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# SHAPE OPTIMIZATION OF RUBBER BUSHING

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**Abstract.** The present paper describes a solution to a non-parametric shape optimization problem of a rubber bushing in order to adjust a function of reaction force with respect to static displacement to a desired function. A main problem is defined as a static hyper-elastic problem considering large deformation and non-linear constitutive equation. A squared error norm of the work done by compulsory displacement and the volume are chosen as cost functions. The shape derivatives of the cost functions are derived theoretically. An iterative algorithm based on the $H^1$ gradient method is used to solve the shape optimization problem.

## 1 Introduction

The rubber bushing is used as a vibration isolator in vehicle suspension systems in order to prevent the vibration of an engine or the tire from transferring into the guest room. The rubber bushing has been modeled as a hyper-elastic continuum deforming largely and following a non-linear constitutive equation. For the constitutive equation, many equations have been proposed using non-linear elastic potentials[1]. Numerical analyses of the rubber bushing using the finite element method have been reported[2, 3].

Moreover, numerical solutions to parametric shape optimization problems of the rubber bushing have been presented[4, 5]. In these researches, in order to adjust a function of the reaction force with respect to static displacement to a desired function, a squared error norm of the reaction force function has been chosen as a cost function.

In the present paper, we present the solution to the non-parametric shape optimization problem of a rubber bushing. Domain variation from an initial domain is chosen as a design variable. Main problem, which we refer to as a boundary value problem of a partial differential equation in which the domain is defined as a design variable, is formulated as a hyper-elastic problem considering large deformation and non-linear constitutive equation. We choose a squared error norm of the work done by compulsory displacement as an objective function, and the volume as a constraint function. The shape derivatives of the cost functions are derived theoretically following the standard procedure using the $H^1$ gradient method[6], but the geometrical and material non-linearities are considered in the present paper.

## 2 Admissible set of design variable

First, let us define the admissible set of design variable for the shape optimization problem. Let $\Omega_0 \subset \mathbb{R}^d$ be a $d \in \{2, 3\}$ dimensional domain with Lipschitz boundary, which is denoted by $\partial\Omega_0$. On $\partial\Omega_0$, $\Gamma_{\mathrm{D}0} \subset \partial\Omega_0$ and $\Gamma_{\mathrm{N}0} = \partial\Omega_0 \setminus \bar{\Gamma}_{\mathrm{D}0}$ ($\bar{\Gamma}_{\mathrm{D}0} = \Gamma_{\mathrm{D}0} \cup \partial\Gamma_{\mathrm{D}0}$) denote the Dirichlet boundary and the homogeneous Neumann boundary, respectively.

We assume that $\Omega_0$ is fixed and that domain is created by continuous one-to-one mapping $\boldsymbol{i} + \boldsymbol{\phi} : \Omega_0 \to \mathbb{R}^d$ as $\Omega(\boldsymbol{\phi}) = \{(\boldsymbol{i} + \boldsymbol{\phi})(\boldsymbol{x}) \mid \boldsymbol{x} \in \Omega_0\}$, where $\boldsymbol{i}$ is used as the identity mapping. In the same manner, the notation $(\,\cdot\,)(\boldsymbol{\phi})$ is used as $\{(\boldsymbol{i} + \boldsymbol{\phi})(\boldsymbol{x}) \mid \boldsymbol{x} \in (\,\cdot\,)_0\}$ in the present paper. In order to define the Fréchet derivatives with respect to domain variaiton, we use

$$X = \left\{ \boldsymbol{\phi} \in H^1\left(\mathbb{R}^d; \mathbb{R}^d\right) \;\middle|\; \boldsymbol{\phi} = \boldsymbol{0}_{\mathbb{R}^d} \text{ on } \Gamma_{\mathrm{D}0} \right\} \tag{1}$$

as the Banach space for $\boldsymbol{\phi}$. In (1), the domain of $\boldsymbol{\phi}$ is extended to $\mathbb{R}^d$ by Calderón's extension theorem. Moreover, to keep continuous one-to-one mapping property, we define the admissible set of $\boldsymbol{\phi}$ as

$$\mathcal{D} = \left\{ \boldsymbol{\phi} \in X \cap Y \mid \|\boldsymbol{\phi}\|_Y < \sigma \right\}, \tag{2}$$

where $Y$ is defined by $W^{1,\infty}\left(\mathbb{R}^d; \mathbb{R}^d\right)$, and $\sigma > 0$ is chosen such that $\boldsymbol{\phi}$ is a bijection.

## 3 Main problem

For $\boldsymbol{\phi} \in \mathcal{D}$, let us define main problem. Let $(0, t_{\mathrm{T}}) \subset \mathbb{R}$ be a time domain with a positive constant $t_{\mathrm{T}}$, and $\boldsymbol{u}_{\mathrm{D}} : (0, t_{\mathrm{T}}) \times \mathbb{R}^d \to \mathbb{R}^d$ be a given function denoting a quasi-static compulsion displacement which is increasing monotonically with respect to $t \in (0, t_{\mathrm{T}})$.

