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A SIMULATION OF CAT'S CRADLE BY GEOMETRICALLY NONLINEAR ANALYSIS WITH SLIDING NODES

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Abstract. This study proposes a new type of element which can treat frictionless contact by involving slide nodes. Authors have been used a geometrical nonlinear theory "the tangent stiffness method" which realizes sure and rapid convergence of unbalanced forces

1 INTRODUCTION

There already exist many studies about geometrically non-linear analysis of cable structures. Cables have sag under gravity, and do not have stiffness against compressional strain. In general, catenary cable elements[1] or truss elements with hyperbolic stiffness[2] to describe behaviour of cables. Furthermore, for example, in case of deformation analysis for catwalk nets of tensegrity structure and so on, contact behaviour between cables should be evaluated. Therefore, we have to consider two different types of nonlinearity simultaneously. Previously, there are some studies about contact and sliding of cable elements. (for example[3])

Authors have been investigated geometrical nonlinearity of soft structures such as thin frames cables, and membranes using the tangent stiffness method. The method has two major characteristics. One is that the unbalanced forces are calculated using the strict compatibility between nodal displacements and element edge deformations, and the other is that the geometrical stiffness is completely separated from elements' own stiffness. Therefore, converged solutions are in perfect equilibrium state and computational process is so rapid as well as Newton Raphson method. Namely, the method has a different idea from the general type of FEM, which has geometrical stiffness formulated form nonlinear strain and shape functions.

In this study, a new type of element which can treat frictionless contact involving slide nodes is proposed, and numerical examples simulate process of cat's cradle which is very popular game among kids as "AYATORI" also in Japan. Cat's cradle has so many complex procedures including contact, sliding and separation, and computation by general type of FEM will be very tough. Therefore, the success of cat' cradle simulation by the tangent stiffness method shall suggest the advantage of the method, and possibility of usage on the method for strong and complex nonlinear problem will be expected.

2 THEORY FOR GEOMETRICALLY NONLINEAR ANALYSIS

The tangent stiffness method uses the geometrical stiffness which is completely separated from elements' own stiffness

Let the vector of the element edge forces independent of each other be indicated by **S**, and let the matrix of equilibrium which relates **S** to the general coordinate system by **J**. Then the nodal forces **U** expressed in the general coordinate follow the equation:

$$\mathbf{U} = \mathbf{JS} \tag{1}$$

The tangent stiffness equation is expressed as the deferential calculus of Eq. (1),

$$\boldsymbol{\delta}\mathbf{U} = \mathbf{J}\boldsymbol{\delta}\mathbf{S} + \boldsymbol{\delta}\mathbf{JS} = (\mathbf{K_0} + \mathbf{K_G})\boldsymbol{\delta}\mathbf{u} \tag{2}$$

In which, $\mathbf{K_0}$ is the element stiffness which provide the element behavior in element (local) coordinate, and $\mathbf{K_G}$ is the tangent geometrical stiffness. $\boldsymbol{\delta}\mathbf{u}$ is nodal displacement vector in general coordinate.

3 CABLE ELEMTNTS WITH SLIDING NODES

3.1 Element force equation

Cable cannot resist compression axial-forces. To simulate this characteristic of "compression-free", there are two different approaches. One is to use the catenary cable element which considers the self-weight as the element's material property. However, the catenary cable has a disadvantage that the nonlinearity inside the element has to be considered when the contact between elements. The other is to assume the elongation stiffness obeys a hyperbolic function which is asymptotic to linear function of Hook's law in tensional direction(Figure1).

In the tension strain area the stiffness closes to E_1, and in the compression strain area the stiffness closes to E_2. Then the stress and the Young's modulus are indicated as following.

$$\sigma = \frac{-\sigma_1\varepsilon + \sigma_0\sqrt{\varepsilon^2 + \varepsilon_1^{\,2}}}{\varepsilon_1} \tag{3}$$

$$\sigma_1 = \frac{E_1 + E_2}{E_1 - E_2}\sigma_0 \ , \quad \varepsilon_1 = \frac{2\sigma_0}{E_1 - E_2} \tag{4),(5}$$

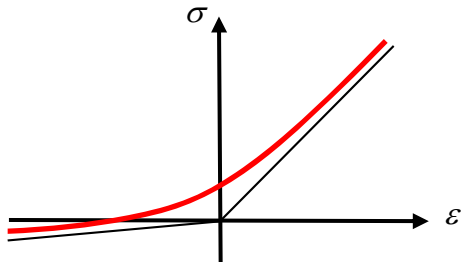


Figure1 Hyperbolic stiffness for compression-free

$$E(\varepsilon) = \frac{1}{\varepsilon_1}\left(\sigma_1 - \frac{\sigma_0 \varepsilon}{\sqrt{\varepsilon^2 + \varepsilon_1^{\,2}}}\right) \tag{6}$$

Therefore, both of σ and $E(\varepsilon)$ can be always calculated from current nodal positions explicitly.

