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THE POYNTING TYPE EFFECT AND NON-HOMOGENEOUS RADIAL DEFORMATION IN THE PROBLEM OF TORSION OF HYPERELASTIC CIRCULAR CYLINDER

IGOR A. BRIGADNOV

National Mineral Resource University (University of Mines)
Line 21-2 V.O., St. Petersburg, 199106 Russian Federation
e-mail: brigadnov@mail.ru

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Abstract. The boundary value problem of torsion of solid cylinder is analyzed for a class of hyperelastic materials that exhibit the power law type dependence of the strain energy density on the magnitude of the deformation gradient. The Saint Venant hypotheses are generalized by including the non-homogeneous longitudinal and radial deformations. A non-linear variational problem with respect to the function of the radial/surface deformation, the function of the longitudinal deformation, the normalized torque and the normalized axial force is formulated. The asymptotic analytical solutions are obtained for the hard device torsion and for large angles of twist. They illustrate the power law type dependencies of the axial force and the reaction torque on the angle of twist with the exponents p and $p-1$, respectively. For Treloar (neo Hookean) materials with $p=2$ the classical results can be obtained. The finite element analysis of the hard device torsion is performed by Matlab. The results indicate that for a homogeneous class of materials and large angles of twist non-homogeneous radial/surface deformation can be observed.

1 INTRODUCTION

The classical Saint Venant's theory of torsion does not describe the well-known Poynting effect [1]-[5], that is the appearance of the axial force if the circular specimen is subjected to a twist in a hard device or the axial elongation in the case of soft loading by a torque. This effect can be analyzed within the non-linear theory of elasticity by the use of Rivlin's universal solution for pure shear deformation of a homogeneous and isotropic non-linear elastic material [2]. The results show that the axial force and the change in the length of a cylindrical specimen are proportional to the the square of the specific twisting angle (twisting angle over the unit length of the specimen) and agree well with experimental data [1], [5].

Not only the axial but also the radial deformation effects can be observed as the twisting becomes essential. The variational asymptotic solutions given in [6] indicate that the radial strain is proportional to the square of the twist and is non-linearly distributed over the radial coordinate for solid and hollow circular cylinders. However, the analysis presented in [6] is limited to linear-elastic materials. For large angles of twist, for non-linear elastic materials and for the hard device torsion both the radial and the axial strains are expected to be non-uniform with respect to axial coordinate such that ring-type wrinkles may arise on the surface of the specimen. Non-uniform radial deformation effect is not analyzed in the available theories of torsion [7]. Although cross section warpings are considered for prismatic solids, [7]-[9], these effects are not essential for cylinders and cannot be related to the radial deformation.

In this paper we recall governing equations of the non-linear theory of elasticity to study the torsion of circular solid bars. We generalize Saint Venant's hypotheses by including the non-homogeneous longitudinal and radial deformation. Applying the general variational principle we reduce a three-dimensional problem to the one-dimensional one with respect to the axial coordinate. The investigated variational functional includes two independent functions - the function of radial/surface deformation and the function of the longitudinal deformation and two independent variables - the normalized torque and the normalized axial force. We consider a class of non-linear elastic materials that exhibit the power law type dependence of the strain energy density on the magnitude of the deformation gradient. For different values of the exponent p several classical potentials can be considered. As an extension of the previous work to the non-linear torsion of circular solid bars we analyze the following problems

- According to the classical results the axial force or the change in the length of are related to the square of the specific twisting angle. The classical solution is limited for Treloar (neo Hookean) materials with $p = 2$. Below we address an analysis for a class of materials with $p > 1$. For hard device torsion derive an asymptotic formula that estimates the axial force and the reaction torque for large angles of twist.

- For hard device torsion we analyze the radial deformation effects. For $p = 2$, i.e. for Treloar (neo Hookean) materials we derive the second order essentially nonlinear differential equation for the radial deformation function. To study the general case we perform a numerical finite element analysis by Matlab.

