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GOAL-ORIENTED ERROR ESTIMATION AND MESH ADAPTIVITY IN THREE-DIMENTIONAL ELASTICITY PROBLEMS

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Abstract. In finite element simulation of engineering applications, accuracy is of great importance considering that generally no analytical solution is available. Conventional error estimation methods aim to estimate the error in energy norms or the global L_2-norm. These values can be used to estimate the accuracy of the model or to guide how to adapt the model to achieve more accuracy. However, in engineering applications specific quantities are required to be accurate. The novel error estimation approach which is called Dual-Weighted Residual (DWR) error estimation, approximates the error with respect to the quantity of interest which can be mean stress or displacement in a subspace or the solution ('s gradient) on a specific point, etc. DWR error estimation is a dual-based scheme which requires an adjoint (dual) problem. The dual problem is described by defining the quantity of interest in a functional form. Then by solving the primal and dual problems, errors in terms of the specified quantities are calculated. In this paper the DWR error estimation besides the conventional residual-based error estimation and a recovery-based error estimation are applied in a three-dimensional elasticity problem. Local estimated errors are exploited in order to accomplish the mesh adaptivity procedure. The goal-oriented mesh adaptivity control the local errors in terms of the prescribed quantities. Both refinement and coarsening processes are applied to raise the efficiency. The convergence rates are plotted to illustrate the superiorities of the goal-oriented adaptivity over the traditional ones.

1 INTRODUCTION

Nowadays complicated engineering applications can be analyzed by numerical methods running on available computers. One of the most robust and reliable computational approaches is Finite Element Method (FEM) [1]. Although, many other numerical methods such as meshfree methods [2, 3, 4], isogeometric analysis method and its extensions [5, 6, 7, 8, 9, 10, 11] have been introduced and successfully applied in different fields, FEM still plays an important role in computational mechancs field.

In the FEM, mesh discretization highly affects the solution accuracy and obviously the computational effort. Therefore, it is of great importance to be able to minimize the computational cost while the expected solution accuracy is gained. Mesh adaptation is a profitable approach to achieve this goal. As a criterion for mesh configuration and updating, estimation of discretization error is required.

A good error estimtor plays a very important role to implement an efficient refinement procedure in numerical methods. An error estimate should be performed to locate the situations of error distribution in the problem domain. The error estimation methods based on classical energy norm are categorized into two classes namely the residuals-based [12, 13, 14] and the recovery-based [15] methods. In a residual-based error estimator, the residuals of a governing differential equation and its boundary conditions are considered as an error criteria while, the gradient of solutions is utilized in recovery-based methods [16, 17].

It is practically important to be able to estimate the error in the so-called quantity of interest (QoI) rather than the global energy norm. A new type of error estimation procedure called goal-oriented error estimation (GOEE) has been proposed to estimate the error with respect to the QoI [18, 19, 20, 21, 22, 23, 24]. It results in quantifying the effect of local errors on the accuracy of the solution with respect to the specific quantities. Therefore, this methodology is so beneficial for adaptivity schemes and quality assessment in engineering applications.

In this paper, the Dual-Weighted Residual error estimation besides the conventional residual-based error estimation and a recovery-based error estimation developed by Kelly et al. [25] are applied in a three-dimensional elasticity problem. Local estimated errors are exploited in order to accomplish the local mesh adaptivity, refinement/coarsening, considering hanging nodes. The simulations are carried out by using the open source FEM library, deal.II [26]. The goal-oriented mesh adaptivity control the local errors in terms of the prescribed quantity. The convergence rates are plotted to illustrate the superiorities of the goal-oriented mesh adaptivity methodology over the traditional methods.

The rest of the paper is organized as follows: In section 2, the goal-oriented error estimation based on dual-weighted residual is described. In section 3, we present a 3D elasticity numerical example where adaptivity procedure is performed by adopting different criteria. Finally, some concluding remarks are outlined in section 4.

2 GOAL-ORIENTED ERROR ESTIMATION

In engineering applications the entire solution of the problem may not be interested, but rather some certain aspects of it. This is the main idea of applying the Goal-Oriented Error Estimation (GOEE). For example, in an elasticity problem one might want to know about values of the stress at certain points to predict whether maximal load values of joints are safe.

In GOEE procedure, solution of a dual/auxiliary/adjoint problem is also required.