3.2 Tangent Geometrical Stiffness

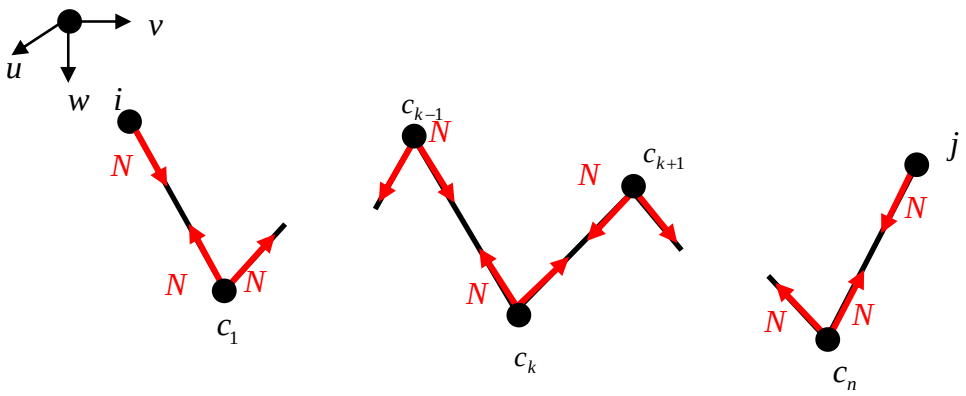


Figure2 Axial force element with plural sliding nodes

Figure2 shows an axial force element in which i and j are the both ends and c_1-c_n are the sliding nodes which are produced at the same time with contact. Let the nodal force at every node **U**, and cosine vector of a segment **α**, the equilibrium equation between **U** and axial force N is expressed as Eq.(9).

$$\mathbf{U}_{icj} = \begin{bmatrix} \mathbf{U}_i & \cdots & \mathbf{U}_{c_k} & \cdots & \mathbf{U}_j \end{bmatrix}^T \tag{7}$$

$$\boldsymbol{\alpha}_{icj} = \begin{bmatrix} -\boldsymbol{\alpha}_{\mathbf{ic_1}} & \boldsymbol{\alpha}_{\mathbf{ic_1}} - \boldsymbol{\alpha}_{\mathbf{c_1c_2}} & \boldsymbol{\alpha}_{\mathbf{c_{k-1}c_k}} - \boldsymbol{\alpha}_{\mathbf{c_kc_{k-1}}} & \boldsymbol{\alpha}_{\mathbf{c_{n-1}}} - \boldsymbol{\alpha}_{\mathbf{c_nj}} & \boldsymbol{\alpha}_{\mathbf{c_nj}} \end{bmatrix}^T \tag{8}$$

$$\mathbf{U}_{icj} = \boldsymbol{\alpha}_{icj} \cdot N \tag{9}$$

And when differentiating both sides of Eq.(9), the tangent stiffness equation becomes as following.

$$\delta \mathbf{U}_{icj} = \delta \boldsymbol{\alpha}_{icj} \cdot N + \boldsymbol{\alpha}_{icj} \delta N = (\mathbf{K}_0 + \mathbf{K}_G)\delta \mathbf{u} \tag{10}$$

$$\mathbf{K}_G = N \begin{bmatrix} \mathbf{k}_{i,1} & -\mathbf{k}_{i,1} & & & & \\ -\mathbf{k}_{i,1} & \mathbf{k}_{i,1} + \mathbf{k}_{2,3} & -\mathbf{k}_{2,3} & & & \\ & -\mathbf{k}_{2,3} & \mathbf{k}_{2,3} + \mathbf{k}_{3,4} & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \mathbf{k}_{n,j} + \mathbf{k}_{n-1,n} & -\mathbf{k}_{n,j} \\ & & & & -\mathbf{k}_{n,j} & \mathbf{k}_{n,j} \end{bmatrix} \tag{11}$$

$$\mathbf{k}_{\mathbf{k},\mathbf{k}+1} = \frac{1}{L_{k,k+1}} \left(\mathbf{e} + \boldsymbol{\alpha}_{\mathbf{k},\mathbf{k}+1}{}^{\mathrm{T}} \boldsymbol{\alpha}_{\mathbf{k},\mathbf{k}+1} \right) \tag{12}$$

3.3 Judgment of contact

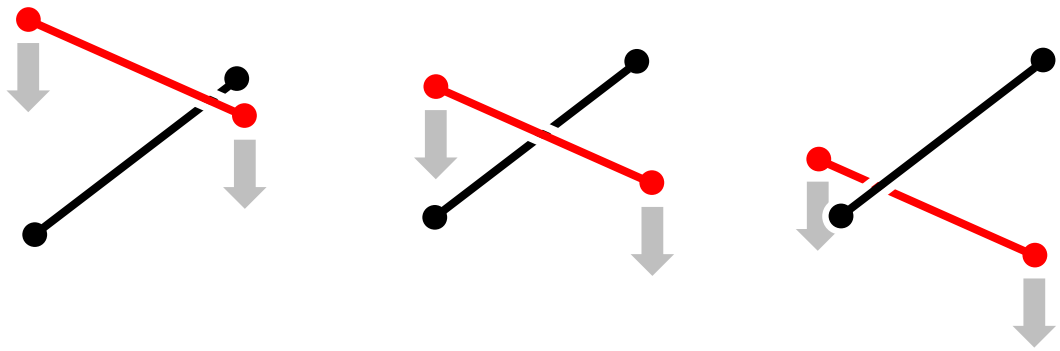


Figure3 Passing through between two elements

When the "passing through" between two elements have detected on the process of developed algorithm, it is assumed that contact has happened. Namely, the contact is judged when the sign of the scalar triple product whose four apexes are two couples of element ends have changed. After that, the sliding node as a contact point is created at the center of gravity of the four apexes, and the equilibrium position of the contact point should be calculated by the tangent stiffness method.