Let $\boldsymbol{u}(\boldsymbol{\phi}) : (0, t_{\mathrm{T}}) \times \mathbb{R}^d \to \mathbb{R}^d$ be a displacement obtained as a solution to a hyper elastic problem shown later in Problem 1. To construct this problem, we need to define the constitutive equation of the hyper elastic continuum. Let $\boldsymbol{y}(\boldsymbol{\phi}) = \boldsymbol{i} + \boldsymbol{u}(\boldsymbol{\phi}) : (0, t_{\mathrm{T}}) \times \mathbb{R}^d \to \mathbb{R}^d$ be the mapping for the large deformation, and

$$\boldsymbol{F}(\boldsymbol{u}) = \left(\boldsymbol{\nabla} \boldsymbol{y}^{\mathrm{T}}\right)^{\mathrm{T}} = \boldsymbol{I} + \left(\boldsymbol{\nabla} \boldsymbol{u}^{\mathrm{T}}\right)^{\mathrm{T}} \tag{3}$$

be the deformation gradient tensor, where $\boldsymbol{I}$ denotes the unit matrix of $d$-th order. Using the definition, the Green-Lagrange strain is defined as

$$\boldsymbol{E}(\boldsymbol{u}) = \frac{1}{2}\left(\boldsymbol{F}(\boldsymbol{u})\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{u}) - \boldsymbol{I}\right) = \boldsymbol{E}_{\mathrm{L}}(\boldsymbol{u}) + \frac{1}{2}\boldsymbol{E}_{\mathrm{BL}}(\boldsymbol{u},\boldsymbol{u}), \tag{4}$$

where

$$\begin{aligned}
\boldsymbol{E}_{\mathrm{L}}(\boldsymbol{u}) &= \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u}^{\mathrm{T}} + \left(\boldsymbol{\nabla}\boldsymbol{u}^{\mathrm{T}}\right)^{\mathrm{T}}\right),\\
\boldsymbol{E}_{\mathrm{BL}}(\boldsymbol{u},\boldsymbol{v}) &= \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u}^{\mathrm{T}}\left(\boldsymbol{\nabla}\boldsymbol{v}^{\mathrm{T}}\right)^{\mathrm{T}} + \boldsymbol{\nabla}\boldsymbol{v}^{\mathrm{T}}\left(\boldsymbol{\nabla}\boldsymbol{u}^{\mathrm{T}}\right)^{\mathrm{T}}\right).
\end{aligned}$$

The constitutive equation for hyper elastic material is defined by assuming the existence of a nonlinear elastic potential $\pi : \mathbb{R}^{d\times d} \to \mathbb{R}$ that gives the second Piola-Kirchhoff stress tensor as

$$\begin{aligned}
\boldsymbol{S}(\boldsymbol{u}) &= \frac{\partial \pi(\boldsymbol{E}(\boldsymbol{u}))}{\partial \boldsymbol{E}(\boldsymbol{u})} = \boldsymbol{D}(\boldsymbol{E}(\boldsymbol{u})) : \boldsymbol{E}(\boldsymbol{u})\\
&= \left(\sum_{(k,l)\in\{1,\cdots,d\}^2} d_{ijkl}(\boldsymbol{E}(\boldsymbol{u}))\, e_{kl}(\boldsymbol{u})\right)_{ij}.
\end{aligned} \tag{5}$$

Here, $\boldsymbol{D}(\boldsymbol{E}(\boldsymbol{u}))$ is the stiffness. For $\pi$, in the present paper, we use the Yeoh model given as

$$\begin{aligned}
\pi(\boldsymbol{E}(\boldsymbol{u})) &= e_1(i_1(\boldsymbol{u}) - 3) + e_2(i_1(\boldsymbol{u}) - 3)^2 + e_3(i_1(\boldsymbol{u}) - 3)^3\\
&\quad + \frac{1}{d_1}(i_3(\boldsymbol{u}) - 1)^2 + \frac{1}{d_2}(i_3(\boldsymbol{u}) - 1)^4 + \frac{1}{d_3}(i_3(\boldsymbol{u}) - 1)^6,
\end{aligned}$$

where $e_1$, $e_2$, $e_3$, $d_1$, $d_2$ and $d_3$ denote material parameters, $i_1(\boldsymbol{u})$ and $i_3(\boldsymbol{u})$ denote the first and third-invariants defined by

$$\begin{aligned}
i_1(\boldsymbol{u}) &= i_3^{-2/3}(\boldsymbol{u})\left(c_1^2(\boldsymbol{u}) + c_2^2(\boldsymbol{u}) + c_3^2(\boldsymbol{u})\right),\\
i_3(\boldsymbol{u}) &= \det \boldsymbol{F}(\boldsymbol{u}),
\end{aligned}$$

and $c_1(\boldsymbol{u})$, $c_2(\boldsymbol{u})$ and $c_3(\boldsymbol{u})$ are the principal values of the right Cauchy-Green deformation tansor $\boldsymbol{C}(\boldsymbol{u}) = \boldsymbol{F}(\boldsymbol{u})\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{u}) = 2\boldsymbol{E}(\boldsymbol{u}) + \boldsymbol{I}$.