2.1 Primal problem

Let us consider the following elastic eqaution:

$$-\partial_j \sigma_{ij} = f_i \ , \quad i,j,k,l = 1,2,3 \tag{1}$$

where the stress is defined as $\sigma_{ij} = c_{ijkl}\partial_k u_l$

The displacement and traction boundary conditions can be written as

$$u_i = \bar{u}_i, \qquad i = 1,2,3, \qquad \text{on } \Gamma_d \tag{2}$$

$$\bar{t}_i = \left(c_{ijkl}\partial_k u_l\right) n_j, \qquad i,j,k,l = 1,2,3, \qquad \text{on } \Gamma_n \tag{3}$$

where the $\bar{\mathbf{u}}$ and $\bar{\mathbf{t}}$ are prescribed displacement and traction imposed on the boundaries Γ_d and Γ_n, respecively. The values c_{ijkl} are the stiffness coefficients. In isotropic case, by introduction of the Lamé's parameters, λ and μ, the coefficient tensor is defined as,

$$c_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{4}$$

Then, the elastic equation can be rewritten in the following form:

$$-\partial_i \lambda \partial_j u_j - \partial_j \mu \partial_i u_j - \partial_j \mu \partial_j u_i = f_i \ , \quad i,j = 1,2,3 \tag{5}$$

and its corresponding variational form of this problem is to find $u \in V$ such that

$$a(\mathbf{u},\mathbf{v}) = l(\mathbf{v}) \qquad \mathbf{u} \in \mathbf{V} \tag{6}$$

where

$$a(\mathbf{u},\mathbf{v}) = \sum_{k,l} \left(\lambda \partial_l u_l, \partial_k v_k\right)_\Omega + \sum_{k,l} \left(\mu \partial_k u_l, \partial_k v_l\right)_\Omega + \sum_{k,l} \left(\mu \partial_k u_l, \partial_l v_k\right)_\Omega \tag{7}$$

$$l(\mathbf{v}) = \sum_l (f_l, v_l)_\Omega \tag{8}$$

where $(a,b)_\Omega$ denotes $\int_\Omega ab d\Omega$ and $\mathbf{V}$ is the Hilbert space

$$\mathbf{V} = \left\{ \mathbf{v} \in \mathbf{H}^1;\ \mathbf{v} = 0,\ \text{on } \Gamma_d \right\} \tag{9}$$

2.2 Dual problem

A Quantity of Interest (QoI) can be characterized as a continuous linear functional on the space of admissible functions. If the QoI functional is non-linear, it may be linearized and then be used [23]. Also, sometimes the linear functional may not be continuous. For instance, when we are interested in the error in the solution at a particular point in the domain.

In GOEE, the evaluation of a functional of the solution, $J(\mathbf{u})$, is of interest rather than the solution values, $\mathbf{u}$, everywhere. Since the exact solution, $\mathbf{u}$, is not available, but only its numerical approximation $\mathbf{u}_h$, it is sensible to check if the computed value $J(\mathbf{u}^h)$ is within certain limits of the exact value $J(\mathbf{u})$. The aim would be attaining the bounds on the error, $J(\mathbf{e}) = J(\mathbf{u}) - J(\mathbf{u}^h)$.

Errors in QoI are calculated by means of solving a dual problem. Let us denote the solution of the following auxiliary problem by $\mathbf{z}$,

$$a(\mathbf{v}, \mathbf{z}) = J(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V} \tag{10}$$

where $a(\cdot,\cdot)$ is the bilinear form associated with the differential equation. Then, by considering $\mathbf{v} = \mathbf{e} = \mathbf{u} - \mathbf{u}^h$ the error, we have

$$J(\mathbf{e}) = a(\mathbf{e}, \mathbf{z}) \tag{11}$$

which it can be rewritten as follow using the Galerkin orthogonality,

$$J(\mathbf{e}) = a(\mathbf{e}, \mathbf{z} - \mathbf{z}^h) = a\left(\mathbf{e}, \mathbf{e}^Q\right) \tag{12}$$

where $\mathbf{z}^h \in \mathbf{V}^h$ is an approximation of $\mathbf{z}$ which is here considered as point interpolation of the dual solution, $z^h = I^h z$.

For elasticity, the error identity reads

$$J(\mathbf{e}) = \sum_{k,l} \left(\lambda \partial_l e_l, \partial_k e_k^Q\right)_\Omega + \sum_{k,l} \left(\mu \partial_k e_l, \partial_k e_l^Q\right)_\Omega + \sum_{k,l} \left(\mu \partial_k e_l, \partial_l e_k^Q\right)_\Omega \tag{13}$$

By splitting the scalar products into terms on all elements and integrating by parts on each of them, we have

$$J(\mathbf{e}) \;=\; \sum_K \left\{ \left(\mathbf{R}^h, \mathbf{e}^Q\right)_K + \left(\mathbf{r}^\mathbf{h}, \mathbf{e}^Q\right)_{\partial K} \right\} \tag{14}$$