4 NUMERICAL EXAMPLES

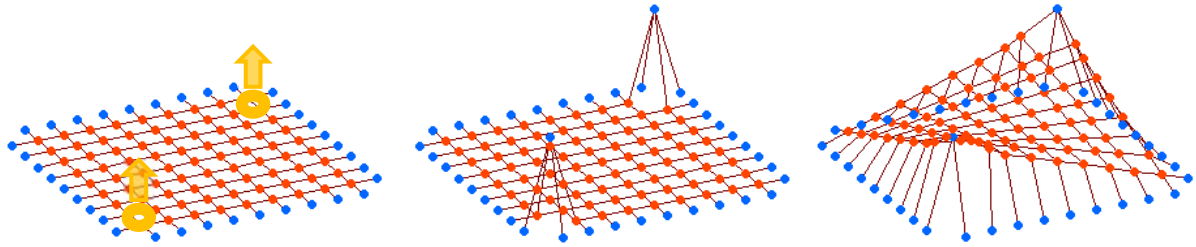
4.1 Verification of convergence property

Figure4(a) shows a primary shape of the example model where eighteen cables are located just like as net form in an alternation. Table1 shows the material and machanical properties of this model. The blue dots shows the fixed nodes. In primary state, contact has detected at all the nodes indicated by the red dots. 10[cm] of compulsory displacement has given to two nodes surrounded by yellow circles to the direction of arrow.

Then the final equilibrium solution has been obtained as Figure5 via unbalanced state of Figure4(b),(c). Figure6 shows the convergence process of unbalanced foreces, and surely and rapid convergence has been recognized even if so large displacement case with sliding nodes.

Table1 Materical and mechanical properties

Length of a cable	30[cm]
Area of cross section	1.0×10^{-4}[cm^2],
Young's modulus (tension)	3.0[GPa]
Young's modulus (tension)	1.0×10^{-9}[GPa]
Total number of nodes	117



(a) primary state (b) compulsory displacement (c) unbalanced state

Figure4 Deformation process of the model

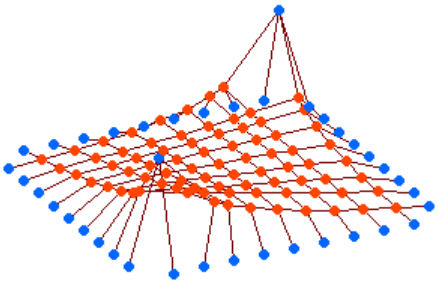


Figure5 Equilibrium shape

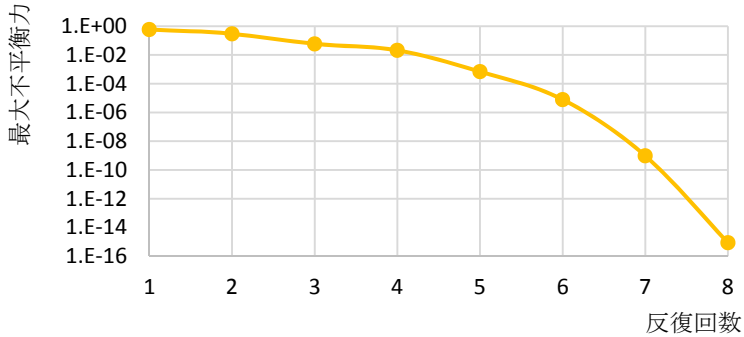


Figure6 Convergence process of unbalanced forces

4.2 Cat's cradle

As shown in Figure7, we start from the shape in imitation of "river". In this example, only one cable element connects node1 and node1 surrounding via sliding nodes of node2-8. Specifically, 1-2-3-4-1-2-5-6-7-8-5-6-7-9-3-4-1 is the connectivity of this cable element with the hyperbolic stiffness. Also, all the nodes have the elastic supports which simulate the passive

