this aim, Navier-Stokes equations are time-averaged in the following way:

$$\bar{U}(t,x,y,z;\eta T_c) = \frac{1}{2\eta T_c}\int_{t-\eta T_c}^{t+\eta T_c} U(r,x,y,z)dr \tag{1}$$

where T_C is a characteristic time and η is a non dimensional small parameter. Then we use asymptotic analysis techniques (where the small parameter considered is the quotient between the characteristic depth and the diameter of the domain). The model obtained in this way is:

$$\frac{\partial H}{\partial t} + \nabla\cdot\left[H\left(\vec{\bar{U}} - \frac{\eta^2 T_C^2}{6}\frac{\partial^2\vec{\bar{U}}}{\partial(t)^2}\right)\right] = 0 \tag{2}$$

$$
\begin{aligned}
&\frac{\partial \vec{\bar{U}}}{\partial t} + \nabla(\vec{\bar{U}}) \cdot \vec{\bar{U}} + \frac{\eta^2 T_C^2}{3} \nabla \left(\frac{\partial \vec{\bar{U}}}{\partial t} \right) \cdot \frac{\partial \vec{\bar{U}}}{\partial t} - \nu \Delta(\vec{\bar{U}}) - 3\nu \nabla \left(\nabla \cdot \vec{\bar{U}} \right) \\
&\quad - \frac{\nu}{H} \left\{ \bar{R} \cdot \nabla(H) + \frac{\eta^2 T_C^2}{6} \left[2 \frac{\partial \bar{R}}{\partial t} \cdot \nabla \left(\frac{\partial H}{\partial t} \right) + \bar{R} \cdot \nabla \left(\frac{\partial^2 H}{\partial (t)^2} \right) \right] \right\} \\
&\quad = -\frac{1}{\rho_0} \nabla(\bar{P}_s) + 2\Phi \left(\sin \varphi \right) \begin{pmatrix} \bar{V} \\ -\bar{U} \end{pmatrix} + \frac{1}{\rho_0 H} \left(\vec{\bar{F}}_W + \vec{\bar{F}}_R \right) \\
&\quad + \left(2\Phi (\cos \varphi) \bar{U} - g \right) \left[\nabla(S) + \frac{\eta^2 T_C^2}{3H} \frac{\partial H}{\partial t} \nabla \left(\frac{\partial H}{\partial t} \right) \right] \\
&\quad + 2\Phi (\cos \varphi) \left[\frac{1}{2} H \nabla(\bar{U}) + \begin{pmatrix} -\vec{\bar{U}} \cdot \nabla(B) + \frac{1}{2} H \nabla \cdot \vec{\bar{U}} \\ 0 \end{pmatrix} \right]
\end{aligned} \tag{3}
$$

where H is the length of the water column, B is the bathymetry, $S = H + B$ is the free surface, $\vec{\bar{U}} = (\bar{U}, \bar{V})$ is the time-averaged horizontal velocity, $\bar{P}_s$ is the time-averaged atmospheric pressure at the surface, g is the gravitational acceleration (assumed constant), $\Phi = 7.29 \times 10^{-5}$ rad/s, φ is the North latitude, $\vec{\bar{F}}_W$, $\vec{\bar{F}}_R$ are the wind and friction forces and

$$
\bar{R} = \begin{pmatrix} 4\dfrac{\partial \bar{U}}{\partial x} + 2\dfrac{\partial \bar{V}}{\partial y} & \dfrac{\partial \bar{U}}{\partial y} + \dfrac{\partial \bar{V}}{\partial x} \\ \dfrac{\partial \bar{U}}{\partial y} + \dfrac{\partial \bar{V}}{\partial x} & 2\dfrac{\partial \bar{U}}{\partial x} + 4\dfrac{\partial \bar{V}}{\partial y} \end{pmatrix}
$$

Numerical experiments confirm that this new model is able to obtain, in most of the cases, a given accuracy using time steps larger than the time step needed by classical shallow water models. In some cases the time step can be even a hundred times larger.

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