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GENERALIZED FINITE DIFFERENCES ON STRUCTURED CONVEX GRIDS FOR IRREGULAR PLANAR DOMAINS

FRANCISCO DOMÍNGUEZ-MOTA, ERIKA RUIZ-DÍAZ, GERARDO TINOCO-GUERRERO, JOSÉ G. TINOCO-RUIZ AND ALEJANDRA VALENCIA

Facultad de Ciencias Físico-Matemáticas
Universidad Michoacana de San Nicolás de Hidalgo
Edificio B, Ciudad Universitaria, 58040, Morelia México
e-mail: dmota@umich.mx, dukie31@gmail.com, jtinoco@umich.mx, jstinoco@gmail.com, lexa.lvls@gmail.com

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Abstract. In structured grids on irregular regions, when the scheme coefficients are selected in order to satisfy an optimality condition given by the definition of consistency, the finite difference method is an intuitive and competitive option for the numerical solution of Poisson-like equations subject to Dirichlet boundary conditions. In this paper, we show the application of this finite difference scheme to unstructured grids which can be used as an alternative to linear finite elements on triangles.

1 INTRODUCTION

In this paper, we show the variational finite difference scheme for the numerical solution of Poisson-like problems using unstructured grids on irregular planar domains with Dirichlet boundary conditions. Finite difference schemes can be generalized by considering a finite set of nodes $p_0, p_1, ..., p_q$, for which it is required to find coefficients $\Gamma_0, \Gamma_1, ..., \Gamma_q$ such that

$$\frac{\partial^m u(p_0)}{\partial x^l \partial y^{m-l}} \approx \sum_{i=0}^{q} \Gamma_i u(p_i). \tag{1}$$

In spite of the fact that the basic idea behind the equation (1) is quite simple, there are few efficient schemes for such kind of regions (See, for instance, Castillo *et al* [7, 8], Shashkov [11], and Tinoco *et al* [1]).

In this paper, we use a variational finite difference scheme which can derived as follows: let A, B, C, D and E be smooth functions of two variables. In order to approximate the second order linear operator in the left hand side of

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = f \tag{2}$$

at the grid point p_0 by means of the difference scheme

$$L_0(p_0) \equiv \sum_i \Gamma_i(p_0, p_i)u(p_i),$$

we impose the consistency condition given by [10]

$$[Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y]_{p_0} - L_0(p_0) \to 0$$

when $p_1, ...p_q \to p_0$, which implies that the local truncation error tends to zero.

At every inner grid point node $p_0 = (x_0, y_0)$, let us consider its q neighbors $p_1, p_2, \cdots, p_q$ with cartesian coordinates $p_i = (x_i, y_i)$. The coefficients $\Gamma_i(p_0, p_i)$ in $L_0(p_0)$ required to approximate the derivatives at the left hand side of (2) up to second order accuracy satisfy the linear system

$$\begin{pmatrix} 1 & 1 & ... & 1 \\ 0 & \Delta x_1 & ... & \Delta x_q \\ 0 & \Delta y_1 & ... & \Delta y_q \\ 0 & (\Delta x_1)^2 & ... & (\Delta x_q)^2 \\ 0 & \Delta x_1 \Delta y_1 & ... & \Delta x_q \Delta y_q \\ 0 & (\Delta y_1)^2 & ... & (\Delta y_q)^2 \end{pmatrix} \begin{pmatrix} \Gamma_0(p_0, p_0) \\ \Gamma_1(p_0, p_1) \\ \Gamma_2(p_0, p_2) \\ . \\ . \\ . \\ \Gamma_q(p_0, p_q) \end{pmatrix} = \begin{pmatrix} F(p_0) \\ D(p_0) \\ E(p_0) \\ 2A(p_0) \\ B(p_0) \\ 2C(p_0) \end{pmatrix}, \tag{3}$$

where $\Delta x_i = x_i - x_0$, $\Delta y_i = y_i - y_0$.