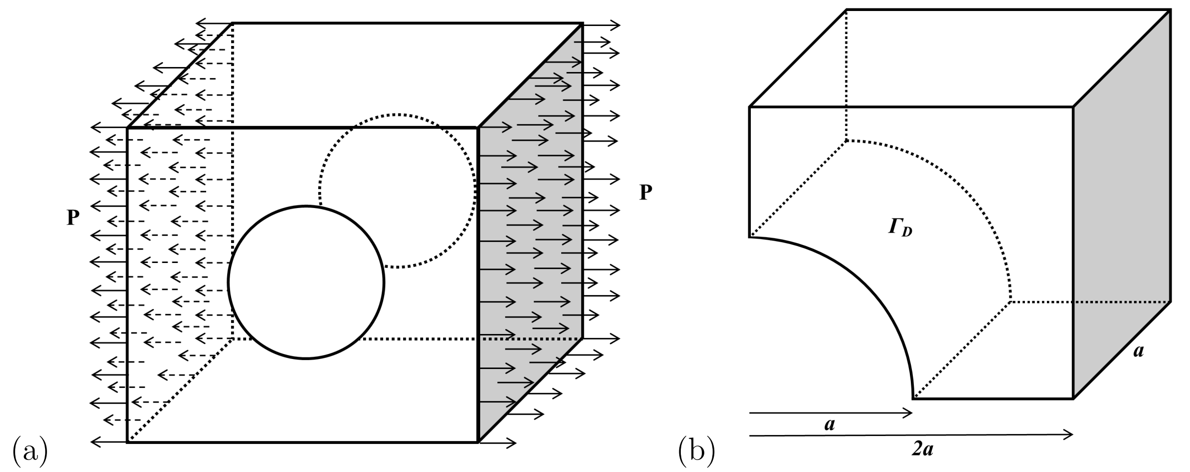

Figure 1: (a) A three-dimentional infinite plate with a circular hole, (b) Quarter of 3D plate and surfaces for QoIs.

with the cell residual

$$R_i^h = f_i + \partial_i \lambda \partial_j u_j^h + \partial_j \mu \partial_i u_j^h + \partial_j \mu \partial_j u_i^h \ , \quad i,j = 1,2,3 \tag{15}$$

The edge residual, $\mathbf{r}^h$, is obtained by exchanging half of the edge integral of cell K with the neighbor element K' and considering the opposite sign of their normal vectors.

$$r_i^h \mid_\Gamma = \begin{cases} \frac{1}{2} n_j \left(c_{ijkl} \partial_k u_l^h \mid_K - c_{ijkl} \partial_k u_l^h \mid_{K'} \right) & \text{if } \Gamma \subset \partial K \setminus \partial \Omega \\ 0 & \text{if } \Gamma \subset \Gamma_d \\ \bar{t}_i - \left(c_{ijkl} \partial_k u_l^h \right) n_j & \text{if } \Gamma \subset \Gamma_n \end{cases} \tag{16}$$

As a result of the above equations, the error of the finite element discretization with respect to arbitrary (linear) functionals $J(\cdot)$ is represented. This GOEE is a weighted form of a residual estimator, where $e^Q = z - z^h$ are weights indicating how important the residuals on a certain cell is for the evaluation of the given functional. Since it is a element-wise quantity, it can be used as a mesh refinement criterion. However, the GOEE requires knowledge of the dual solution z, which carries the information about the quantity we want to know to best accuracy. For this purpose, we compute the dual solution numerically, and approximate z by some numerically obtained $\tilde{z}$. It is noted that it is not sufficient to compute this approximation $\tilde{z}$ using the same method as used for the primal solution u^h, since then $\tilde{z} - I_h \tilde{z} = 0$, and the overall error estimate would be zero. Rather, the approximation $\tilde{z}$ has to be from a larger space than the primal finite element space. There are various ways to obtain such an approximation. In this paper, we compute it one higher order finite element space.

2.3 Mean stress on a face

In this contribution, we consider the mean value of a component of stress σ on a surface Γ_D as a quantity of interest. Therefore, the corresponding functional is defined as

$$J(\mathbf{u}) = \frac{1}{|\Gamma_D|} \int_{\Gamma_D} \sigma_{ij}(\mathbf{u}) d\Gamma \tag{17}$$

3 NUMERICAL EXAMPLE

In this section, a three-dimentional linear elasticity problem is considered. A solid with a cylindrical hole is subjected to a uniaxial traction P, as shown in Fig. 1(a). Due to symmetry, only a quarter of the solid is modeled (See Fig. 1(b)). The edge lengths of the solid are $2a$ and its thickness is a, where a is the radius of the cylindrical hole. The following constant values are considered: modulus of elasticity $E = 200GPa$, Poissons ratio $\nu = 0.3$ and $P = 100MPa$.