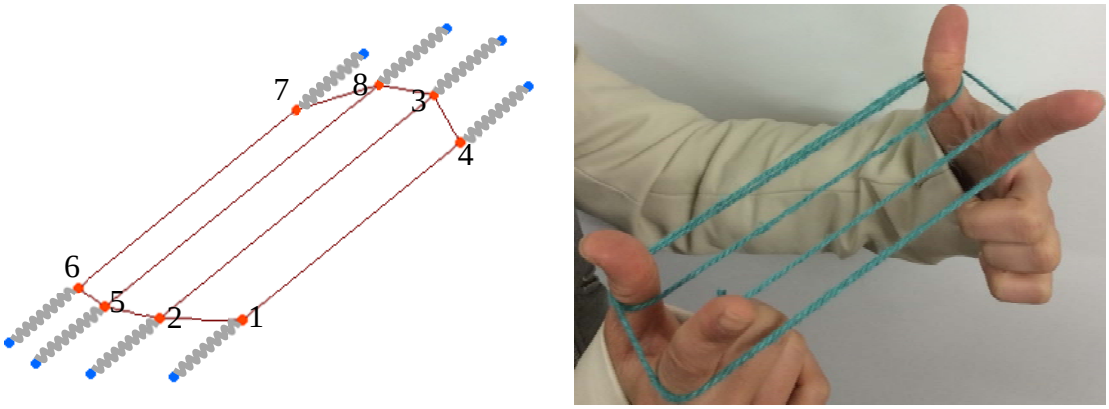


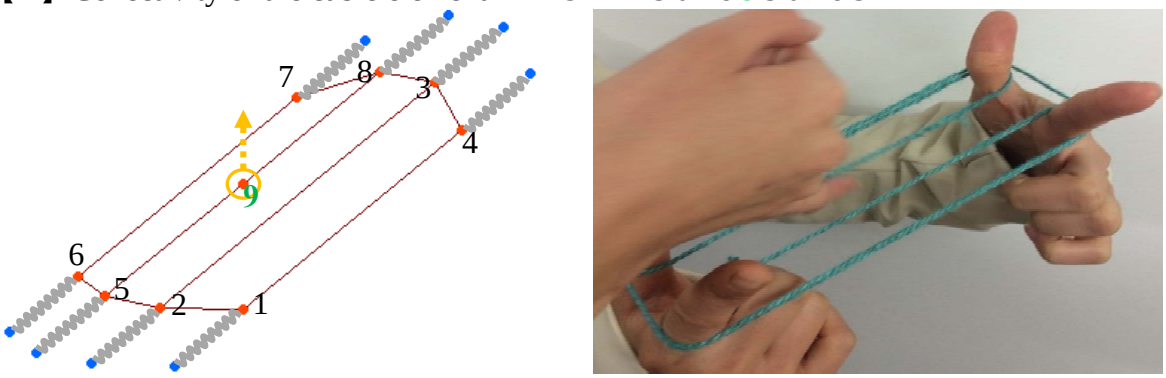
Figure7 Primary shape of a cat's cradle

Table2 Materical and mechanical properties of the element for cat' cradle

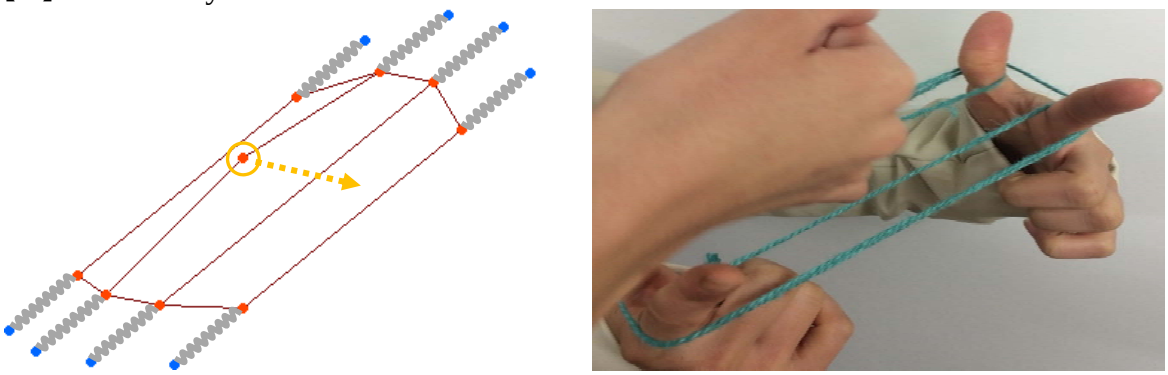
Length of a cable	131[cm]
Area of cross section	1.0×10^{-4}[cm^2],
Young's modulus (tension)	3.0[GPa]
Young's modulus (tension)	1.0×10^{-9}[GPa]
Total number of nodes (primary)	8

movement of fingers. In Figure7, the nodes indicated by blue dots are fixed, and the nodes indicated by red dots are the free sliding nodes. In Figure8 to 12 all the process are shown.