Hereinafter, for the sake of brevity, $\Gamma(p_0, p_i)$ will be denoted simply as Γ_i.

One must note that, in general, system (3) is underdetermined. Thus, in order to calculate the coefficients $\Gamma_0, \Gamma_1, ..., \Gamma_q$ at the inner grid points, several alternatives can be considered. For instance, the third order residuals can also be included and the local optimization problem

$$\begin{array}{cc} \min & z = R_6^2 + R_7^2 + R_8^2 + R_9^2, \\ \text{subject to} & \\ & R_k = 0, \; k = 0, ..., 5. \end{array}$$

solved, where

$$\begin{array}{ll} R_0 = \sum_{i=0}^{q} \Gamma_i - F, & R_3 = \sum_{i=1}^{q} \Gamma_i (\Delta x_i)^2 - 2A, \\ R_1 = \sum_{i=1}^{q} \Gamma_i (\Delta x_i) - D, & R_4 = \sum_{i=1}^{q} \Gamma_i (\Delta x_i)(\Delta y_i) - B, \\ R_2 = \sum_{i=1}^{q} \Gamma_i (\Delta y_i) - E, & R_5 = \sum_{i=1}^{q} \Gamma_i (\Delta y_i)^2 - 2C, \end{array}$$

$$\begin{array}{l} R_6 = \sum_{i=1}^{q} \Gamma_i (\Delta x_i)^3 \\ R_7 = \sum_{i=1}^{q} \Gamma_i (\Delta x_i)^2 (\Delta y_i) \\ R_8 = \sum_{i=1}^{q} \Gamma_i (\Delta x_i)(\Delta y_i)^2 \\ R_9 = \sum_{i=1}^{q} \Gamma_i (\Delta y_i)^3. \end{array}$$

(See [3, 4]).

This problem must be solved at every inner grid node to produce the optimal local approximation to the left hand side of (2). Another approach was given in [12], where we proposed an efficient heuristic scheme based on an unconstrained optimization problem which is closely related to the constrained one. First, we separate the first equation of the matrix system (3)

$$\sum_{i=1}^{q} \Gamma_i - F = 0 \tag{4}$$

and then we solve the least squares problem defined by

$$\begin{pmatrix} \Delta x_1 & ... & \Delta x_q \\ \Delta y_1 & ... & \Delta y_q \\ (\Delta x_1)^2 & ... & (\Delta x_q)^2 \\ \Delta x_1 \Delta y_1 & ... & \Delta x_q \Delta y_q \\ (\Delta y_1)^2 & ... & (\Delta y_q)^2 \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ . \\ . \\ . \\ \Gamma_q \end{pmatrix} = \begin{pmatrix} D(p_0) \\ E(p_0) \\ 2A(p_0) \\ B(p_0) \\ 2C(p_0) \end{pmatrix}. \tag{5}$$

through the Cholesky factorization of its normal equations

$$M^T M \ \Gamma = M^T \beta, \tag{6}$$

where

$$M = \begin{pmatrix} \Delta x_1 & ... & \Delta x_q \\ \Delta y_1 & ... & \Delta y_q \\ (\Delta x_1)^2 & ... & (\Delta x_q)^2 \\ \Delta x_1 \Delta y_1 & ... & \Delta x_q \Delta y_q \\ (\Delta y_1)^2 & ... & (\Delta y_q)^2 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ . \\ . \\ . \\ \Gamma_q \end{pmatrix}, \qquad \beta = \begin{pmatrix} D(p_0) \\ E(p_0) \\ 2A(p_0) \\ B(p_0) \\ 2C(p_0) \end{pmatrix}.$$