Using (5) as the constitutive equation, the hyper elastic problem can be defined using the first Piola-Kirchhoff stress tensor defined by

$$\boldsymbol{\Pi}(\boldsymbol{u}) = \boldsymbol{S}(\boldsymbol{u})\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{u}).$$

In the present paper, $\boldsymbol{\nu}$ denotes the outer unit normal on boundary.

**Problem 1 (Hyper elastic problem)** *For $\boldsymbol{\phi} \in \mathcal{D}$ and $t \in (0, t_{\mathrm{T}})$, let $\boldsymbol{u}_{\mathrm{D}}(t) : \mathbb{R}^d \to \mathbb{R}^d$ be a given function. Find $\boldsymbol{u}(\boldsymbol{\phi}, t) : \Omega(\boldsymbol{\phi}) \to \mathbb{R}^d$ such that*

$$\begin{aligned} -\boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{\Pi}(\boldsymbol{u}(\boldsymbol{\phi}, t)) &= \boldsymbol{0}_{\mathbb{R}^d}^{\mathrm{T}} \text{ in } \Omega(\boldsymbol{\phi}), \\ \boldsymbol{\Pi}(\boldsymbol{u}(\boldsymbol{\phi}, t)) \boldsymbol{\nu} &= \boldsymbol{0}_{\mathbb{R}^d} \text{ on } \Gamma_{\mathrm{N}}(\boldsymbol{\phi}), \\ \boldsymbol{u}(\boldsymbol{\phi}, t) &= \boldsymbol{u}_{\mathrm{D}}(t) \text{ on } \Gamma_{\mathrm{D0}}. \end{aligned}$$

If $\boldsymbol{u}_{\mathrm{D}}(t)$ is given appropriately, for the weak solution $\boldsymbol{u}(\boldsymbol{\phi}, t)$ to Problem 1, $\tilde{\boldsymbol{u}}(\boldsymbol{\phi}, t) = \boldsymbol{u}(\boldsymbol{\phi}, t) - \boldsymbol{u}_{\mathrm{D}}(t)$ lies within

$$U = \left\{ \boldsymbol{u} \in H^1\left(\mathbb{R}^d; \mathbb{R}^d\right) \;\middle|\; \boldsymbol{u} = \boldsymbol{0}_{\mathbb{R}^d} \text{ on } \Gamma_{\mathrm{D0}} \right\}, \tag{6}$$

since the domain of $\boldsymbol{u}(\boldsymbol{\phi}, t)$ can be extended to $\mathbb{R}^d$ by Calderón's extension theorem. Moreover, in the present paper, we define the admissible set of $\tilde{\boldsymbol{u}}(\boldsymbol{\phi}, t)$ by

$$\mathcal{S} = U \cap W^{2,4q}\left(\mathbb{R}^d; \mathbb{R}^d\right) \tag{7}$$

for $q > d$ in order to obtain the domain variation in $Y$ without singular points by the $H^1$ gradient method[6]. For the simplicity, $\boldsymbol{u}(\boldsymbol{\phi}, t)$ is denoted by $\boldsymbol{u}(t)$ or $\boldsymbol{u}$, and $\boldsymbol{u}_{\mathrm{D}}(t)$ is denoted by $\boldsymbol{u}_{\mathrm{D}}$ from here.

In order to use later, we define the Legrange function for Problem 1 as

$$\begin{aligned} \mathscr{L}_{\mathrm{M}}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}) &= \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{\Pi}(\boldsymbol{u}) \boldsymbol{v} \; \mathrm{d}x \\ &\quad + \int_{\Gamma_{\mathrm{D0}}} \left\{ (\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}) \cdot \boldsymbol{\Pi}(\boldsymbol{v}) \boldsymbol{\nu} + \boldsymbol{v} \cdot \boldsymbol{\Pi}(\boldsymbol{u}) \boldsymbol{\nu} \right\} \mathrm{d}\gamma, \end{aligned} \tag{8}$$

where $\boldsymbol{v} \in U$ is introduced as the Lagrange multiplier. The first term on the right-hand side of (8) can be rewritten as

$$\begin{aligned} \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{\Pi}(\boldsymbol{u}) \boldsymbol{v} \; \mathrm{d}x &= \int_{\Omega(\boldsymbol{\phi})} \left\{ \boldsymbol{\nabla} \cdot (\boldsymbol{\Pi}(\boldsymbol{u}) \boldsymbol{v}) + \boldsymbol{\Pi}(\boldsymbol{u}) \cdot \left(\boldsymbol{\nabla} \boldsymbol{v}^{\mathrm{T}}\right)^{\mathrm{T}} \right\} \mathrm{d}x \\ &= \int_{\partial\Omega(\boldsymbol{\phi})} (\boldsymbol{\Pi}(\boldsymbol{u}) \boldsymbol{v}) \cdot \boldsymbol{\nu} \; \mathrm{d}\gamma - \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\Pi}(\boldsymbol{u}) \cdot \boldsymbol{F}'(\boldsymbol{u})[\boldsymbol{v}] \, \mathrm{d}x, \end{aligned}$$