In addition to the proposed goal-oriented mesh adaptivity (GOMA) procedure, mesh adaptivity based on global refinement, kelly error indicator [25] and conventional residual-based error estimation are performed and the results are compared. For this purpose, the local refinement/coarsening procedures are done applying hanging nodes.

3.1 Global mesh adaptivity

Initial and refined mesh configurations with 675, 4131, 28611 and 212355 number of degrees of freedoms (DoFs) with 3 DoFs per node (u_x, u_y, u_z) are shown in Fig. 2. In this refinement procedure, in each step, an element is uniformly devided to 8 new elements.

3.2 Mesh adaptivity based on Kelly error indicator

Initial and refined mesh configurations with 675, 2094, 5667, 16902, 51285 and 145527 Dofs are shown in Fig. 3. The meshes are refined based on the kelly error indicator [25].

3.3 Mesh adaptivity based on residual-based error estimation

Initial and refined nodal configurations with 675, 2178, 6786, 21567, 69081 and 229275 Dofs are shown in Fig. 4. The meshes are refined based on residual-based error estimator.

3.4 Goal-oriented mesh adaptivity for mean σ_{xx} on Γ_D

The mean stress σ_{xx} on the curved surface Γ_D (see Fig. 1(b)) is considered as quantity of interest for the solution. The propsoed dual-weighted residual is applied for mesh adaptivity. Initial and refined mesh configurations with 675, 2292, 6771, 18282 and 58323 Dofs are shown in Fig. 5.

3.5 Comparison of convergence rates

The mean σ_{xx} on Γ_D in the finest solution of the goal-oriented mesh adaptivity procedure is considered as the approximation of the exact QoI. Considering that, the approximated errors of QoI for different mesh configurations obtained by appling different

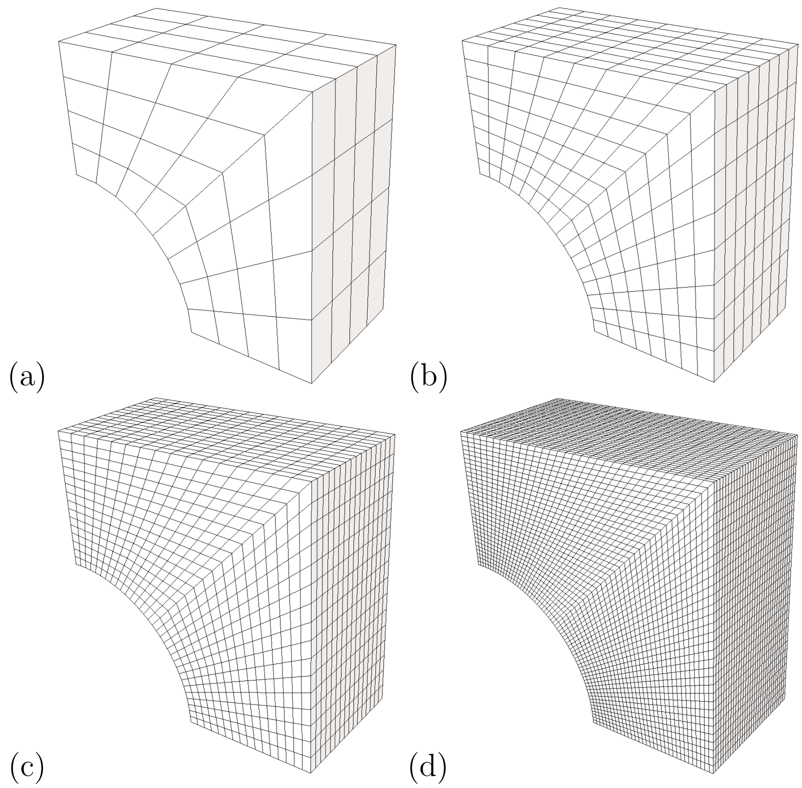

Figure 2: Global mesh adaptivity: (a) Initial mesh with 675 DoFs, (b) Refined mesh (1st step) with 4131 DoFs, c) Refined mesh (2nd step) with 28611 DoFs, d) Refined mesh (3rd step) with 212355 DoFs.

mesh adaptivity procedures are illustrated in Fig. 6(a). The corresponding convergence rates are also plotted in Fig. 6(b). While the obtained convegence rate of the GOMA procedure is 0.65, other refinement procedures based on residual error estimator, Kelly error indicator and global refinement results in the convergence rates 0.45, 0.37 and 0.33, respectively. The results clearly show the superiority of the Dual-Weighted Residual error estimation in mesh adaptation over other conventional error estimation approaches where a specified quantity is of interest.