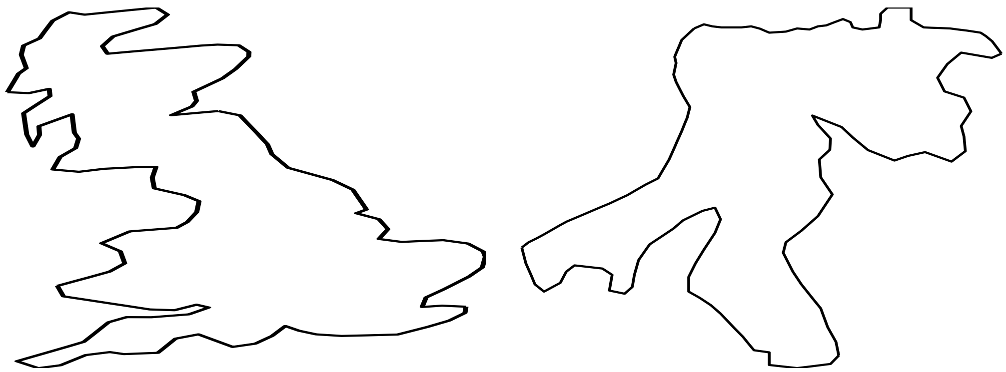

Figure 1: Regions ENG and HAB.

Next, Γ_0 is obtained from (4).

In summary, to calculate the approximation to the solution of equation (2), the following algorithm is applied:

1. The boundary of Ω is approximated by a polygonal Jordan curve.
2. A grid G for Ω is generated.
3. L is discretized using G; *i.e.*, is replaced by a linear combination of the values of u at the inner grid nodes in G as described above.
4. The algebraic system is solved using a suitable method: Gauss-Seidel, SOR, Sparse Gaussian Elimination.

2 NUMERICAL TESTS

For the numerical tests, we have selected 6 polygonal regions, most of them approximations to actual geographical locations: will be denoted as ENG, HAB, MEX, MIC, SWA and UCH (Figures 1, 2, 3), allof them scaled and shifted in order to lie inside $[0, 1] \times [0, 1]$. For these regions, Delaunay triangulations were generated using an adaptation of DistMesh [17]. The resulting on structured grids were used to calculate the Γ_i coefficients applying the heuristic scheme as described in the previous section and then the equation $-\nabla^2 u = f$ was solved numerically.

In all cases, the algebraic systems for finite differences obtained frox m (2) were assembled and solved by Sparse Gaussian elimination. Function u was selected as (See [11])

$$u = 2\exp(2x + y),$$

and f inside the domain was chosen in such a way that u was the exact solution. The boundary conditions were selected accordingly.

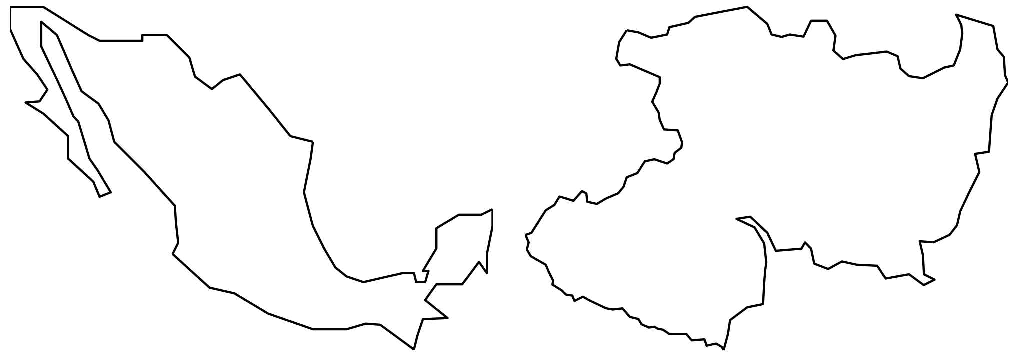

Figure 2: Regions MEX and MIC.

