11th World Congress on Computational Mechanics (WCCM XI)
5th European Conference on Computational Mechanics (ECCM V)
6th European Conference on Computational Fluid Dynamics (ECFD VI)
E. Oñate, J. Oliver and A. Huerta (Eds)

THE ANALYSIS OF THE EFFIECIENCY OF AN ADAPTATION METHOD BASED ON THE GRID GENERATOR

Jan Kucwaj*

* Institute for Informatics,
Department of Physics, Mathematics and Informatics,
Cracow University of Technology,
ul. Warszawska 24, 31-155 Cracow, Poland,,
e-mail: jkucwaj@pk.edu.pl, web page: http://www.pk.edu.pl/ jkucwaj/

†Polish Association for Computer Methods in Mechanics

Key words: adaptivity, mesh generation, error estimation, finite element methods, finite difference methods, plasticity.

Abstract. The main goal of presented the paper is the numerical analysis of convergence of the adaptation algorithm [6] developed by the author based on remeshing. The main feature of the considerd algorithm is an application of the mesh generator with a mesh size function [6]. The proposed method uses a sequence of meshes obtained with successive modification of the mesh size function. The rate is obtained numerically conidering known solution and unknown solution as well. In the second case some properties are known, so they can be observed on the approximate solutions.

1 INTRODUCTION

The purpose of the paper is to apply the adaptive remeshing [8, 9] algorithm to the elastic-plastic twisting of bars with hardening [5]. The problem problem is posed in a form of the search for the stationary point of the functional defined on function space of infinite dimension satisfying homogenous boundary conditions. For the sake of the numerical solution the infinite space is approximated by finite dimensional space spanned by a given set of basis functions [2, 12, 13], the approximated solution to the problem is equal to a linear combination of the basis functions. The coeffiecients of the linear combination are found from the nonliner algebraic system of equations. The system is denoted b the stationarity conditions. The algebraic system of nonliner algebraic equations is solved by the Newton-Raphson method. In consequtive steps of the algorithm the values of the mesh size function taken at the nodes are so modified that at the points with greatest values of an error indicators the values of mesh size function are the most diminished. Having the values of the mesh size function at nodes the new mesh size function is defined by the linear interpolation. The process is performed till the error indicator obtains satisfied value. The error indicator is found at every node as a measure of discontinuity of derivatives values at the node, which are calculated at elements coming from the node.

The presented numerical analysis of convergence suggests better than linear dependence between number of degrees of freedom and error norm for derivatives. In further development it is planned to generalize the method to apply anisotropic meshes. The proposed method was applied to both problems, in which the solutions is known and unknown. The obtained results were in consistence with physical interpretations [6].

The adapted mesh for an example problem, where the strict solution is known, is presented. It can be observed that the rapid change of the mesh size function corresponds to the great gradient of the solution. Additionally it can be said that the final adapted depends on both the solution and the assumed error indicator. The proposed method was applied to the elastic-plastic twisting of bars with hardening.

2 Problem formulation

2.1 The Poisson equation

The boundary problem for the Poisson equation is formulated as follows:

$$\mathbf{\Delta u = f(x,y), \quad in \quad \Omega,} \tag{1}$$

$$\mathbf{u = 0 \quad in \quad \partial\Omega.} \tag{2}$$

This equation is used for error indicators calculation:

$$\mathbf{e_i = \tilde{\Delta} u_h(P_i) - f(x,y)} \text{ at } i\text{-th node} \tag{3}$$

where $\tilde{\Delta}$ is an approximation of Δ. hspace5mm In this case of the Poisson equation this problem is equivalent to the search for a stationary point of the following functional:

$$\mathbf{I(u) = \int_\Omega (u_x^2(x,y) + u_y^2(x,y) + 2u(x,y)f(x,y)) d\Omega} \tag{4}$$