where $\boldsymbol{F}'(\boldsymbol{u})[\boldsymbol{v}] = \partial \boldsymbol{v} / \partial \boldsymbol{x}^{\mathrm{T}}$. Moreover, considering $\boldsymbol{S}(\boldsymbol{u}) = \boldsymbol{S}^{\mathrm{T}}(\boldsymbol{u})$,

$$\begin{aligned} -\int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\Pi}(\boldsymbol{u}) \cdot \boldsymbol{F}'(\boldsymbol{u})[\boldsymbol{v}] \, \mathrm{d}x &= -\int_{\Omega(\boldsymbol{\phi})} \left( \boldsymbol{S}(\boldsymbol{u}) \boldsymbol{F}^{\mathrm{T}}(\boldsymbol{u}) \right) \cdot \boldsymbol{F}'(\boldsymbol{u})[\boldsymbol{v}] \, \mathrm{d}x \\ &= -\int_{\Omega(\boldsymbol{\phi})} \boldsymbol{F}'(\boldsymbol{u})[\boldsymbol{v}] \cdot (\boldsymbol{F}(\boldsymbol{u}) \boldsymbol{S}(\boldsymbol{u})) \, \mathrm{d}x = -\int_{\Omega(\boldsymbol{\phi})} \boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{E}'(\boldsymbol{u})[\boldsymbol{v}] \; \mathrm{d}x \end{aligned} \tag{9}$$

holds, where

$$\begin{aligned}\boldsymbol{E}'(\boldsymbol{u})[\boldsymbol{v}] &= \frac{1}{2}\Big\{\boldsymbol{F}'^{\mathrm{T}}(\boldsymbol{u})[\boldsymbol{v}]\,\boldsymbol{F}(\boldsymbol{u}) + \boldsymbol{F}^{\mathrm{T}}(\boldsymbol{u})\,\boldsymbol{F}'(\boldsymbol{u})[\boldsymbol{v}]\Big\}\\ &= \boldsymbol{E}_{\mathrm{L}}(\boldsymbol{v}) + \boldsymbol{E}_{\mathrm{BL}}(\boldsymbol{u},\boldsymbol{v}).\end{aligned}$$

Then, using (9), (8) can be rewritten as

$$\begin{aligned}\mathscr{L}_{\mathrm{M}}(\boldsymbol{\phi},\boldsymbol{u},\boldsymbol{v}) = &-\int_{\Omega(\boldsymbol{\phi})} \boldsymbol{S}(\boldsymbol{u})\cdot\boldsymbol{E}'(\boldsymbol{u})[\boldsymbol{v}]\ \mathrm{d}x\\ &+\int_{\Gamma_{\mathrm{D0}}}\big\{(\boldsymbol{u}-\boldsymbol{u}_{\mathrm{D}})\cdot\boldsymbol{\Pi}(\boldsymbol{v})\,\boldsymbol{\nu} + \boldsymbol{v}\cdot\boldsymbol{\Pi}(\boldsymbol{u})\,\boldsymbol{\nu}\big\}\mathrm{d}\gamma. \end{aligned}\tag{10}$$

If $\boldsymbol{u}$ is the solution to Problem 1,

$$\mathscr{L}_{\mathrm{M}}(\boldsymbol{\phi},\boldsymbol{u},\boldsymbol{v}) = 0 \tag{11}$$

holds for all $\boldsymbol{v}\in U$. Then, (11) agrees with the weak form of Problem 1.

## 4 Shape optimization problem

Using $\boldsymbol{u}$, we define a shape optimization problem as follows. Let $\alpha_1, \cdots, \alpha_m$ be the constants denoting the desired value of $\boldsymbol{u}_{\mathrm{D}}\cdot(\boldsymbol{\Pi}(\boldsymbol{u})\,\boldsymbol{\nu})$ at $t_1, \cdots, t_m \in (0, t_{\mathrm{T}}]$, respectively. In the present paper, we defined

$$f_0(\boldsymbol{\phi},\boldsymbol{u}) = \sum_{i\in\{1,\cdots,m\}} f_{0i}(\boldsymbol{\phi},\boldsymbol{u}(t_i)) \tag{12}$$

as the objective cost function, where

$$f_{0i}(\boldsymbol{\phi},\boldsymbol{u}(t_i)) = \int_{\Gamma_{\mathrm{D0}}}\big|\boldsymbol{u}_{\mathrm{D}}(t_i)\cdot(\boldsymbol{\Pi}(\boldsymbol{u}(t_i))\,\boldsymbol{\nu}) - \alpha_i\big|^2\mathrm{d}x.$$

Moreover, we define

$$f_1(\boldsymbol{\phi}) = \int_{\Omega(\boldsymbol{\phi})}\mathrm{d}x - c_1 \tag{13}$$

as a constraint cost function, where $c_1$ is a positive constant for which there exists $\boldsymbol{\phi}\in\mathcal{D}$ such that $f_1(\boldsymbol{\phi})\le 0$.

Using these cost functions, we construct the shape optimization problem as follows.