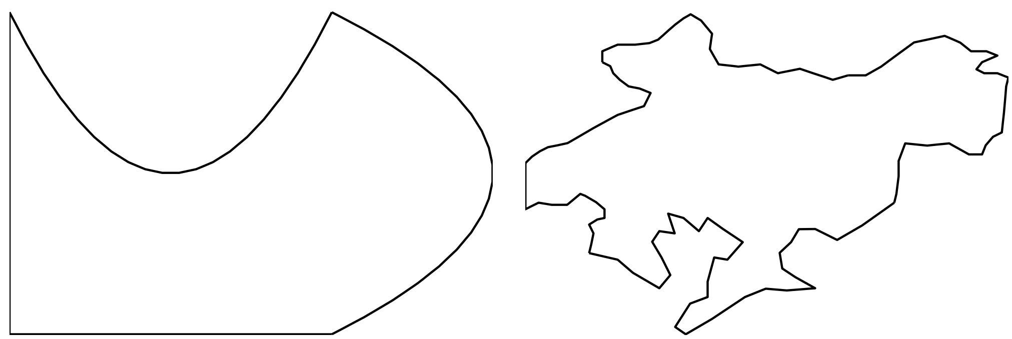

Figure 3: Regions SWA and UCH.

To compare the accuracy of the calculated solutions, linear finite elements $\phi_i(x, y) = a_i + b_i x + c_i y$ on the i^{th} triangular elements were used [18]. The $\| \cdot \|_2$ error norm for the tests is summarized in table (1). It was calculated as the grid function

$$\|u - U\|_2 = \sqrt{\sum_i (u_i - U_i)^2 \mathcal{A}_i} \quad ,$$

where u and U are the exact and the approximated solution calculated at the i^{th}-element respectively, and $\mathcal{A}_i$ is the area of the triangular element.
In table (1), N stands for the number of triangular elements in the grid, FE is the quadratic error for the finite element approximation, and FD is the corresponding for finite differences; MFE and MFD are the maximum errors for the finite elements and finite differences, respectively. We also present in figures (4), (5), (6) and (7), the sketch of the calculated solution for the regions ENG, HAB, MIC and UCH.

From the numerical results that the proposed approach using finite differences on the unstructured grids for the selected test regions produces satisfactory results.

3 CONCLUSIONS AND FUTURE WORK

In this paper, we considered a general difference scheme which produces satisfactory

Table 1: Quadratic errors.

Region	N	FD	FE	MFD	MFE
eng21s	276	1.1476×10^{-04}	1.7351×10^{-03}	1.8296×10^{-3}	4.6956×10^{-2}
eng41s	999	1.3214×10^{-05}	1.4789×10^{-04}	3.5972×10^{-4}	2.0269×10^{-2}
eng81s	3763	2.4046×10^{-06}	6.9386×10^{-04}	1.0980×10^{-4}	5.3931×10^{-2}
hab21s	384	1.9895×10^{-04}	2.1748×10^{-03}	2.1109×10^{-3}	2.2651×10^{-2}
hab41s	1382	3.5269×10^{-05}	4.9405×10^{-04}	4.8706×10^{-4}	9.3081×10^{-3}
hab81s	5326	2.2400×10^{-04}	1.1263×10^{-04}	2.6384×10^{-2}	2.5096×10^{-3}
mex21s	183	1.2667×10^{-04}	9.3252×10^{-04}	7.9489×10^{-4}	1.2292×10^{-2}
mex41s	571	3.6579×10^{-05}	3.1195×10^{-04}	4.9381×10^{-4}	4.4307×10^{-3}
mex81s	2013	8.6614×10^{-06}	9.2237×10^{-05}	2.9020×10^{-4}	1.6004×10^{-3}
mic21s	428	6.2461×10^{-05}	4.1359×10^{-04}	1.7779×10^{-3}	8.8014×10^{-3}
mic41s	1560	1.1604×10^{-05}	1.0389×10^{-04}	5.2738×10^{-4}	2.0318×10^{-3}
mic81s	5961	2.2333×10^{-06}	2.5900×10^{-05}	2.1352×10^{-4}	1.0621×10^{-3}
swa21s	415	7.2817×10^{-04}	1.7317×10^{-03}	1.6404×10^{-2}	2.1067×10^{-2}
swa41s	1491	8.7149×10^{-05}	5.5218×10^{-04}	6.7184×10^{-4}	1.1256×10^{-2}
swa81s	5689	1.9908×10^{-05}	8.2795×10^{-05}	6.4480×10^{-5}	9.2299×10^{-4}
uch21s	341	1.5511×10^{-04}	2.1848×10^{-03}	1.4383×10^{-3}	2.6050×10^{-2}
uch41s	1187	3.6082×10^{-05}	3.1879×10^{-04}	6.0554×10^{-4}	4.7423×10^{-3}
uch81s	4536	1.4963×10^{-05}	1.2540×10^{-04}	9.6960×10^{-4}	3.2938×10^{-3}