Let us recall the basic equations of the non-linear theory of elasticity [8], [9]. Consider a solid that occupies a domain $\Omega \subset \mathbb{R}^3$ in the reference (undeformed) configuration. Every point $\mathbf{X}$ takes the position $\mathbf{x} = \mathbf{X} + \mathbf{u}$ in the actual (deformed) configuration, where $\mathbf{x}$ is the mapping and $\mathbf{u}$ is the displacement. The mappings are assumed to be invertible and orientation-preserving with the gradient $\mathbf{F} = (\nabla \mathbf{x})^T$ such, that $\det \mathbf{F} > 0$ in Ω, where $\nabla = \partial/\partial \mathbf{X}$ is the Hamilton operator and the superscript T denotes the transpose.

The finite deformation of elastic materials is characterized by the energetic pair $(\mathbf{F}, \mathbf{\Sigma})$ in the reference configuration, where $\mathbf{\Sigma}$ is the first (non-symmetric) Piola-Kirchhoff stress tensor.

An elastic material is described by a response function $\mathbf{\Sigma} = \hat{\mathbf{\Sigma}}(\mathbf{X}, \mathbf{F})$ [8], [9]. For a hyperelastic material the non-negative scalar potential W exists, such that $W(\mathbf{X}, \mathbf{I}) = 0$, where $\mathbf{I}$ is the second rank unit tensor, and $\hat{\mathbf{\Sigma}}(\mathbf{X}, \mathbf{F}) = \partial W(\mathbf{X}, \mathbf{F})/\partial \mathbf{F}^T$ for all tensors $\mathbf{F}$ and almost all points $\mathbf{X} \in \Omega$. For an isotropic material the potential is a function of principal invariants of a strain measure, for example the left Cauchy-Green tensor $\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T$ [8], [9], the Hencky logarithmic tensor $\mathbf{H} = \ln \sqrt{\mathbf{B}}$ [10], [11] or principal values of the left stretch tensor $\mathbf{\Lambda} = \mathbf{B}^{1/2}$. All descriptions are equivalent. The potential necessarily satisfies the material indifference principle [9]. For homogeneous materials $\hat{\mathbf{\Sigma}}, W = const(\mathbf{X})$.

For incompressible materials $\det \mathbf{F} = 1$, while for compressible materials $|\hat{\mathbf{\Sigma}}(\mathbf{X}, \mathbf{F})| \to \infty$, $W(\mathbf{X}, \mathbf{F}) \to +\infty$ if $\det \mathbf{F} \to +0$, i.e. to compress a volume to a point an infinite load and energy are required [8], [9]. Here and below $|\mathbf{F}| = (F_i^\alpha F_i^\alpha)^{1/2}$, the Latin subscripts refer to initial configuration, while the Greek superscripts refer to actual configuration, $i, \alpha = 1, 2, 3$. Here and furthermore the Einstein summation convention is applied.

2 BOUNDARY VALUE PROBLEM OF TORSION OF CYLINDRICAL BAR

Consider the following boundary value problem. The solid $\Omega \subset \mathbb{R}^3$ with the Lipshiz boundary is subjected to the following quasi-stationary loads: the volume force with the density $\mathbf{g}$ in Ω, the surface force with the density $\mathbf{P}$ on a part of the boundary Γ^2 and the displacement $\mathbf{u}_\gamma$ on the part of the boundary Γ^1, while $\Gamma^1 \cup \Gamma^2 = \partial\Omega$, $\Gamma^1 \cap \Gamma^2 = \emptyset$ and $area(\Gamma^1) > 0$.

As shown in [12] and [13] the weak solution of the boundary value problem for a hyperelastic solid is a displacement that provides a global minimum of the total energy functional

$$\mathbf{u}^* = \arg\inf\{I(\mathbf{u}) : \ \mathbf{u} \in V\} \ , \tag{1}$$

$$I(\mathbf{u}) = \int\limits_\Omega W(\mathbf{X}, \nabla\mathbf{u}(\mathbf{X}) + \mathbf{I}) \, d\Omega - A(\mathbf{u}) \ ,$$

$$A(\mathbf{u}) = \int\limits_\Omega \langle \mathbf{g}, \mathbf{u} \rangle(\mathbf{X}) \, d\Omega + \int\limits_{\Gamma^2} \langle \mathbf{P}, \mathbf{u} \rangle(\mathbf{X}) \, d\gamma \ ,$$