2.2 The elastic-plastic twisting of bars with hardening

In this section the elastic-plastic twisting of bars with hardening is formulated. According to [5] the problem can be led to search for the extremum of the following functional:

$$I(u) = \int \int_\Omega [\int_0^T sg(s)ds - 2\omega u] d\Omega, \tag{5}$$

where T is the stress intensity

$$T = \sqrt{(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2}, \qquad \tau_{13} = \frac{\partial u}{\partial y}, \qquad \tau_{23} = -\frac{\partial u}{\partial x}$$

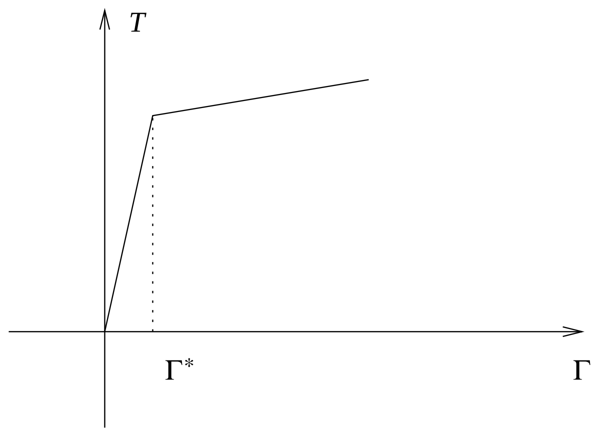

Figure 1: The dependence between the strain and stress intesity

.

The function g defines the dependence between the effective stress and the effective strain: $T = g(\Gamma)\Gamma$ (Fig. 1), where $\Gamma = \sqrt{\epsilon_{ij}\epsilon_{ij}}$, ϵ_{ij} is strain tensor and ω is the torsion angle.

After the substitution $s = \sqrt{r}$, it is obtained:

$$I(u) = \int\int_{\Omega}[\int_{0}^{T^2} g(\sqrt{r})\frac{1}{2}dr - 2\omega u]d\Omega. \qquad (6)$$

3 The unstructured grid generation with mesh size function in arbitrary domains

Grid generation with arbitrary size is performed by 2-D generator [7, 8]. The main idea of grid generation is based upon the algorithm of the advancing front technique and generalization of Delaunay triangulation for wide class of $2-D$ domains. It is assumed, that the domain is multiconnected with arbitrary numbers of internal loops. The boundary of the domain may be composed of the following curves:

- a straight line segment,
- an arc of circle,
- a B-spline curve.

In case of the advancing front technique combined with Delaunay triangulation the point insertion and triangulation can be divided into the following steps:

1. points generations on the boundary components of the boundary of the domain,
2. internal points generation by the advancing front technique,

3. Delaunay triangulation of the previously obtained set of points,

4. Laplacian smoothing of the obtained mesh.

The algorithm for boundary points generation depends upon the type of boundary segment.

4 Algorithm of remeshing

The whole algorithm of the adaptation is realized in successive generation of a sequence of meshes $\{\mathbf{T}_\nu\}$, where $\nu = 0, 1, 2, \ldots$ with a modified mesh size function. By using every mesh of the sequence the problem is solved and then appropriate error indicator at every element is calculated. The values of the error indicator are reduced to the nodes by an averaging method. Having values of errors at nodes a continuous error function in the whole domain is constructed by using piecewise linear interpolation at all the elements. The error function is appropriately transformed to obtain a multiplier for mesh size function.

The proposed approach gives the possibility to solve the considered problem on well-conditioned meshes and to obtain optimal graded meshes.

4.1 Remeshing Scheme

The algorithm of remeshing can be divided into the following steps:

1. preparation of the information about the geometry and boundary conditions of the problem to be solved,

2. fixing an initial mesh size function,

3. mesh generation with the mesh size function,

4. solution to the considered problem on the generated mesh,

5. evaluation of error indicator in every element,

6. calculation of nodal error indicator values by using averaging method,

7. definition of the new mesh size function by using the errors found at every point,

8. if error is not small enough go to the point 3,

9. end of computations.

In the examples solved by the author of the paper it was sufficient to make from 3 to 7 steps of adaptation.