4 CONCLUSIONS

In this paper, a goal-oriented error estimation approach called Dual-Weighted Residual has been applied in adaptive mesh refinement procedure to quantify and control the local error in quantities of interest. A three-dimensional linear elasticity example has been solved and the goal-oriented adaptivity has been applied considering the quantity of mean stress over a face. By comparing the results with other conventional residual- and recovery-based error estimation, the efficiency and advantages of the propsoed approach have been illustrated.

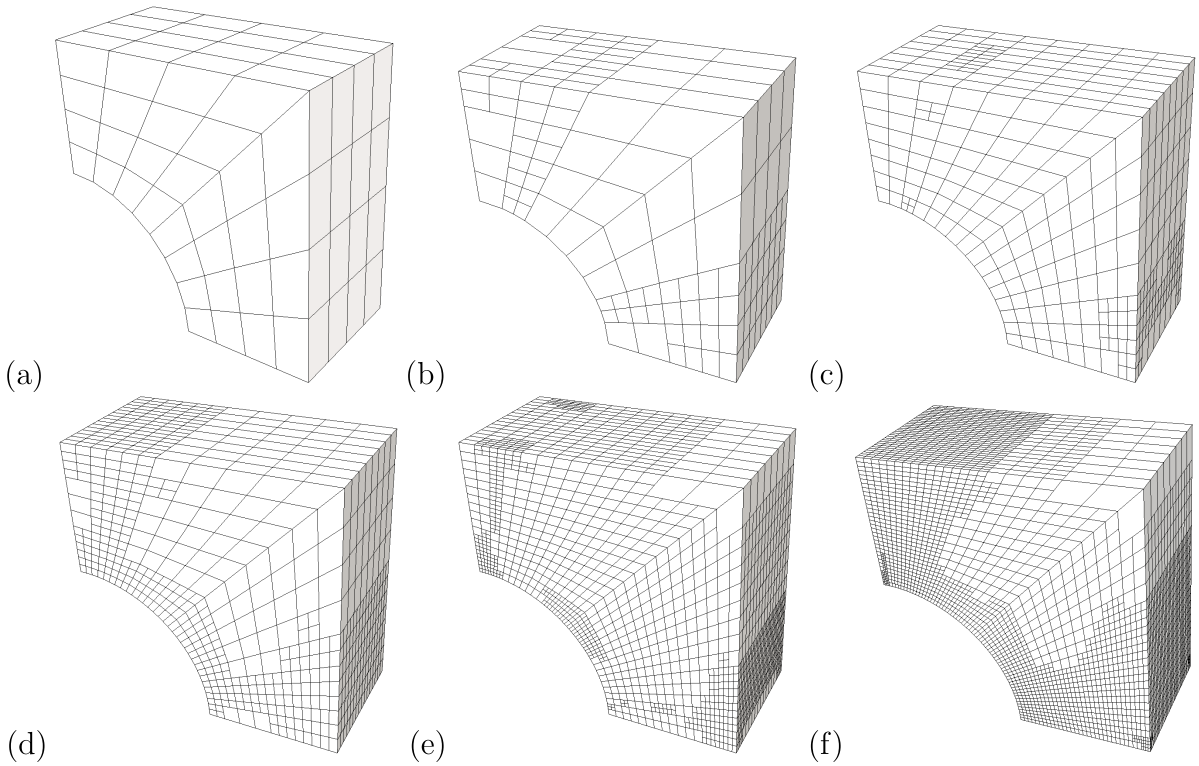

Figure 3: Mesh adaptivity based on Kelly error indicator, (a) Initial mesh with 675 DoFs, (b) Refined mesh (1st step) with 2094 DoFs, c) Refined mesh (2nd step) with 5667 DoFs, d) Refined mesh (3rd step) with 16902 DoFs, e) Refined mesh (4th step) with 51285 DoFs, f) Refined mesh (5th step) with 145527 DoFs.

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REFERENCES

[1] Zienkiewicz, O.C. and Taylor, R.L. *The finite element method*. McGraw Hill, Vol. I., (1989), Vol. II., (1991).

[2] Nguyen, V.P., Rabczuk, T., Bordas, S. and Duflot, M. Meshless methods: a review and computer implementation aspects. *Math. Comput. Simulat.* (2008) **79**(3):763-813.

[3] Rabczuk, T., Belytschko, T. Cracking particles: A simplified meshfree method for arbitrary evolving cracks. *Int. J. Numer. Meth. Eng.* (2204) **61**(13):2316-2343.