【A】Conectivity of the cable element：1-2-3-4-1-2-5-6-7-8-9-5-6-7-8-3-4-1



【B】Conectivity of the cable element：1-2-3-4-1-2-5-6-7-8-9-5-6-7-8-3-4-1



【C】Conectivity of the cable element：1-2-3-4-1-2-5-6-7-8-9-5-6-7-8-3-4-1

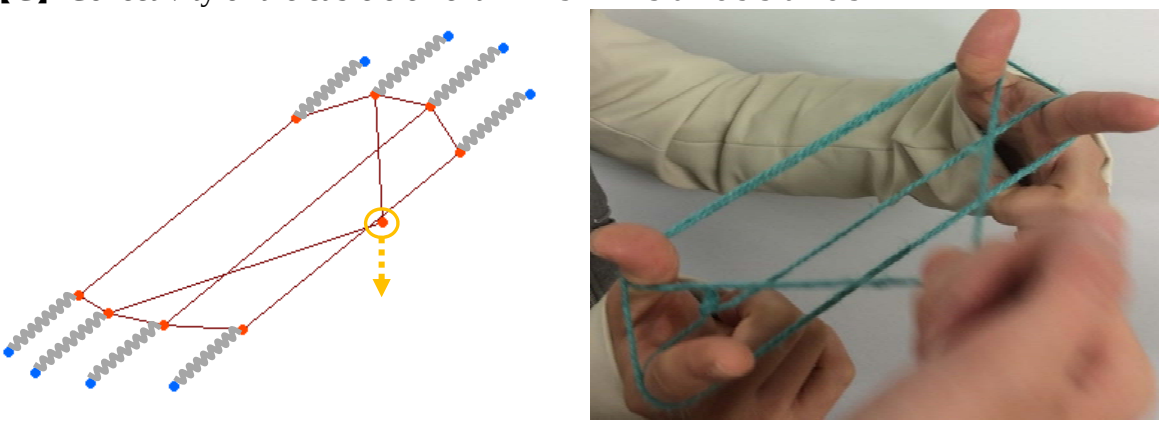
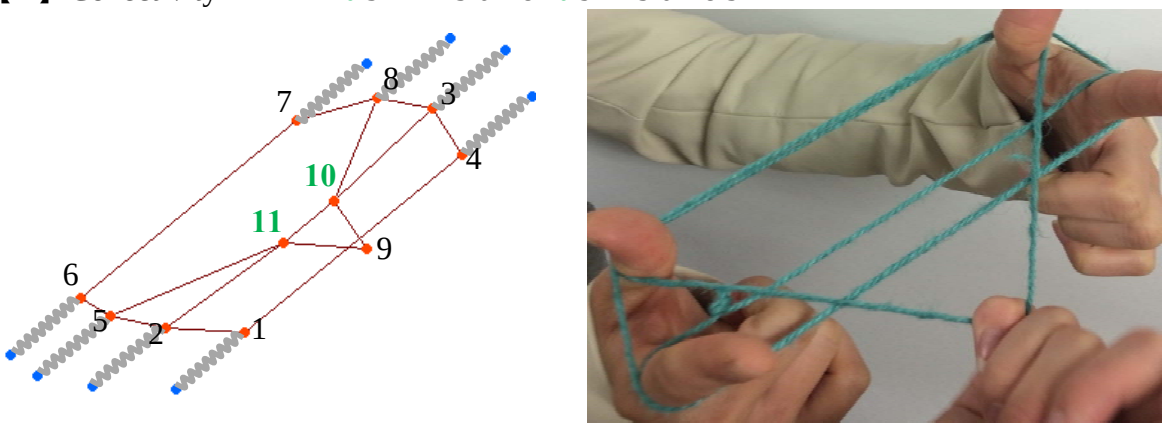


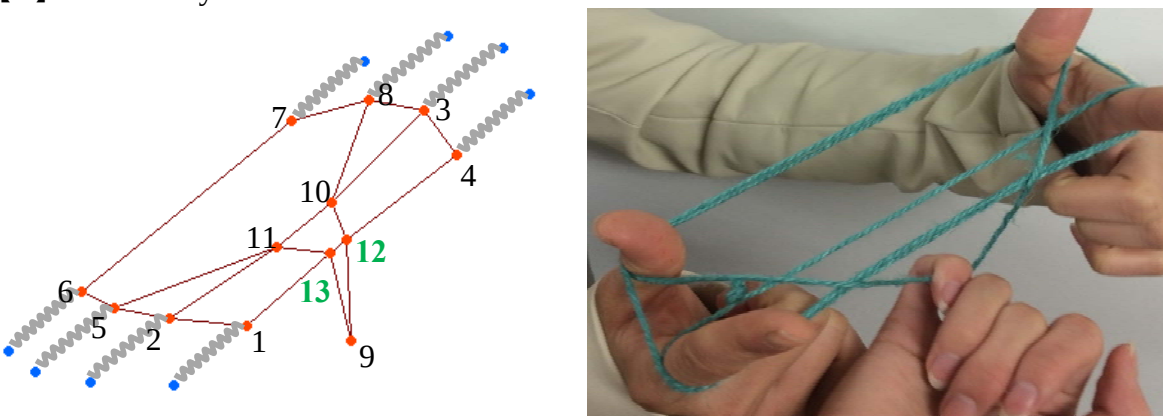
Figure8 Deformation process of [A] to[C]

[A]: A new sliding node9 is created between node5 and node8. and compulsory displacement up to vertical direction is given.
[B]: Let it move to horizontal direction to position of [C]. Then down to vertical direction.
[D]: Contacts between node2 and node3 are detected and new nodes of 10 and 11 are created.
[E]: Contacts between node1 and node4 are detected and new nodes of 12 and 13 are created
[F]: A new sliding node14 is created between node10and node11, and let it up to vertical direction.