**Problem 2 (Squared error norm minimization)** *Let $f_0(\boldsymbol{\phi},\boldsymbol{u})$ and $f_1(\boldsymbol{\phi})$ be defined as in* (12) *and* (13)*, respectively. Find $\boldsymbol{\phi}$ such that*

$$\min_{\boldsymbol{\phi}\in\mathcal{D}}\Big\{f_0(\boldsymbol{\phi},\boldsymbol{u})\ \big|\ f_1(\boldsymbol{\phi})\le 0, \boldsymbol{u}(t)\in\mathcal{S},\ t\in(0,t_{\mathrm{T}}),\ \textit{Problem 1}\Big\}.$$

## 5 Shape derivative of the cost functions

To solve Problem 2 by the gradient method, the Fréchet derivatives of $f_0$ and $f_1$ with respect to domain variation, which we refer to as the shape derivative, are required. Let $\boldsymbol{\varphi} \in X$ be the domain variation from $\boldsymbol{\phi}$. If there exist $\boldsymbol{g}_0$ and $\boldsymbol{g}_1$ such that $f_0'(\boldsymbol{\phi}, \boldsymbol{u}(\boldsymbol{\phi}))[\boldsymbol{\varphi}] = \langle \boldsymbol{g}_0, \boldsymbol{\varphi} \rangle$ and $f_1'(\boldsymbol{\phi})[\boldsymbol{\varphi}] = \langle \boldsymbol{g}_1, \boldsymbol{\varphi} \rangle$ for all $\boldsymbol{\varphi} \in X$, we call $\boldsymbol{g}_0$ and $\boldsymbol{g}_1$ the the shape derivatives of $f_0$ and $f_1$, respectively. Here, $\langle \cdot\,, \cdot \rangle$ denotes the dual product.

Since $f_0$ is a functional of $\boldsymbol{u}$, $\boldsymbol{g}_0$ is obtained by using the Lagrange multiplier method as follows. We define

$$\begin{aligned}\mathscr{L}_0(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}) &= \sum_{i \in \{1, \cdots, m\}} \mathscr{L}_{0i}(\boldsymbol{\phi}, \boldsymbol{u}(t_i), \boldsymbol{v}_{0i}) \\ &= \sum_{i \in \{1, \cdots, m\}} \left( f_{0i}(\boldsymbol{\phi}, \boldsymbol{u}(t_i)) + \mathscr{L}_{\mathrm{M}}(\boldsymbol{\phi}, \boldsymbol{u}(t_i), \boldsymbol{v}_{0i}) \right)\end{aligned}$$

as the Lagrangian for $f_0$, where $\boldsymbol{v}_{0i} : \mathbb{R}^d \to \mathbb{R}^d$ is introduced as the Lagrange multipliers for Problem 1 at $t = t_i$. The shape derivative of $\mathscr{L}_{0i}$ can be written as

$$\begin{aligned}\mathscr{L}_{0i}'(\boldsymbol{\phi}, \boldsymbol{u}(t_i), \boldsymbol{v}_{0i})[\boldsymbol{\varphi}, \boldsymbol{u}^*(t_i), \boldsymbol{v}_{0i}^*] &= \mathscr{L}_{0i\boldsymbol{\phi}}(\boldsymbol{\phi}, \boldsymbol{u}(t_i), \boldsymbol{v}_{0i})[\boldsymbol{\varphi}] \\ &+ \mathscr{L}_{0i\boldsymbol{u}(t_i)}(\boldsymbol{\phi}, \boldsymbol{u}(t_i), \boldsymbol{v}_{0i})[\boldsymbol{u}^*(t_i)] + \mathscr{L}_{0i\boldsymbol{v}_{0i}}(\boldsymbol{\phi}, \boldsymbol{u}(t_i), \boldsymbol{v}_{0i})[\boldsymbol{v}_{0i}^*], \end{aligned} \tag{14}$$

where $\boldsymbol{u}^*(t_i)$ and $\boldsymbol{v}_{0i}^*$ are the partial shape derivatives of $\boldsymbol{u}(t_i)$ and $\boldsymbol{v}_{0i}$, respectively[6].

Here, if $\boldsymbol{u}$ is the solution of Problem 1, the third term on the right-hand side of (14) becomes 0. The second term on the right-hand side of (14) becomes

$$\begin{aligned}&\mathscr{L}_{0i\boldsymbol{u}(t_i)}(\boldsymbol{\phi}, \boldsymbol{u}(t_i), \boldsymbol{v}_{0i})[\boldsymbol{u}^*(t_i)] \\ &= -\int_{\Omega(\boldsymbol{\phi})} \big( \boldsymbol{S}'(\boldsymbol{u}(t_i))[\boldsymbol{u}^*(t_i)] \cdot \boldsymbol{E}'(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}] + \boldsymbol{S}(\boldsymbol{u}(t_i)) \cdot \boldsymbol{E}''(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}, \boldsymbol{u}^*(t_i)] \big) \mathrm{d}x \\ &\quad + \int_{\Gamma_{\mathrm{D}0}} \Big[ \boldsymbol{u}^*(t_i) \cdot (\boldsymbol{\Pi}(\boldsymbol{v}_{0i}) \boldsymbol{\nu}) \\ &\quad + \big\{ \boldsymbol{v}_{0i} + 2 (\boldsymbol{u}_{\mathrm{D}}(t_i) \cdot (\boldsymbol{\Pi}(\boldsymbol{u}(t_i)) \boldsymbol{\nu}) - \alpha_i) \boldsymbol{u}_{\mathrm{D}}(t_i) \big\} \cdot (\boldsymbol{\Pi}'(\boldsymbol{u}(t_i))[\boldsymbol{u}^*(t_i)] \boldsymbol{\nu}) \Big] \mathrm{d}\gamma \end{aligned} \tag{15}$$