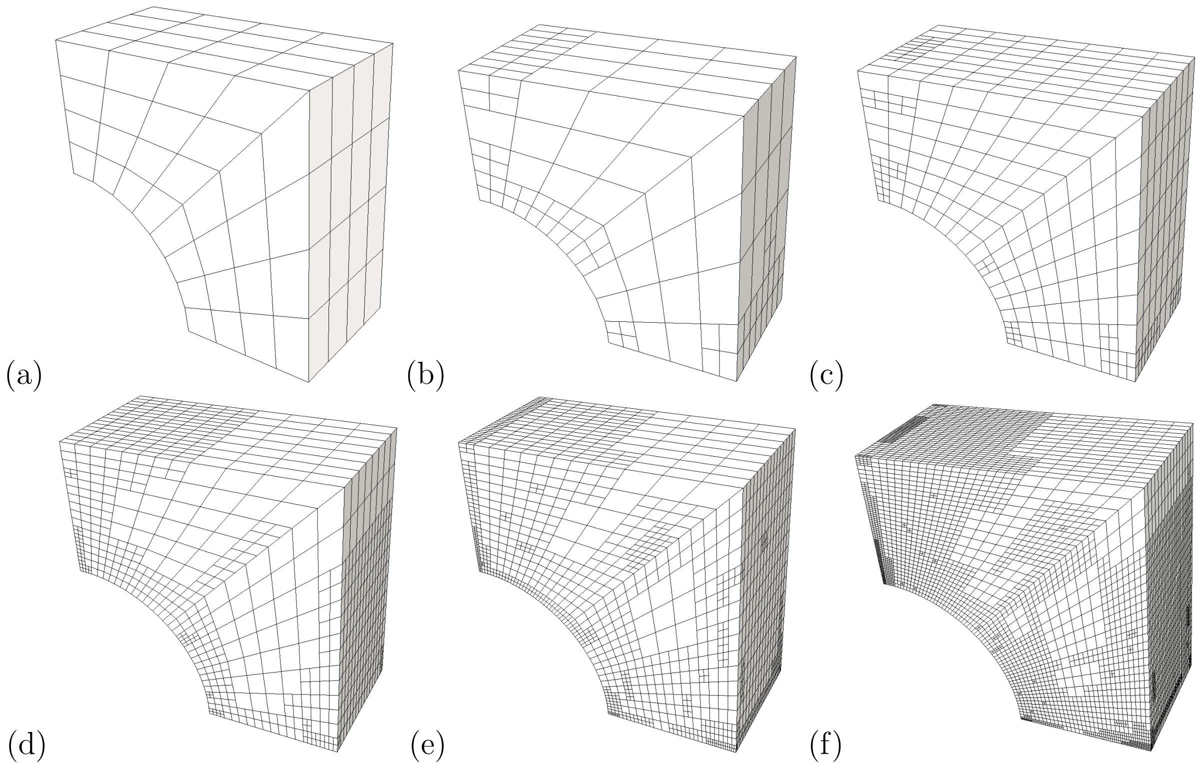
(a) (b) (c) (d) (e) (f)

Figure 4: Mesh adaptivity based on residual-based error estimation, (a) Initial mesh with 675 DoFs, (b) Refined mesh (1st step) with 2178 DoFs, c) Refined mesh (2nd step) with 6786 DoFs, d) Refined mesh (3rd step) with 21567 DoFs, e) Refined mesh (4th step) with 69081 DoFs, f) Refined mesh (5th step) with 229275 DoFs.

[4] Ghorashi, S.Sh., Mohammadi, S. and Sabbagh-Yazdi, S.-R. Orthotropic enriched element free Galerkin method for fracture analysis of composites *Eng. Fract. Mech.* (2011) **78**(9):1906–1927.

[5] Hughes, T.J.R., Cottrell, J.A. and Bazilevs, Y. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Comput. Method. Appl. M.* (2005) **194**:4135-4195.

[6] Valizadeh, N., Ghorashi, S.Sh., Yousefi, H., Bui, T.Q. and Rabczuk, T. Transient analysis of laminated composite plates using isogeometric analysis. *Proceedings of the Eighth International Conference on Engineering Computational Technology*, Civil-Comp Proceedings, (2012) Paper 43.

[7] Ghorashi, S.Sh., Valizadeh, N., Mohammadi, S., Ghasemzadeh, H. and Shojaee, S. Enhancement of isogeometric analysis method for analyzing 2D cracked problems using extrinsic enrichment functions. *Proceedings, Annual Conference - Canadian Society for Civil Engineering*, (2011) **2**: 1238-1247.