$$\langle \mathbf{Q}, \mathbf{u} \rangle(\mathbf{X}) = \int\limits_{\mathbf{X}}^{\mathbf{u}(\mathbf{X})} Q^\alpha(\mathbf{X}, \mathbf{v}) dv^\alpha \ ,$$

where $V = \{\mathbf{u} : \overline{\Omega} \to \mathbb{R}^3;\ \mathbf{u}(\mathbf{X}) = \mathbf{u}_\gamma(\mathbf{X}),\ \mathbf{X} \in \Gamma^1\}$ is a set of kinematically admissible displacements, $\langle *, \mathbf{u}\rangle$ and $A(\mathbf{u})$ is the specific and the total work of external forces for the displacement $\mathbf{u}$, respectively.

Let us place the origin of the coordinate system in the midpoint of the cross section in left edge of the cylindrical bar. Following [14] we consider mapping in normalized cylindrical coordinates of the reference state

$$\mathbf{x}(a\rho, \varphi, lz) = \mathbf{X}(ar, \varphi + \psi, lz + lw)\ ,$$

where $\rho \in [0, 1]$ is the radial coordinate, $\varphi \in [0, 2\pi)$ is the circumferential coordinate, $z \in [0, 1]$ is the axial coordinate, a is the radius of the cross section and l is the length of the bar.

Let us apply the semi-inverse Saint Venant's method with the following generalized hypotheses:

1) $\psi = \alpha z$, where α angle of cross-section rotation of the right edge (angle of twist),

2) $r = \rho f(z)$, where the function f describes the surface deformation,

3) $w = w(z)$ describes the longitudinal deformation

4) the cylindrical surface is load-free and the volume forces are absent,

5) the material is incompressible.

In what follows we will use the angle of twist α (*rad*), as well as specific angle of twist $\alpha_0 = \alpha/l$ (*rad/m*).

Let us recall that the Saint Venant theory of torsion is based on the same hypotheses with $f(z) \equiv 1$ and $w(z) \equiv 0$ [3]. In [7] and [10] the functions $f(z) = const(z)$ and $w(z) = Cz$ with $C = const$ are applied. The hypotheses formulated in this paper generalize the classical approach toward the consideration of the radial and the axial deformations while the basic (first) Saint Venant hypothesis of the plane cross sections is retained.

With the formulated hypotheses the components of the deformation gradient are

$$\mathbf{F}(f, w) = \begin{pmatrix} f & 0 & \eta\rho f' \\ 0 & f & \alpha\eta\rho f \\ 0 & 0 & 1 + w' \end{pmatrix}\ ,$$

where $\eta = a/l$ and $(\cdot)'$ denotes the derivative with respect to the coordinate z.

In the case of the soft device torsion (the torque is applied in the right edge) the boundary conditions for the introduced functions are: $f(0) = 1$, $f(1) = 1$ and $w(0) = 0$. For the hard device torsion the additional condition $w(1) = 0$ must be satisfied.

Without loss of generality let us consider the following potential [15]-[17]

$$W_p(\mathbf{F}) = 3\mu\, p^{-1} \left(\left|\mathbf{F}/\sqrt{3}\right|^p - 1 \right)\ , \qquad \det \mathbf{F} = 1\ , \tag{2}$$

where $\mu > 0$ is the constant shear modulus for infinitesimal strains. The basic feature of the potential (2) is the power law type dependence of the strain energy density on the magnitude of the deformation gradient with the exponent p.

Let us compare (2) with several well-known potentials of the elasticity theory. To this end we recall that the principal invariants of the left Cauchy-Green tensor are

$$I_1(\mathbf{B}) = \mathrm{tr}\mathbf{B} = |\mathbf{F}|^2 \ , \ I_2(\mathbf{B}) = \frac{1}{2}\left[(\mathrm{tr}\mathbf{B})^2 - \mathrm{tr}\left(\mathbf{B}^2\right)\right] \sim |\mathbf{F}|^4 \ ,$$

$$I_3(\mathbf{B}) = \det \mathbf{B} = (\det \mathbf{F})^2 \ .$$

For example, the very popular Mooney-Rivlin potential [8], [9]