where

$$\begin{aligned}\boldsymbol{S}'(\boldsymbol{u})[\boldsymbol{v}] &= \boldsymbol{D}(\boldsymbol{E}(\boldsymbol{u})) \boldsymbol{E}'(\boldsymbol{u})[\boldsymbol{v}], \\ \boldsymbol{E}''(\boldsymbol{u})[\boldsymbol{v}, \boldsymbol{w}] &= \boldsymbol{E}_{\mathrm{BL}}(\boldsymbol{v}, \boldsymbol{w}), \\ \boldsymbol{\Pi}'(\boldsymbol{u})[\boldsymbol{v}] &= \boldsymbol{S}'(\boldsymbol{u})[\boldsymbol{v}] \boldsymbol{F}^{\mathrm{T}}(\boldsymbol{u}) + \boldsymbol{S}(\boldsymbol{u}) \boldsymbol{F}'^{\mathrm{T}}(\boldsymbol{u})[\boldsymbol{v}].\end{aligned}$$

If we use the same relation used in (9), (15) becomes

$$\begin{aligned}&-\int_{\Omega(\boldsymbol{\phi})} \big( \boldsymbol{S}'(\boldsymbol{u}(t_i))[\boldsymbol{u}^*(t_i)] \cdot \boldsymbol{E}'(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}] + \boldsymbol{S}(\boldsymbol{u}(t_i)) \cdot \boldsymbol{E}''(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}, \boldsymbol{u}^*(t_i)] \big) \mathrm{d}x \\ &\quad = -\int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\Pi}'(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}] \cdot \boldsymbol{F}'(\boldsymbol{u}(t_i))[\boldsymbol{u}^*(t_i)]\, \mathrm{d}x\end{aligned}$$

Moreover, we assume the relations such as $\boldsymbol{u}^*(t_i) = \mathbf{0}_{\mathbb{R}^d}$ on $\Gamma_{\mathrm{D}0}$ and $\boldsymbol{\Pi}'(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}]\boldsymbol{\nu} = \mathbf{0}_{\mathbb{R}^d}$ on $\Gamma_{\mathrm{N}}(\boldsymbol{\phi})$, we have

$$\int_{\partial\Omega(\boldsymbol{\phi})} (\boldsymbol{\Pi}'(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}]\boldsymbol{u}^*(t_i)) \cdot \boldsymbol{\nu}\, \mathrm{d}\gamma - \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\Pi}'(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}] \cdot \boldsymbol{F}'(\boldsymbol{u}(t_i))[\boldsymbol{u}^*(t_i)]\, \mathrm{d}x$$
$$= \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\nabla}^{\mathrm{T}}\boldsymbol{\Pi}'(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}]\boldsymbol{u}^*(t_i)\ \mathrm{d}x$$

From the relations above, (15) can be rewritten as

$$\mathscr{L}_{0i\boldsymbol{u}(t_i)}(\boldsymbol{\phi}, \boldsymbol{u}(t_i), \boldsymbol{v}_{0i})[\boldsymbol{u}^*(t_i)] = \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\nabla}^{\mathrm{T}}\boldsymbol{\Pi}'(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}]\boldsymbol{u}^*(t_i)\ \mathrm{d}x$$
$$+ \int_{\Gamma_{\mathrm{D}0}} \Big[\boldsymbol{u}^*(t_i) \cdot (\boldsymbol{\Pi}(\boldsymbol{v}_{0i})\boldsymbol{\nu}) + \big\{\boldsymbol{v}_{0i} + 2(\boldsymbol{u}_{\mathrm{D}}(t_i) \cdot (\boldsymbol{\Pi}(\boldsymbol{u}(t_i))\boldsymbol{\nu}) - \alpha_i)\boldsymbol{u}_{\mathrm{D}}(t_i)\big\}$$
$$\cdot (\boldsymbol{\Pi}'(\boldsymbol{u}(t_i))[\boldsymbol{u}^*(t_i)]\boldsymbol{\nu})\Big]\mathrm{d}\gamma$$

for all $\boldsymbol{u}^*(t_i)$ such that $\boldsymbol{u}^*(t_i) = \mathbf{0}_{\mathbb{R}^d}$ on $\Gamma_{\mathrm{D}0}$. Then, the (15) becomes 0 if $\boldsymbol{v}_{0i}$ is the solution of the following adjoint problem.