4.2 Error indicators

The applied indicators are calculated for every element or directly at the nodes [8, 15]: Let e_i for $i = 1, \ldots, n_0$ be an error indicator at *i-th* apex of the grid $\mathcal{T}_0$, and $\mathcal{P}_0$={ P_i, i=1,..., n_P} − set of nodes. We define a patch of elements incidental for the given node P_i as:

$$L_i = \{k : P_i \in \overline{T}_k\} \text{ for } i = 1, \ldots, n_P. \tag{7}$$

1. Let N_i be a set of neighbours of *i-th* element e.q.:

   $$N_i = \{k : \ T_k \ \textit{has a common edge with} \ T_i\}, \tag{8}$$

   $$\text{then} \qquad \tilde{e}_i = \sqrt{\sum_{k \in N_i} (\frac{\partial u_i}{\partial n_k} - \frac{\partial u_k}{\partial n_k}^2)^2}, \tag{9}$$

   where u_i is the restriction of the solution to *t-th* element and n_k is unit normal to the the edge common with *k-th* and *i-th* element.

2. In this case it is suggested to introduce directlty values of error indicator at every node of the mesh. The error indicator is suggested by the author. From the numerical analyses it follows, that the usage of this error indicator causes generation similar meshes to the firstly defined.

   $$e_i = \sqrt{\sum_{k \in L_i,\ l \in L_i,\ l \neq k} (\frac{\partial u_i}{\partial x} - \frac{\partial u_k}{\partial x}^2)^2 + (\frac{\partial u_i}{\partial y} - \frac{\partial u_k}{\partial y}^2)^2}, \tag{10}$$

   where L_i is the set of numbers of elements meeting at *i-th* node.

4.3 Modification of the mesh size function

The modification of the mesh size function is performed at every adaptation step for the realization of the next one. The main idea of this part of the algorithm relies on a multiplication of the values of the mesh size function by an appropriately chosen function. The chosen function is continuous, linear and has the smallest value at node where the value of the error indicator is maximal and the greatest where the value of the error is minimal. It increases when error decreases. To describe the algorithm of the mesh size function modification it is necessary to reduce the values of the error indicators to nodes. For every node P_i the weighted averaged value of the indicator is defined as follows:

$$\tilde{e}_i = \frac{\sum_{k \in L_i} area(T_k) e_k}{\sum_{k \in L_i} area(T_k)}, \quad where \tag{11}$$

$$L_i = \{k : \ P_i \in T_k\} \ \text{ and } T_k \text{ is the } k\text{-}th \text{ element.} \tag{12}$$

In such a way a set of values of the error at every nodal point is obtained.

$$\alpha = \min_{k=1,2,\ldots,N_{NOD}} \tilde{e}_k, \qquad \beta = \max_{k=1,2,\ldots,N_{NOD}} \tilde{e}_k, \tag{13}$$

where N_{NOD} is the number of nodes. Obviously, $\alpha \leq \tilde{e}_k \leq \beta$ for $k = 1, \ldots, N_{NOD}$.

The following new values are introduced:

λ − a value indicating the greatest mesh size function reduction,

μ − a value indicating the smallest mesh size function reduction.

Usually λ and μ have positive values and usually is less than 1, and additionally $\mu < \lambda$. The following transformation is defined

$$l : [\alpha, \beta] \mapsto [\mu, \lambda] \tag{14}$$

which satisfies the conditions: $l(\alpha) = \lambda$ and $l(\beta) = \mu$. By these assumptions it can be observed that $\mu \leq l(x) \leq \lambda$.