$$W^{MR} = \frac{1}{2}\mu\left(\lambda(I_1 - 3) + (1-\lambda)(I_2 - 3)\right) \ ,$$

where $0 < \lambda \leq 1$ is a dimensionless parameter has only two fixed powers with respect to the magnitude of the deformation gradient, namely $W^{MR} \sim |\mathbf{F}|^4$ for $\lambda < 1$ and $W^{MR} \sim |\mathbf{F}|^2$ for $\lambda = 1$. For $p = 2$ the strain energy function (2) coincides with the Treloar or the neo-Hookean potential [15]. This is the special case of the Mooney-Rivlin model for $\lambda = 1$.

The detailed analysis for the parameter $p = 1$ is presented in author's works [16]-[23]. In this case the limit load (the torque and/or the axial force) exist above which the static equilibrium state is not possible. For such conditions the variational problem (1) provides either discontinuous solutions or no solution.

Below we analyze torsion for $p > 1$. As shown in [15] the range $1 < p < 2$ can be identified for such materials which exhibit large reversible strains up to 400-500 %. Examples include rubber type or foam materials operating in water, oil or liquid hydrocarbon medium.

With the assumed hypotheses and the potential (2) the variational functional (1) takes the following form

$$I_p(f,w) = \frac{3^{1-p/2}}{p}\int\limits_0^1\int\limits_0^1 \left(\eta^2\rho^2 f'^2 + (2+\alpha^2\eta^2\rho^2)f^2 + (1+w')^2\right)^{p/2} \rho\, d\rho dz$$

$$-D_\varphi\alpha - D_z w(1) \ ,$$

where $D_\varphi = M_z/(2\pi a^2 \mu\, l)$ is the normalized torque and $D_z = P_z/(2\pi a^2\mu)$ is the normalized axial force applied to the right edge of the bar. After the integration over the variable ρ we obtain the main variational problem with respect to the functions (f, w)

$$(f_*, w_*) = \arg\inf \{\, I_p(f,w) : \ (f,w) \in V \,\} \ , \tag{3}$$

$$I_p(f,w) = \frac{3^{1-p/2}}{p(p+2)\eta^2}\int\limits_0^1 \left[\left(\eta^2 f'^2 + (2+\alpha^2\eta^2)f^2 + (1+w')^2\right)^{p/2+1}\right.$$

$$\left. - \left(2f^2 + (1+w')^2\right)^{p/2+1}\right]\left(f'^2+\alpha^2 f^2\right)^{-1}\, dz - D_\varphi\alpha - D_z w(1)\ ,$$

$$V = \left\{ (f,w) \in \left(W^{1,p}(0,1)\right)^2 :\ f(0)=1,\ f(1)=1,\ w(0)=0 \right\}\ .$$

Let us show that for $p=1$ limit loads in the variational problem (3) exist. Indeed, let us consider a hard device torsion with $f(z)\equiv 1$ and $w(z)\equiv 0$. Then for $\alpha \gg 1$ the functional in (3) yields the asymptotic relation $I_1 \le \alpha\left(\frac{\sqrt{3}}{3}\eta - D_\varphi\right)$. Therefore, for $D_\varphi > D_\varphi^* = \frac{\sqrt{3}}{3}\eta$ the energy functional does not bounded below, since $I_1 \to -\infty$ as $\alpha \to \infty$. For a pure tension with $D_\varphi = 0$ and, for example, the following sequence $f_k(z) = (1+kz)^{-1/2}$, $w_k(z) = kz^2/2$ ($k \in \mathbb{N}$) the energy functional does not bounded below for $D_z > D_z^* = \frac{\sqrt{3}}{2}$.

The Poynting effect can be characterized by the following relationships [1]-[5]

1) $\alpha \mapsto D_z$ for torsion in a hard device with the given angle α and axially fixed edges, that is for $w(1)=0$;

2) $\alpha \mapsto w(1)$ for torsion in a soft device with the given torque D_φ without the axial force, i.e. for $D_z = 0$. To analyze this case the relationship $\alpha \mapsto D_\varphi$ must be found.

