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WEAKLY NONLINEAR ANALYSIS OF SHALLOW MIXING LAYERS WITH VARIABLE FRICTION

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Methods of linear and weakly nonlinear stability theory are widely used for the analysis of shallow mixing layers. It is known that bottom friction in shallow water, where the horizontal length scale is significantly larger the flow depth, plays an important role for the development of instability. Hydraulic engineers often use Chezy, Manning or Darcy-Weisbach formulas to model the effect of bottom friction. These formulas contain a friction coefficient which depends on Reynolds number and relative roughness of the material that bounds the flow field. The solution of the linear stability problem can be used to determine the set of parameters of the problem for which the base flow becomes unstable to large scale perturbations in shallow flows. In particular, the critical value S_c of the dimensionless bed friction number S (which is proportional to the friction coefficient) can be computed so that the base flow is linearly stable for all $S > S_c$ and unstable if $S < S_c$. In accordance with the method of normal modes the unstable perturbation in this case is exponentially growing with respect to time (at least for short time until nonlinear terms in the equations of motion become important). However, if S is only slightly smaller that S_c, then the growth rate of the unstable perturbation is very small and one can try to analyze the development of instability analytically by constructing an amplitude evolution equation for the unstable mode using methods of weakly nonlinear theory. Such an approach was used in the past for the case of shallow wake flows with constant friction coefficient [1], [2].
In the present paper we perform linear and weakly nonlinear stability analysis of shallow mixing layers with variable friction coefficient. One important case of a situation where friction force varies in the transverse direction is related to shallow flows during floods where water flows through partially vegetated area [3] or in compound and composite channels [4]. In such cases the resistance force in the main channel is usually smaller than in the vegetated area of a composite channel and or in the shallower area of a compound channel. In this paper, the variability of the friction coefficient in the transverse direction is modeled by a smooth differentiable shape function. The basic steps of the procedure for deriving the weakly nonlinear stability equation for composite channels are outlined below.
The system of equations governing the flow is the mass and momentum equations for shallow water flows under the rigid-lid assumption. Introducing the stream function and eliminating

the pressure we transform the system to one equation for the stream function. The base flow solution is represented by a hyperbolic tangent function.
We assume that the bed friction number S is slightly smaller than the critical value, namely, $S = S_c(1-\varepsilon^2)$. Introducing slow time τ and longitudinal coordinate ξ by the relations $t = \varepsilon^2\tau,\ \xi = \varepsilon(x - c_g t),$ where c_g is the group velocity, representing the most unstable mode (in accordance with the linear theory) in the form

$$\psi(x, y, t, \xi, \tau) = A(\xi, \tau)\varphi(y)\exp[ik(x - ct],\qquad (1)$$

where $\varphi(y)$ is the eigenfunction of the linear stability problem, k is the wave number, c is the phase speed of the most unstable perturbation and $A(\xi,\tau)$ is unknown amplitude function, expanding the solution in powers of ε in the form

$$\psi(x, y, t) = \psi_0(y) + \varepsilon\psi_1(x, y, t) + \varepsilon^2\psi_2(x, y, t) + \varepsilon^3\psi_3(x, y, t) + ...$$

and collecting the terms containing the same powers of ε we obtain

$$L_1\psi_1 = 0,\quad L_1\psi_2 = f_1,\ \ L_1\psi_3 = f_2, ...\qquad (2)$$

The operator L_1 is the same at all orders of approximation. Substituting (1) into (2) and using zero boundary conditions at infinity we obtain the linear stability problem. Solving the linear stability problem numerically we obtain the critical values of the parameters of the problem: S_c, k_c and c_c. In order to solve the equations in (2) at orders 2 and 3, the Fredholm's alternative is used. Using the solvability condition at order two we obtain the group velocity c_g . The form of the function f_1 in (2) suggests the following representation of the solution for ψ_2:

$$\psi_2 = AA^*\varphi_2^{(0)}(y) + A_\xi\varphi_2^{(1)}(y)\exp[ik_c(x - c_c t)] + A^2\varphi_2^{(2)}(y)\exp[2ik_c(x - c_c t)].\qquad (3)$$

Substituting (3) into (2) and using zero boundary conditions at infinity we obtain three boundary value problems for the functions $\varphi_2^{(k)}(y),\ k = 0,1,2$, which are solved numerically.
Finally, substituting (1) and (3) into (2) and using solvability condition at order three we obtain the amplitude evolution equation for the most unstable mode in the form of the complex Ginzburg-Landau equation:

$$A_\tau = \sigma A + \delta A_{\xi\xi} - \mu |A|^2 A,\qquad (4)$$

where the complex coefficients σ, δ and μ are evaluated in closed form in terms of integrals containing the components of the solutions (1) and (3).
Numerical values of the coefficients σ, δ and μ are computed for different values of the parameters of the problem. Results of numerical simulations are presented.

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