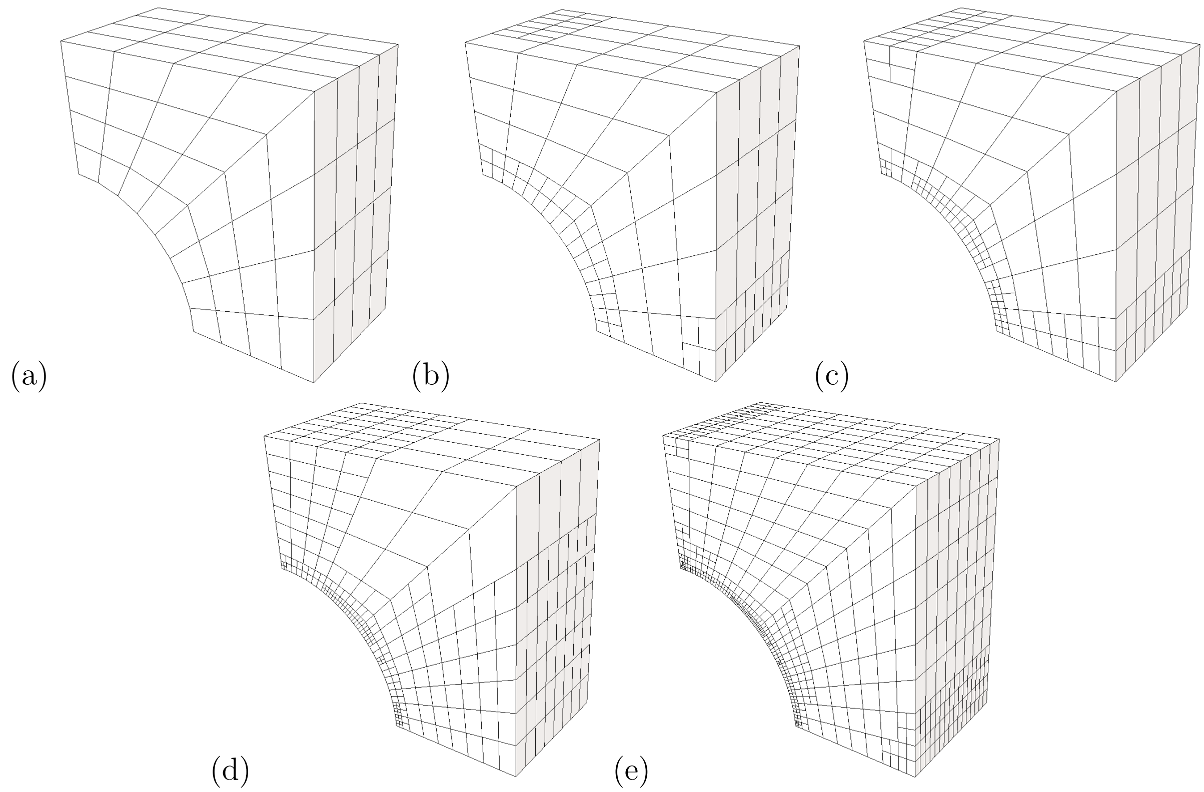

Figure 5: Goal-oriented mesh adaptivity considering mean σ_{xx} on Γ_D as QoI: (a) Initial mesh with 675 DoFs, (b) Refined mesh (1st step) with 2292 DoFs, c) Refined mesh (2nd step) with 6771 DoFs, d) Refined mesh (3rd step) with 18282 DoFs, e) Refined mesh (4th step) with 58323 DoFs.

[8] Ghorashi, S.Sh., Valizadeh, N. and Mohammadi, S. Extended isogeometric analysis for simulation of stationary and propagating cracks *Int. J. Numer. Meth. Eng.* (2012) **89**(9):1069–1101.

[9] Ghorashi, S.Sh., Valizadeh, N. and Mohammadi, S. Orthotropic enriched extended isogeometric analysis for fracture analysis of composites. *Proceedings of the Eighth International Conference on Engineering Computational Technology*, Civil-Comp Proceedings, (2012) Paper 46.

[10] Ghorashi, S.Sh., Valizadeh, N., Mohammadi, S. and Rabczuk, T. Extended isogeometric analysis of plates with curved cracks. *Proceedings of the Eighth International Conference on Engineering Computational Technology*, Civil-Comp Proceedings, (2012) Paper 47.

[11] Jia, Y., Anitescu, C., Ghorashi, S.Sh. and Rabczuk, T. Extended isogeometric analysis for material interface problems. *IMA J. Appl. Math.* (2014) doi:10.1093/imamat/hxu004.