From the incompressibility condition $\det \mathbf{F} = f^2(1+w') = 1$ and the boundary condition $w(0)=0$ the following relations for the axial and the radial displacements can be obtained

$$w'(z) = f^{-2}(z) - 1\ , \qquad w(z) = \int\limits_0^z f^{-2}(q)\, dq - z\ . \tag{4}$$

3 ESTIMATION OF THE AXIAL FORCE FOR HARD DEVICE TORSION

To find the relationship between the axial force and the angle of twist let us apply the principle of virtual displacements. To this end we compute $\delta I_p(f,w) = 0$ for the virtual displacement $\delta w(1)$ [8], [9]. With the variational form of the incompressibility condition (4) $\delta f = -\frac{1}{2} f^3 \delta w'$ and after integration by parts we obtain

$$D_z = -\alpha^2 \Psi_p\left(\alpha, \eta, f'(1)\right)\ , \tag{5}$$

$$\Psi_p = \frac{3^{1-p/2}}{p(p+2)}\,\frac{\eta^2}{R}\left[\left(\frac{p}{2} - \frac{3}{R}\right)(R+3)^{p/2} + \frac{3^{1+p/2}}{R}\right]\ ,$$

where $R = \eta^2\left(\alpha^2 + f'^2(1)\right)$. The value $f'(1) = f'(1-0)$ must be computed by solving the variational problem (3) with the incompressibility condition (4) and the boundary condition $w(1) = 0$ in the form $\int\limits_0^1 f^{-2}(z)\,dz = 1$.

Assuming that the solution of the variational problem (3) is regular, i.e. the value $f'(1-0)$ remains limited for any angle of twist, the following asymptotic form of Eq. (5) for $\alpha \gg 1$ can be obtained

$$D_z \approx -k_z\,\alpha^p\,, \qquad k_z = \frac{3^{1-p/2}}{2(p+2)}\,\eta^p\,. \tag{6}$$

Equation (10) coincides with the relation $D_z \sim -\alpha^p$ found by [17] from numerical tests.

To find a closed form analytical expression for $\alpha \mapsto D_z$ for all values of p the solution of the non-linear variational problem (3) is required. However for $p = 2$, i.e. for the neo-Hookean material this expression can be found without the variational functional (3), as shown in [3], [8], [9]

$$D_z = -\tfrac{1}{8}\eta^2\,\alpha^2\,. \tag{7}$$

The relation (7) provides the estimation of the axial force for the neo-Hooke material for any angle of twist. However, for small strains all available models of the non-linear theory of elasticity can be reduced to the classical quadratic Hooke's potential [8], [9]. Therefore, for small angles of twist Eq. (7) is valid for any elastic material. Let us note that the minus sign in Eq. (7) shows that the axial force is compressible [2], [3], [5], [14].

Let us rewrite Eq. (7) for the axial force under small angles of twist

$$P_z = -k_z\,\alpha_0^2\,, \qquad k_z = \tfrac{1}{4}\mu\,\pi\,a^4\,. \tag{8}$$

As an example let us compute the axial force for the bar with the radius of the circular cross section $a = 5\cdot 10^{-3}$ (m), made from steel with the Young' modulus $E = 210$ (GPa), the Poisson's ratio $\nu = 0.282$ and the shear modulus $\mu = 0.5\,E/(1+\nu) \approx 82$ (GPa) at the room temperature. Equation (8) yields $k_z \approx 40.2$ $(N{\cdot}m^2/rad^2)$.

The reaction torque for a test in a hard device under small angles of twist $|\alpha_0| \ll 1$ is computed by $M_z = k_\varphi \alpha_0$, where $k_\varphi = \frac{1}{2}\,\mu\,\pi\,a^4 \approx 80.4$ $(N{\cdot}m^2/rad)$ [9]. For example, for $\alpha_0 = 10$ $(grad/m) \approx 0.175$ (rad/m) the axial force is $P_z \approx -1.22$ (N), while the torque takes the value $M_z \approx 14.1$ $(N{\cdot}m)$.

For small angles of twist the value of the axial force is negligible. This is the reason why the Poynting effect can only be observed in real and numerical tests only for large angles of twist [5], [14].