【D】Conectivity：1-2-11-10-3-4-1-2-5-6-7-8-10-9-11-5-6-7-8-3-4-1



【E】Conectivity：1-2-11-10-3-4-12-13-1-2-5-6-7-8-10-12-9-13-11-5-6-7-8-3-4-12-13-1



【F】Conectivity：1-2-11-14-10-3-4-12-13-1-2-5-6-7-8-10-12-9-13-11-5-6-7-8-3-4-12-13-1

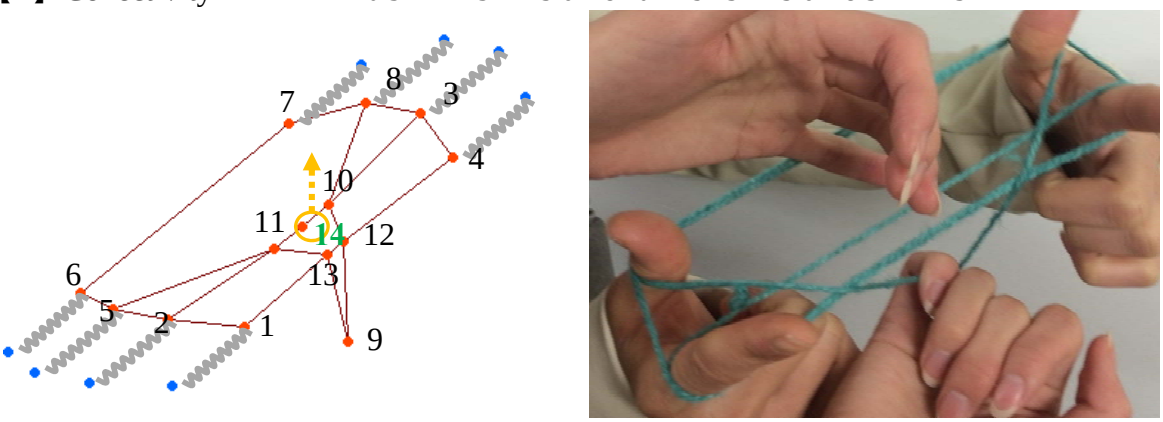
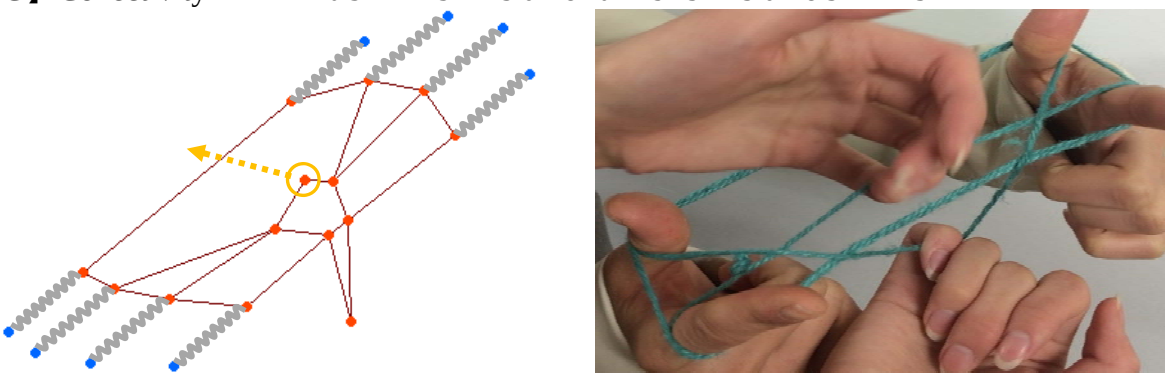


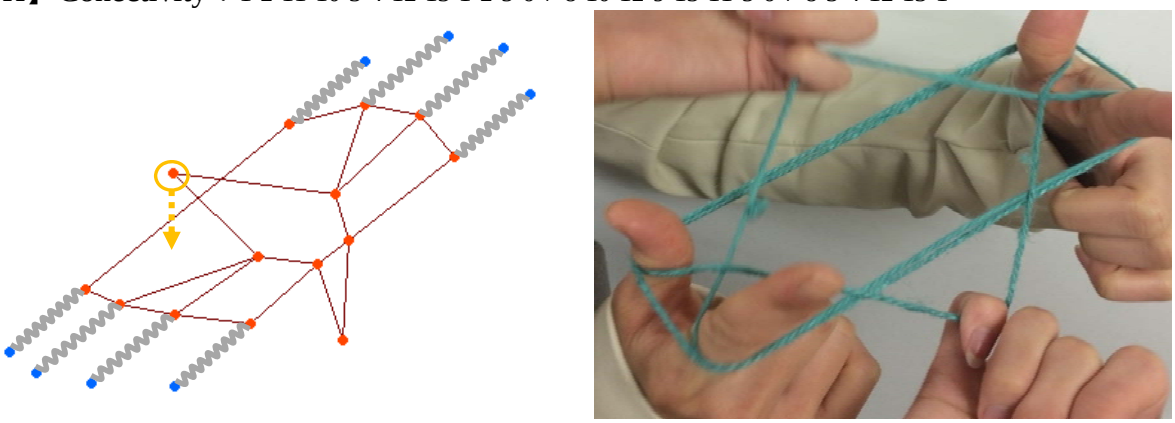
Figure9 Deformation process of [D] to[F]

[G]: Compulsory displacement to horizontal direction is given to node14.
[H]: Let node14 go down to vertical direction.
[I]: Two new sliding nodes of 15 and 16 are created between node6 and node7, and let it down to vertical direction.