**Problem 3 (Adjoint problem for $f_0$)** *Let $\boldsymbol{u}(t_i)$ be the solution of Problem 1. Find $\boldsymbol{v}_{0i} : \Omega(\boldsymbol{\phi}) \to \mathbb{R}^d$ such that*

$$-\boldsymbol{\nabla}^{\mathrm{T}}\boldsymbol{\Pi}'(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}] = \mathbf{0}_{\mathbb{R}^d}^{\mathrm{T}} \text{ in } \Omega(\boldsymbol{\phi}),$$
$$\boldsymbol{\Pi}'(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}]\boldsymbol{\nu} = \mathbf{0}_{\mathbb{R}^d} \text{ on } \Gamma_{\mathrm{N}}(\boldsymbol{\phi}),$$
$$\boldsymbol{v}_{0i} = -2(\boldsymbol{u}_{\mathrm{D}}(t_i) \cdot (\boldsymbol{\Pi}(\boldsymbol{u}(t_i))\boldsymbol{\nu}) - \alpha_i)\boldsymbol{u}_{\mathrm{D}}(t_i) \text{ on } \Gamma_{\mathrm{D}0}$$

In order to obtain the domain variation in $Y$ without singular points by the $H^1$ gradient method, $\tilde{\boldsymbol{v}}_{0i}$ such that $\tilde{\boldsymbol{v}}_{0i} = \boldsymbol{v}_{0i} + 2(\boldsymbol{u}_{\mathrm{D}}(t_i) \cdot (\boldsymbol{\Pi}(\boldsymbol{u}(t_i))\boldsymbol{\nu}) - \alpha_i)\boldsymbol{u}_{\mathrm{D}}(t_i) = \mathbf{0}_{\mathbb{R}^d}$ on $\Gamma_{\mathrm{D}0}$ belongs to $\mathcal{S}$ is required[6].

Let $\boldsymbol{u}(t_i)$ and $\boldsymbol{v}_{0i}$ be solutions of Problem 1 and Problem 3, respectively. Then, (14) becomes

$$\mathscr{L}_{0i\boldsymbol{\phi}}(\boldsymbol{\phi}, \boldsymbol{u}(t_i), \boldsymbol{v}_{0i})[\boldsymbol{\varphi}] = f'_{0i}(\boldsymbol{\phi}, \boldsymbol{u}(\boldsymbol{\phi}))[\boldsymbol{\varphi}] = \int_{\Gamma_{\mathrm{N}}(\boldsymbol{\phi})} \boldsymbol{g}_{0i\mathrm{N}} \cdot \boldsymbol{\varphi}\, \mathrm{d}\gamma = \langle \boldsymbol{g}_{0i}, \boldsymbol{\varphi} \rangle \tag{16}$$

where

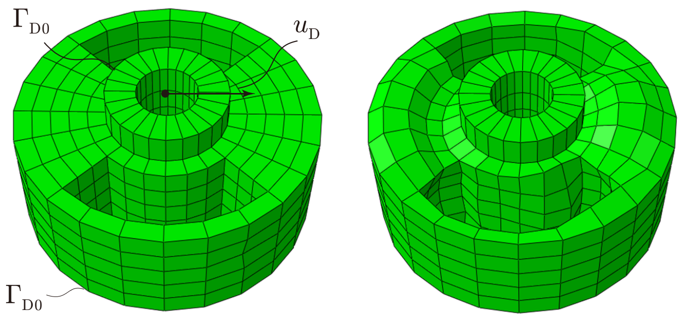
$$\boldsymbol{g}_{0i\mathrm{N}} = \big\{\big|\boldsymbol{u}_{\mathrm{D}}(t_i) \cdot (\boldsymbol{\Pi}(\boldsymbol{u}(t_i))\boldsymbol{\nu}) - \alpha_i\big|^2 - \boldsymbol{S}(\boldsymbol{u}(t_i)) \cdot \boldsymbol{E}'(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}]\big\}\boldsymbol{\nu}.$$

For $f_0$, we have

$$f'_0(\boldsymbol{\phi}, \boldsymbol{u}(\boldsymbol{\phi}))[\boldsymbol{\varphi}] = \sum_{i \in \{1, \cdots, m\}} \langle \boldsymbol{g}_{0i}, \boldsymbol{\varphi} \rangle = \langle \boldsymbol{g}_0, \boldsymbol{\varphi} \rangle. \tag{17}$$

Moreover, for the shape derivative of $f_1$, we have

$$f'_1(\boldsymbol{\phi})[\boldsymbol{\varphi}] = \int_{\Gamma_{\mathrm{N}}(\boldsymbol{\phi})} \boldsymbol{\nu} \cdot \boldsymbol{\varphi}\, \mathrm{d}\gamma = \langle \boldsymbol{g}_1, \boldsymbol{\varphi} \rangle. \tag{18}$$




(a) Initial (b) Optimized

**Figure 1**: Finite element models of simple rubber bushing

## 6 Solution

The algorithm for solving Problem 2 can be shown based on the sequential quadratic programming[6]. In this algorithm, the $H^1$ gradient method is used for reshaping with shape derivatives $\boldsymbol{g}_0$ and $\boldsymbol{g}_1$ in (17) and (18), respectively.