4 ESTIMATION OF THE REACTION TORQUE FOR HARD DEVICE TORSION

Let us consider $D_z = 0$ and an admissible deformation describing the incompressible pure torsion of the rod within the framework of the classical Saint-Venant model of plane cross-sections with $\psi = \alpha z$, $f(z) \equiv 1$ and $w \equiv 0$. In this case the energy functional has the following form

$$I_p = \frac{9}{\alpha^2\eta^2 p(p+2)} \left[\left(1 + \frac{\alpha^2\eta^2}{3} \right)^{p/2+1} - 1 \right] - \alpha\, D_\varphi + const\ .$$

The function I_p is convex with respect to α for $p > 1$. Therefore, from the necessary condition of stationarity $dI_p/d\alpha = 0$ the following relationship can be obtained

$$D_\varphi = \frac{3}{\alpha\, p} \left(1 + \frac{\alpha^2\eta^2}{3} \right)^{p/2} - \frac{18}{\alpha^3\eta^2 p(p+2)} \left[\left(1 + \frac{\alpha^2\eta^2}{3} \right)^{p/2+1} - 1 \right] . \tag{9}$$

For $\alpha \gg 1$ the asymptotic form of (9) is

$$D_\varphi \approx k_\varphi\, \alpha^{p-1}\ , \qquad k_\varphi = \frac{3^{1-p/2}}{p+2}\, \eta^p\ . \tag{10}$$

Equation (10) coincides with the relation $D_\varphi \sim \alpha^{p-1}$ found by [17] from numerical tests.

For $p = 2$ the relation (9) is linear because $D_\varphi(\alpha) = \alpha\, \eta^2/4$ and coincides with the classical formula of Saint-Venant's theory of torsion of prismatic rods [7]. Let us note that the relationship (9) is only approximate because the Poynting effect, i.e. change of the length is ignored.

5 ESTIMATION OF RADIAL DEFORMATION FOR HARD DEVICE TORSION

For the analysis of the radial deformation under hard device conditions, i.e. for the given angle α and the condition $w(1) = 0$, the variational problem (3) with the incompressibility condition $\int_0^1 f^{-2}(z)\, dz = 1$ must be solved. The incompressibility condition can be considered by the use of the Lagrange multiplier [9], [13]. As a result we have to find a stationary value of the following functional

$$(f_*, \lambda_*) = \arg\,\mathrm{stat}\, \{\, K_p(f, \lambda) :\ (f, \lambda) \in V \times \mathbb{R} \,\}\ , \tag{11}$$

$$K_p(f, \lambda) = \int_0^1 \frac{\left(\eta^2 f'^2 + (2 + \alpha^2\eta^2) f^2 + f^{-4}\right)^{p/2+1} - \left(2f^2 + f^{-4}\right)^{p/2+1}}{f'^2 + \alpha^2 f^2}\, dz +$$

$$+\lambda \int_0^1 \left(f^{-2} - 1 \right)\, dz\ ,$$

$$V = \left\{ f \in W^{1,p}(0,1),\ f^{-1} \in L^2(0,1) :\ f(0) = 1,\ f(1) = 1 \right\}\ .$$

For the parameter $p = 2$ (neo-Hooke's material) the necessary condition of the stationarity of $K_2(f, \lambda)$ on the manyfold $V \times \mathbb{R}$ is the following boundary value problem

$$f'' - \left(\omega^2 + \alpha^2\right) f + \omega^2 f^{-5} + \xi f^{-3} = 0 \ , \qquad (12)$$

$$f(0) = 1 \ , \qquad f(1) = 1 \ , \qquad \int_0^1 f^{-2}(z)\, dz = 1 \ ,$$

where $\omega = 2\,\eta^{-1}$ and α are given values and $\xi = \lambda\,\eta^{-4}$ has to be found. It is evident, that the trivial solution $f_*(z) \equiv 1$, $\lambda_* = \alpha^2\eta^4$ (i. e. $\xi_* = \alpha^2$) satisfies Eq. (12).