results on triangulations, and suggests that the use of suitable finite differences can indeed be considered as a reliable alternative for producing reasonable approximations in a simple way.

Our current research deals with solving time-dependent partial differential equations using the proposed scheme and a robust software implementation, as well as with some theoretical considerations regarding (4). The corresponding results will be reported in a future paper.

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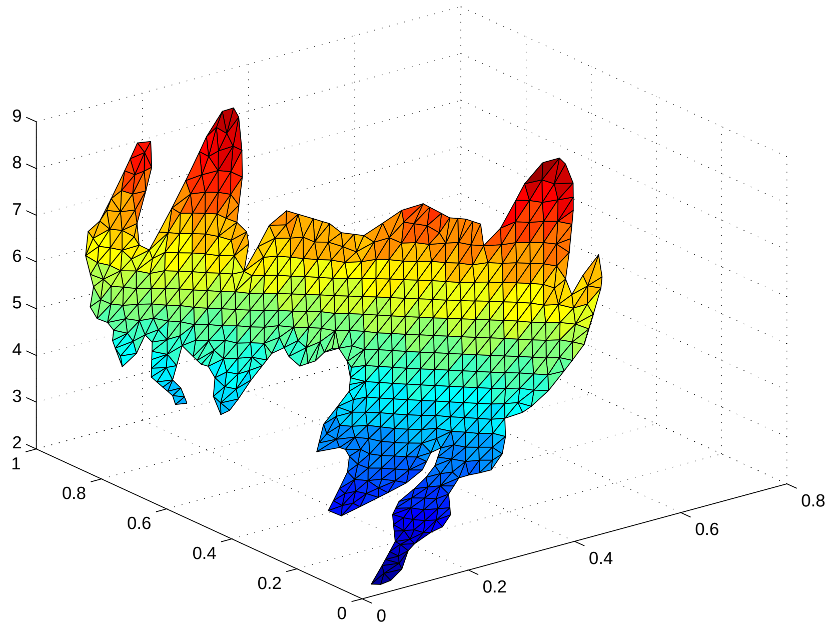
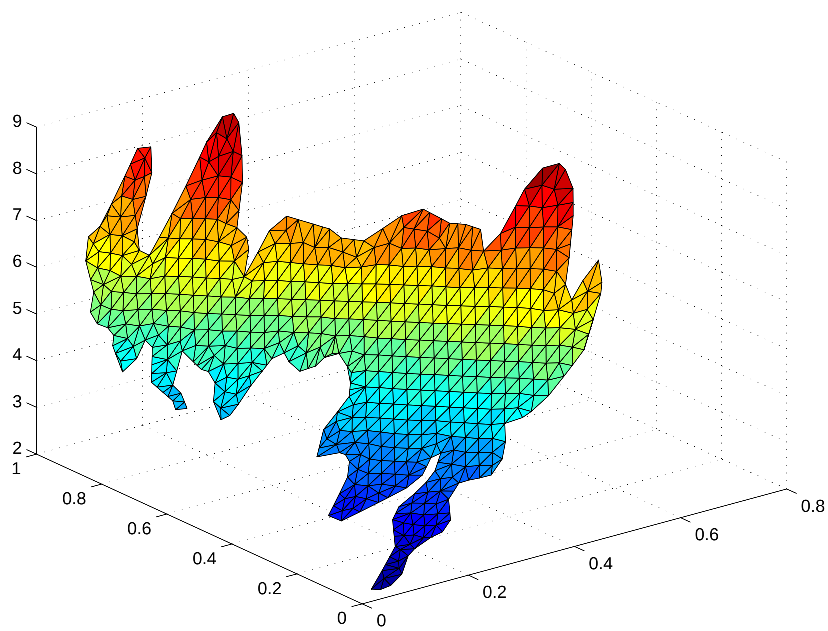
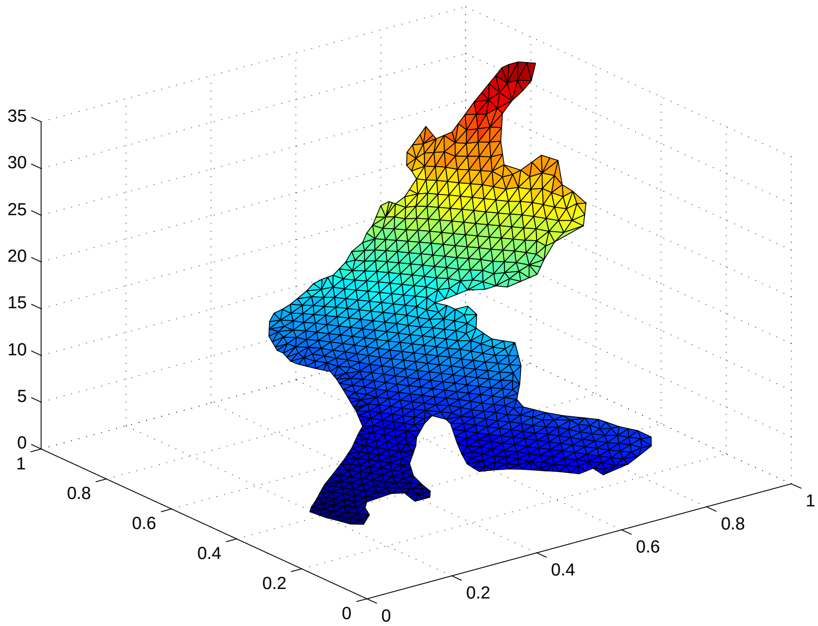