## 7 Numerical example

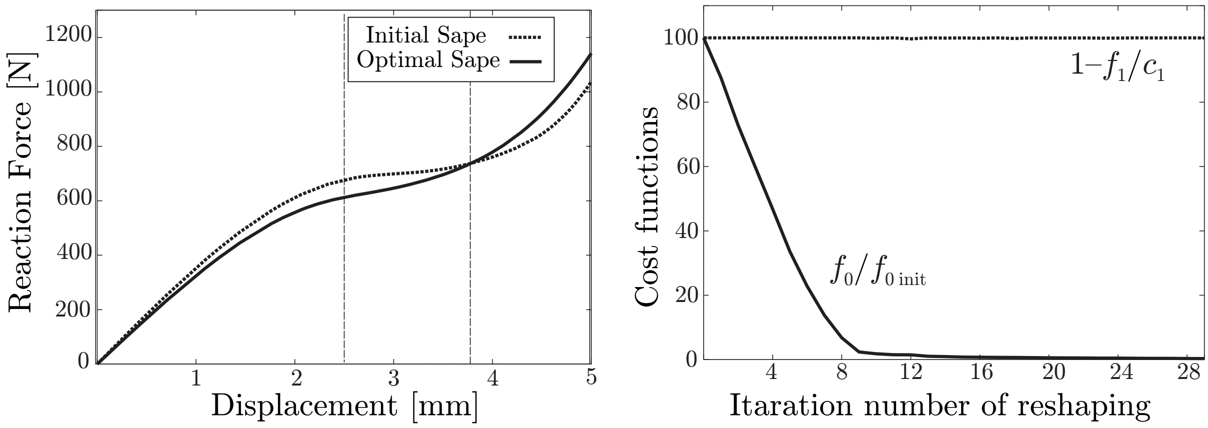
We developed a computer program to solve Problem 2. In the program, a commercial finite element program, Abaqus 6.9 (Dassault Systèmes), is used to solve Problem 1 and Problem 3. Moreover, OPTISHAPE-TS 2011 (Quint Corporation) is employed to solve the boundary value problem in the $H^1$ gradient method.

Figure 1 (a) shows the finite element model of rubber bushing used as an example. The diameter of the outer cylinder is 50.0 [mm]. The outer and inner cylinders are assumed as the homogeneous and non-homogeneous Dirichlet boundaries, respectively. The nodes on the inner cylinder are connected with a rigid element. The arrow of $\boldsymbol{u}_{\mathrm{D}}$ shows the compulsory displacement of the rigid element, which magnitude is 5.0 [mm]. For $f_0$, we assume $m = 3$, and $\{\|\boldsymbol{u}_{\mathrm{D}}(t_1)\|, \|\boldsymbol{u}_{\mathrm{D}}(t_2)\|, \|\boldsymbol{u}_{\mathrm{D}}(t_3)\|\} = \{2.5, 3.75, 5.0\}$ [mm]. For $\alpha_1$, $\alpha_2$ and $\alpha_3$, we use the values of 10 % decrease, even and 10 % increase with respect to the values of $\boldsymbol{u}_{\mathrm{D}} \cdot (\boldsymbol{\Pi}(\boldsymbol{u})\boldsymbol{\nu})$ at $t = t_1$, $t_2$ and $t_3$, respectively.

Figure 1 (b) shows the optimum shape obtained by the developed program. The reaction force of the rigid element defined by $\| \int_{\Gamma_{\mathrm{D0}}} \boldsymbol{\Pi}(\boldsymbol{u}(t))\boldsymbol{\nu}\, \mathrm{d}\gamma\|$ with respect to compulsory displacement $\|\boldsymbol{u}_{\mathrm{D}}(t)\|$ is shown in Fig. 2 (a). Figure 2 (b) shows the iteration histories of the cost functions with respect to the number of reshapings, where $f_{0\,\mathrm{init}}$ and $c_1$ denote the values of $f_0$ and the volume at the initial shape, respectively.

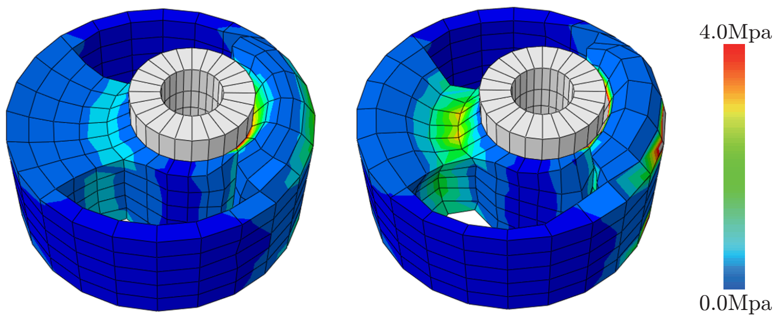
From these results, we see that $f_0$ decreases monotonically under the constraint of $f_1$ (Fig. 2 (b)), and the desired reaction force function is obtained (Fig. 2 (a)).

In addition, Fig. 3 shows the distributions of the von Mises stress of the initial and optimum shapes at $t = t_3$. From the results, it can be confirmed that as the result of increasing the reaction force at $t = t_3$, the von Mises stress increase in optimum shape.




(a) Reaction force functions (b) Iteration histories

**Figure 2**: Graphs for shape optimization analysis




(a) Initial (b) Optimized

**Figure 3**: Comparison of von Mises stress

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