It is natural to assume, that for small angles of twist $|\alpha| \ll 1$ the radial deformations are small and can be represented by the function $f(z) = 1 + q(z)$, where $|q(z)| \ll 1$ is the solution of the following linearized boundary value problem

$$q'' - (6\,\omega^2 + 3\,\xi)q = -\xi \ ,$$

$$q(0) = 0 \ , \qquad q(1) = 0 \ , \qquad \int_0^1 q(z)\, dz = 0 \ .$$

The problem has infinite number of solutions. For $\xi = 0$ the solution is trivial $q(z) \equiv 0$, and for $\xi = -2\omega^2 - \frac{4}{3}x_k^2$ ($k \in \mathbb{N}$) we obtain

$$q_k(z) = \left(\frac{1}{3} + \frac{\omega^2}{2x_k^2}\right) \sin(2x_k z)\,(\tan(x_k z) - x_k) \ ,$$

where x_k are positive roots of the transcendent equation $x = \tan(x)$. We observe, that $x_k \approx (2k+1)\pi/2$ ($k \in \mathbb{N}$). The energy functional takes the global minimum value on the trivial solution.

For the general case the problem (11) is solved numerically applying the finite element method. To this end the interval $[0, 1]$ is divided by n line elements of the same length and the piecewise linear approximation of the unknown function is applied. As a result the problem (11) is reduced to the minimization of the non-linear function of $(n + 1)$ unknown nodal values of $f(z)$ with two linear constraints on the boundaries and one non-linear constraint due to the incompressibility.

The computations were performed by Matlab R2012a (version 7.14.0.739). The finite element mesh was created with $n = 100$. The numerical minimization with constraints was performed by the use of the Matlab function *fmincon* specifying the initial guess.

It was found that with the trivial guess $f^{(0)}(z) \equiv 1$ the constrained minimum of the functional coincides with the trivial solution for all input parameters of the problem. For the guess in the form $f^{(0)}(z) = 1 + 0.1 * \sin(\pi z)$ and for the parameters $p = 1.5$, $\alpha = 2\pi$ and $\eta = 0.05$ the constrained minimum is shown in Fig. 1. The non-uniform radial deformation with the amplitude not greater than 1,5 % of the cross section radius can be clearly observed. The splashes at the ends of the interval can be explained by the strong ravines of the functional and are the computational errors.

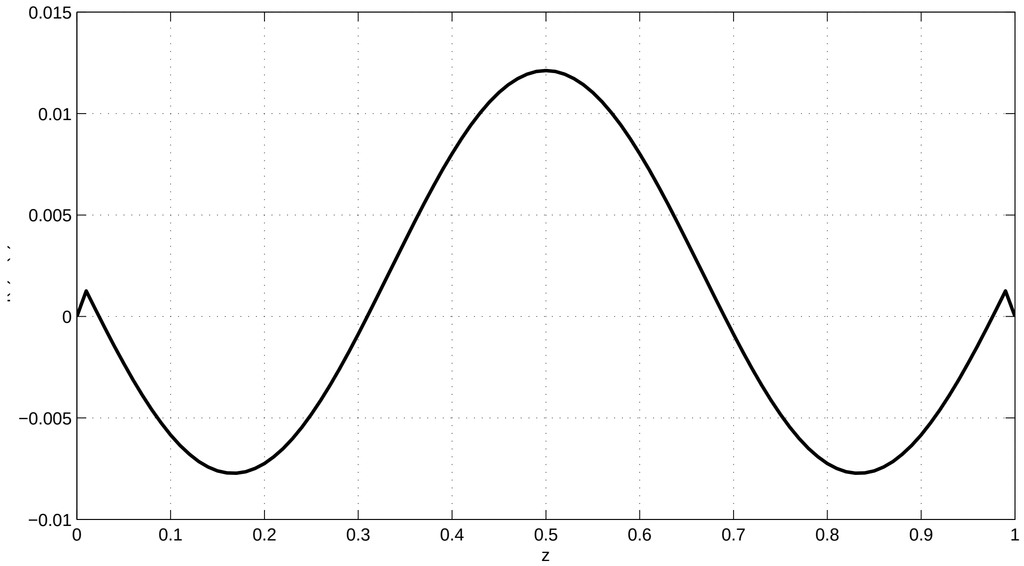

Figure 1: The radial deformation function $q(z) = f(z) - 1$ for parameters $p = 1.5$, $\alpha = 2\pi$ and $\eta = 0.05$.

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