Finite Difference Solution.

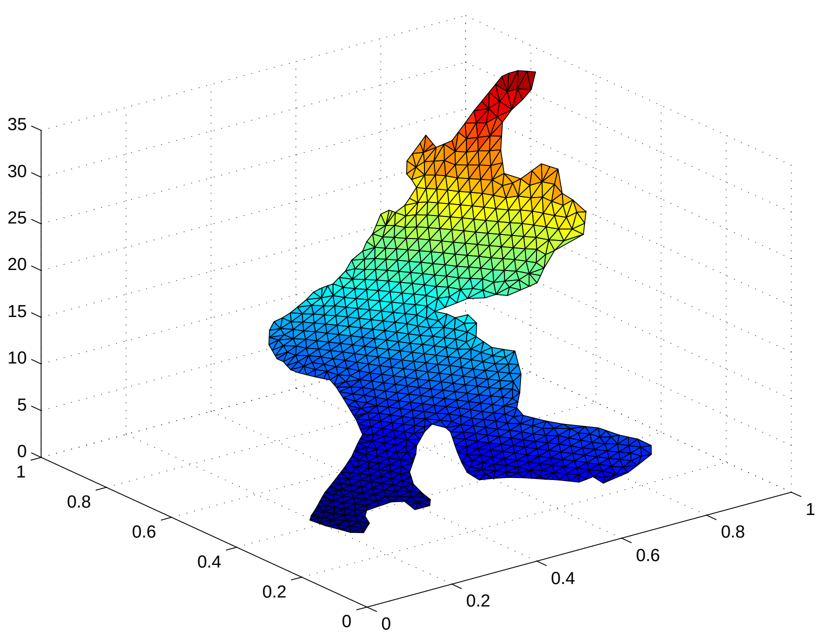


Finite Element Solution.