【G】Conectivity：1-2-11-10-3-4-12-13-1-2-5-6-7-8-10-12-9-13-11-5-6-7-8-3-4-12-13-1



【H】Conectivity：1-2-11-10-3-4-12-13-1-2-5-6-7-8-10-12-9-13-11-5-6-7-8-3-4-12-13-1



【I】Conectivity: 1-2-11-15-14-16-10-3-4-12-13-1-2-5-6-15-16-7-8-10-12-9-13-11-5-6-15-16-7-8-3-4-12-13-1

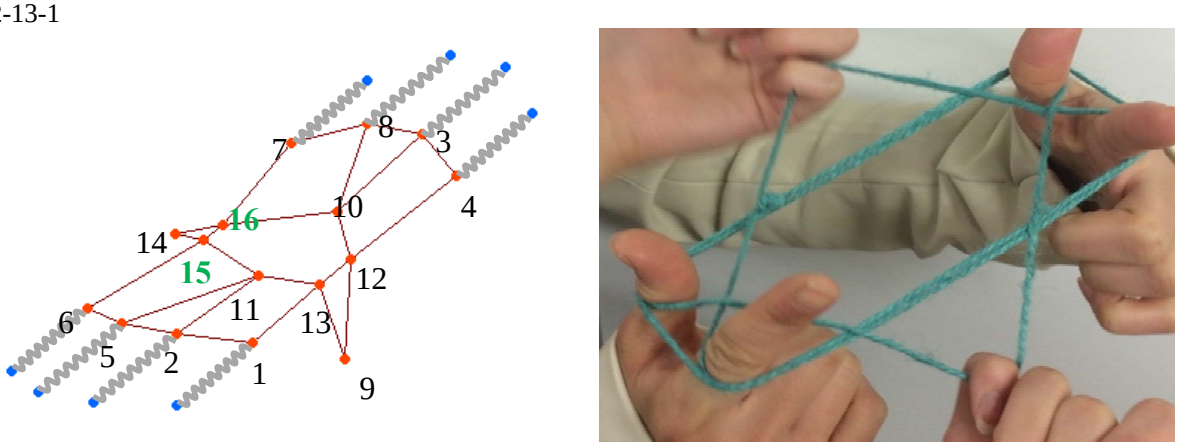
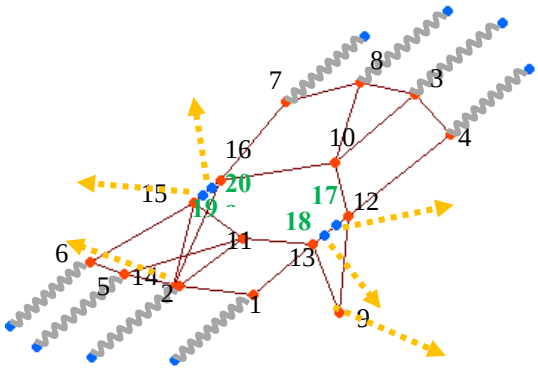
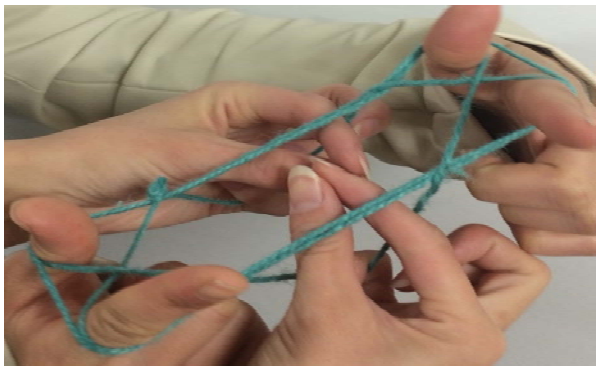


Figure10 Deformation process of [G] to[I]

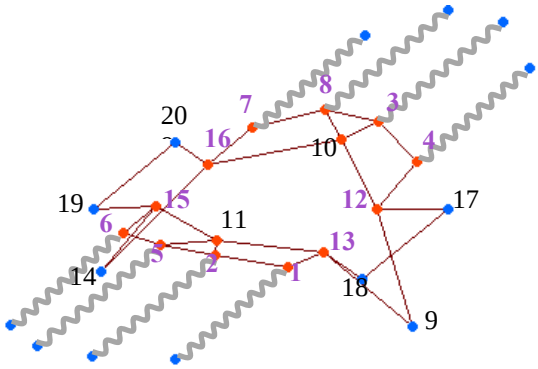
[J]: Four new sliding nodes are created. Node17 and 18 are located between node12 and 13, node19 and 20 are between 15 and 16. Then every node is given compulsory displacement to the directions shown by yellow arrows in the figure.
[L]: The nodes which are indicated in purple letters of [K] are released from the element. In this state, total distance of all the segments becomes smaller than non-stressed length, so that the element has axial strain. As we can find in the photo, the string of cat's cradle is also in the state of relaxation. Therefore, to insert the tension, the compulsory displacements are given to the nodes which surrounded by yellow circles to the direction of the yellow arrows.