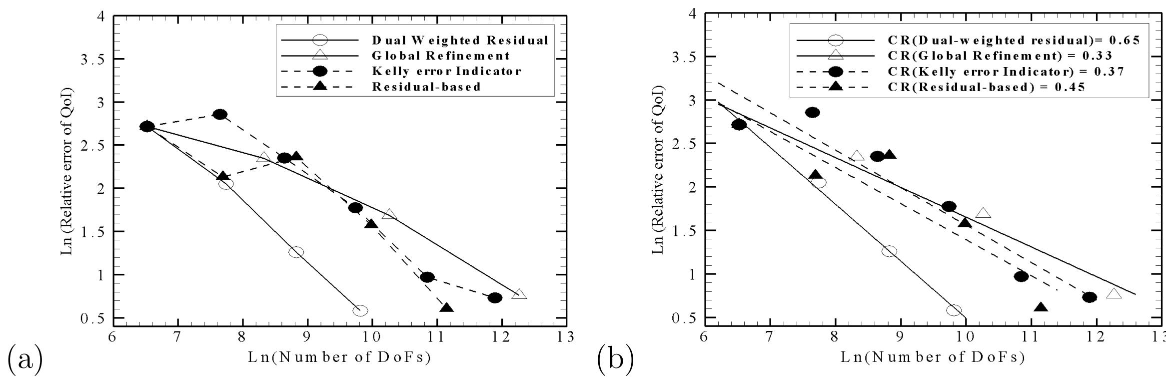

Figure 6: Approximated errors of the QoI, mean σ_{xx} on Γ_D: (a) Convergence curve and (b) Convergence rate.

[12] Babuška, I. and Rheinboldt, W.C. Aposteriori error estimates for the finite element method. *Int. J. Numer. Meth. Eng.* (1978) **12**:1597–1615.

[13] Amani, J., Afshar, M.H. and Naisipour, M. Mixed discrete least squares meshless method for planar elasticity problems using regular and irregular nodal distributions. *Eng. Anal. Bound. Elem.* (2012) **36**:894–902.

[14] Amani, J., Bagherzadeh, A.S. and Rabczuk, T. Error estimate and adaptive refinement in mixed discrete least squares meshless method. *Math. Probl. Eng.* (2014) **2014**, Art. ID 721240, 16 pages. doi:10.1155/2014/721240

[15] Zienkiewicz, O.C. and Zhu, J.Z. A simple error estimator and adaptive procedure for practical engineerng analysis. *Int. J. Numer. Meth. Eng.* (1987) **24**:337–357.

[16] Zienkiewicz, O.C. and Zhu, J.Z. The superconvergent path recovery and a posterriori error estimators. Part 1. The recovery technique. *Int. J. Numer. Meth. Eng.* (1992) **33**:1331–1364.

[17] Zienkiewicz, O.C. and Zhu, J.Z. The Superconvergent patch recovery and a posteriori error estimators. Part 2. Error estimates and adaptivity. *Int. J. Numer. Meth. Eng.* (1992) **33**:1365–1382.

[18] Becker, R. and Rannacher, R. A feed-back approach to error control in finite element methods: Basic analysis and examples. *EastWest J. Numer. Math.* (1996) **4**:237–264.

[19] Becker, R. and Rannacher, R. Weighted a posteriori error control in FE methods. In Bock, H. G. et al., ed.s, ENUMATH 95, pages 621637, Paris, September 1998. World Scientific Publ., Singapore.

[20] Bangerth, W. and Rannacher, R. Adaptive finite element methods for differential equations. Birkhäuser Verlag, Basel, (2003).

[21] Prudhomme, S. and Oden, J.T. On goal-oriented error estimation for local elliptic problems: application to the control of pointwise errors. *Comput. Method. Appl. M.*(1999) **176**:313–331.

[22] Stein, E., Rüter, M. and Ohnimus, S. Error-controlled adaptive goal-oriented modeling and finite element approximations in elasticity. *Comput. Method. Appl. M.* (2007) **196**:3598–3613.

[23] Zaccardi, C., Chamoin, L., Cottereau, R., Ben Dhia, H. Error estimation and model adaptation for a stochastic-deterministic coupling method based on the Arlequin framework. *Int. J. Numer. Meth. Eng.* (2013) **96**:87–109.

[24] González-Estrada, O.A., Nadal, E., Ródenas, J.J., Kerfriden, P., Bordas, S.P.A. and Fuenmayor, F.J. Mesh adaptivity driven by goal-oriented locally equilibrated superconvergent patch recovery. *Comput. Mech.* (2013) in press.

[25] Kelly, D.W., Gago, J.P. De S.R., Zienkiewicz, 0.C. and Babuška, I. A posteriori error analysis and adaptive processes in the finite element method: Part Ierror analysis *Int. J. Numer. Meth. Eng.* (1983) **19**(11):1593-1619.

[26] Bengerth, W., Hartmann, R., Kanschat, G. deal.IIA General-Purpose Object-Oriented Finite Element Library. *ACM Trans. Math. Soft.* (2007) **33**: Article 24.