Figure 4: Solutions for ENG

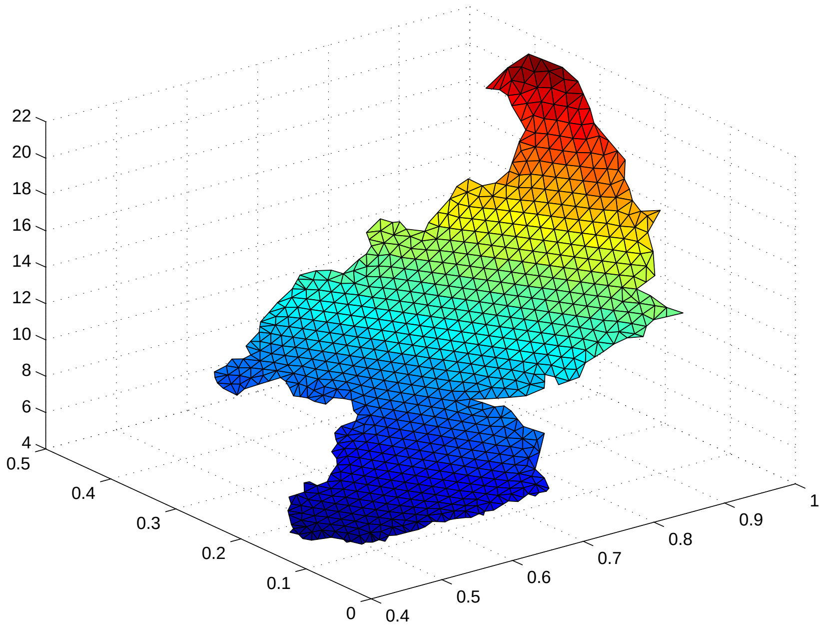


Finite Difference Solution.

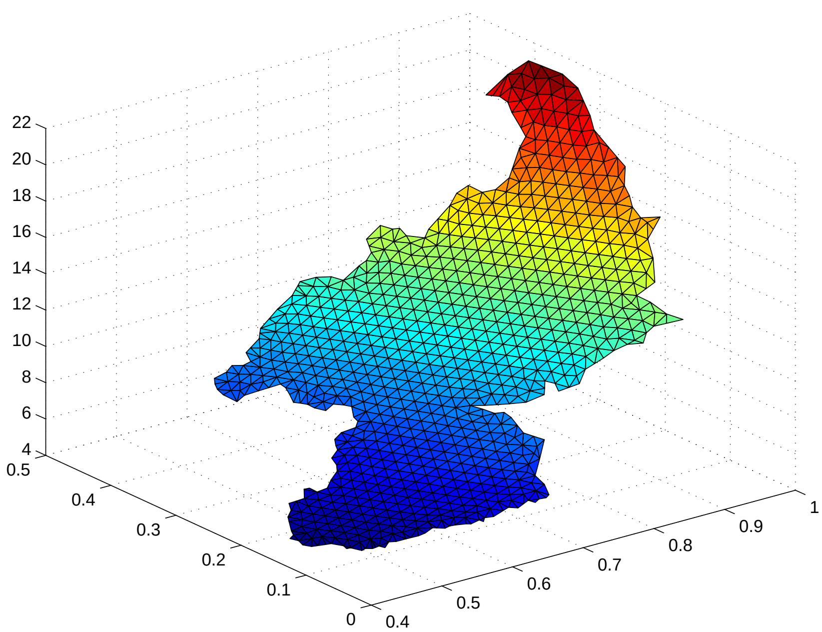


Finite Element Solution.