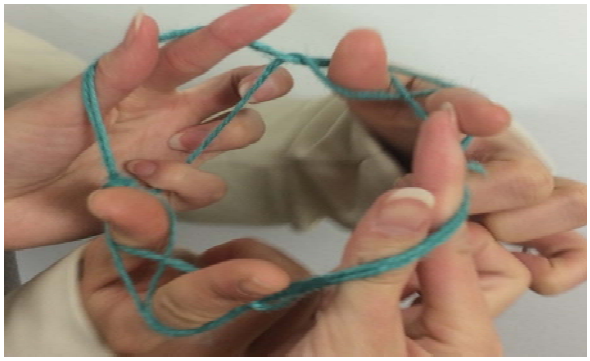
【J】Conectivity: 1-2-11-15-14-16-10-3-4-12-17-18-13-1-2-5-6-15-19-20-16-7-8-10-12-9-13-11-5-6-15-19-20-16-7-8-3-4-12-17-18-13-1



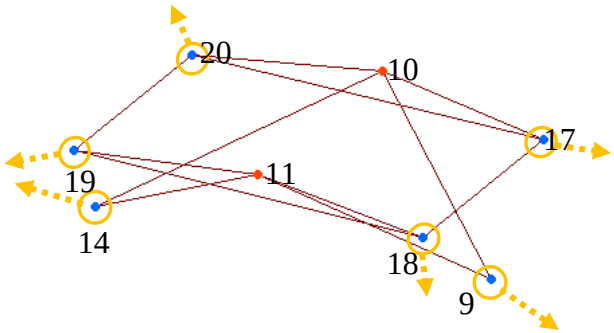


【K】Conectivity:1-2-11-15-14-16-10-3-4-12-17-18-13-1-2-5-6-15-19-20-16-7-8-10-12-9-13-11-5-6-15-19-20-16-7-8-3-4-12-17-18-13-1





【L】Conectivity:11-14-10-17-18-19-20-10-9-11-19-20-17-18-11



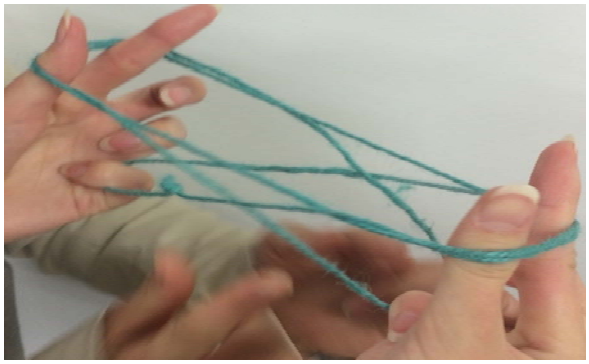
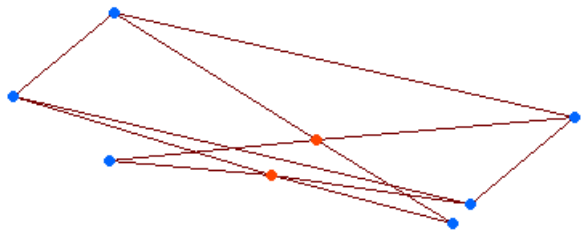


Figure11 Deformation process of [J] to[L]

[M]: This is the final shape of “bridge” shape.

【M】Conectivity:11-14-10-17-18-19-20-10-9-11-19-20-17-18-11



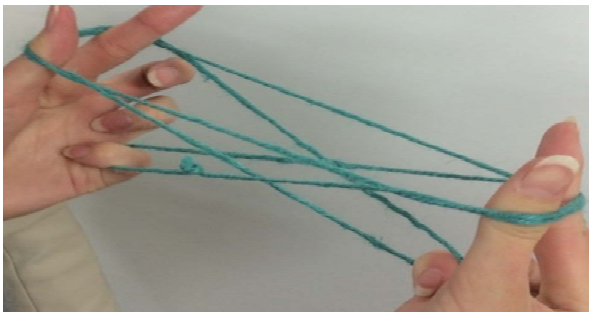


Figure12 Final shape of “bridge”

5 CONCLUSION

As shown in numerical examples of this study, the proposed cable elements gives strict equilibrium solutions with the tangent stiffness method, even if in case of tough compound nonlinear situation that has so many sliding nodes and so complex procedure.

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