Figure 5: Solutions for HAB

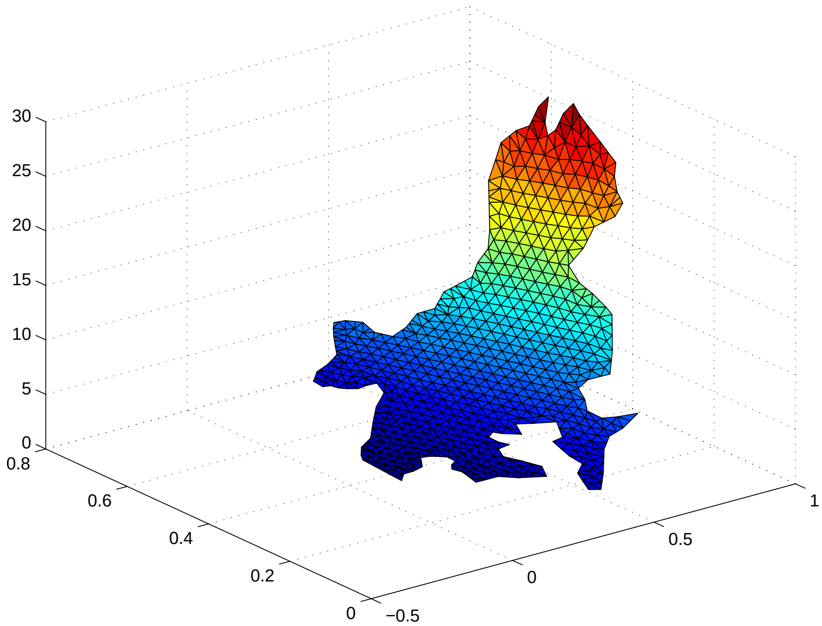


Finite Difference Solution.

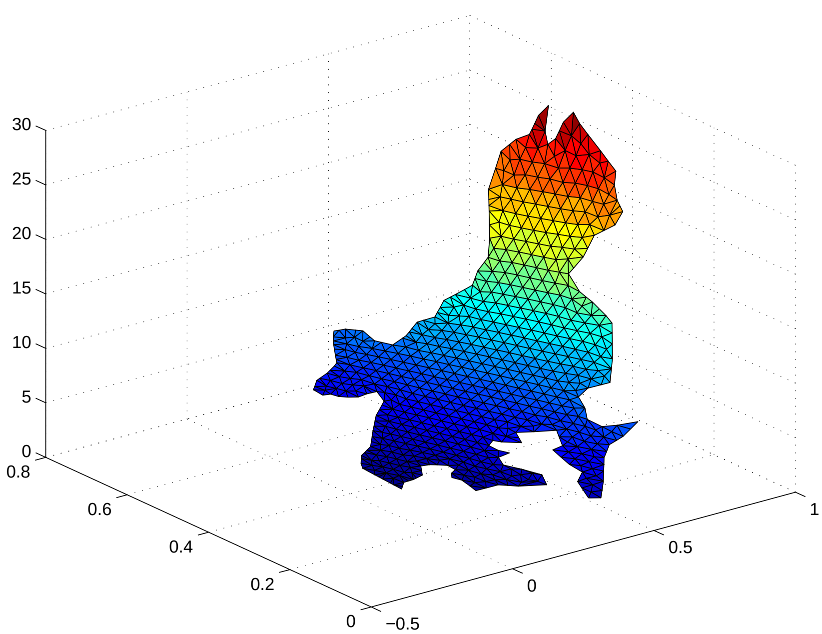


Finite Element Solution.

Figure 6: Solutions for MIC



Finite Difference Solution.



Finite Element Solution.

Figure 7: Solutions for UCH