Provided that

$$Q_i = l(\tilde{e}_i) \ for \ i = 1, \ldots, N_{NOD}, \tag{15}$$

then we have: $\min_{i=1,2,\ldots,N_{NOD}} Q_i = \mu, \quad \max_{i=1,2,\ldots,N_{NOD}} Q_i = \lambda.$

Introducing the function $r : \overline{D} \mapsto I\!R$ as follows: $r(\overline{x}) = \Pi(\overline{x}), \ if \ \overline{x} \in \overline{T}_s$, where Π is an affine mapping of two variables satisfying the following equalities are true:

$$\Pi(P_i) = Q_i \ for \ i = 1, 2, 3, \tag{16}$$

where P_1, P_2, P_3 are the vertices of the triangle T_s of the triangulation of Ω, and appropriately Q_1, Q_2, Q_3 are the values defined by the formula (15). The function $r(\overline{x})$ is defined in the whole domain because the triangles $\{\overline{T}_s\}_{s=1}^{n_e}$ cover it. The new mesh size function is defined as follows:

$$\gamma_{i+1}(\overline{x}) = \gamma_i(\overline{x}) r(\overline{x}). \tag{17}$$

As $\mu \leq r(\overline{x}) \leq \lambda$ then $\mu\gamma_i(\overline{x}) \leq \gamma_{i+1}(\overline{x}) \leq \lambda\gamma_i(\overline{x})$.

It can be checked that: $\exists \overline{x}, \overline{y} \in \overline{\Omega}$ such, that: $\mu\gamma_i(\overline{x}) = \gamma_{i+1}(\overline{x}), \ and \ \gamma_{i+1}(\overline{y}) = \lambda\gamma_i(\overline{y})$.

$$\text{It can be shown, that } \ ||\gamma_{i+1} - \gamma_i||_{\overline{\Omega}, max} \leq ||\gamma_i||_{\Omega, max} \max\{|1-\mu|, |1-\lambda|\} \tag{18}$$

$$\text{where } \ ||\gamma||_{\Omega, max} := \max_{\overline{x} \in \Omega}\{|\gamma(\overline{x})|\}. \tag{19}$$

5 Numerical Examples

A way of size function modification depends on an error indicator and on the coeffiecients λ ,μ, which determine amount mesh size reduction. If the values of the coeffiecients λ ,μ are small less number of adaptation steps is necessary. How quickly an adapted grid will be close enough to an optimal mesh, except of error indicator function, it depends on an initial mesh too. Tn the solved problems it was assumed that $\lambda = 0.6$ i $\mu = 0.9$, what caused performing greater number of iteration. In nonlinear problems sometimes it is useful to perform greater number of adaptation steps, what may lead to a better solution. In the plasticity theory problems it can be observed (3, 4), that how starting from the coarse mesh in consequtive iteration the yiels points are marked on the final mesh. It would rather impossible to obtain the effect by the methods based on mesh enrichment [1, 3, 11].

For the sake of numerical rate of the convergence of the proposed method in the problem defined in 4 the function f was defined in the way the solution to the problem is the the function [15]: $u(x, y) = x(1 - x)y(1 - y)\arctan(a(\frac{x+y}{\sqrt{2}} - \xi))$, where $a = 20$ and $\xi = 0.8$. The figure 5 presents the dependence between number of nodes and norms $||u - u_h||$, $||\frac{\partial u}{\partial x} - \frac{\partial u_h}{\partial x}||$, $||\frac{\partial u}{\partial y} - \frac{\partial u_h}{\partial y}||$.

The figure 6 presents the adaptive mesh.

6 CONCLUSIONS

- The paper presents the application of the grid generator with mesh size function to the adapted solution to the elastic-plastic twisting of bars with hardening.

- After the discretization by the finite element method the problem was led to the solution of nonlinear system of algebraic equations. For the considered problems the applied Newton-Raphson method was always convergent giving the residuum error of an order 10^{-10} in about $10 - 14$ iterations.

- The whole algorithm has two loops, the first external over adaptation steps and the second internal to previous over the Newton-Raphson iterations.

- The optimal mesh size function is obtained iteratively and depends on values of error indicators at nodes.

- The generator based on Delaunay condition and advancing front technique seems very suitable to the class of problems where different zones of the domain are to be appointed.

- For farther investigations the anisotropic mesh generation algorithm will be developed and appropriate anisotropic adaptation algorithms as well too.

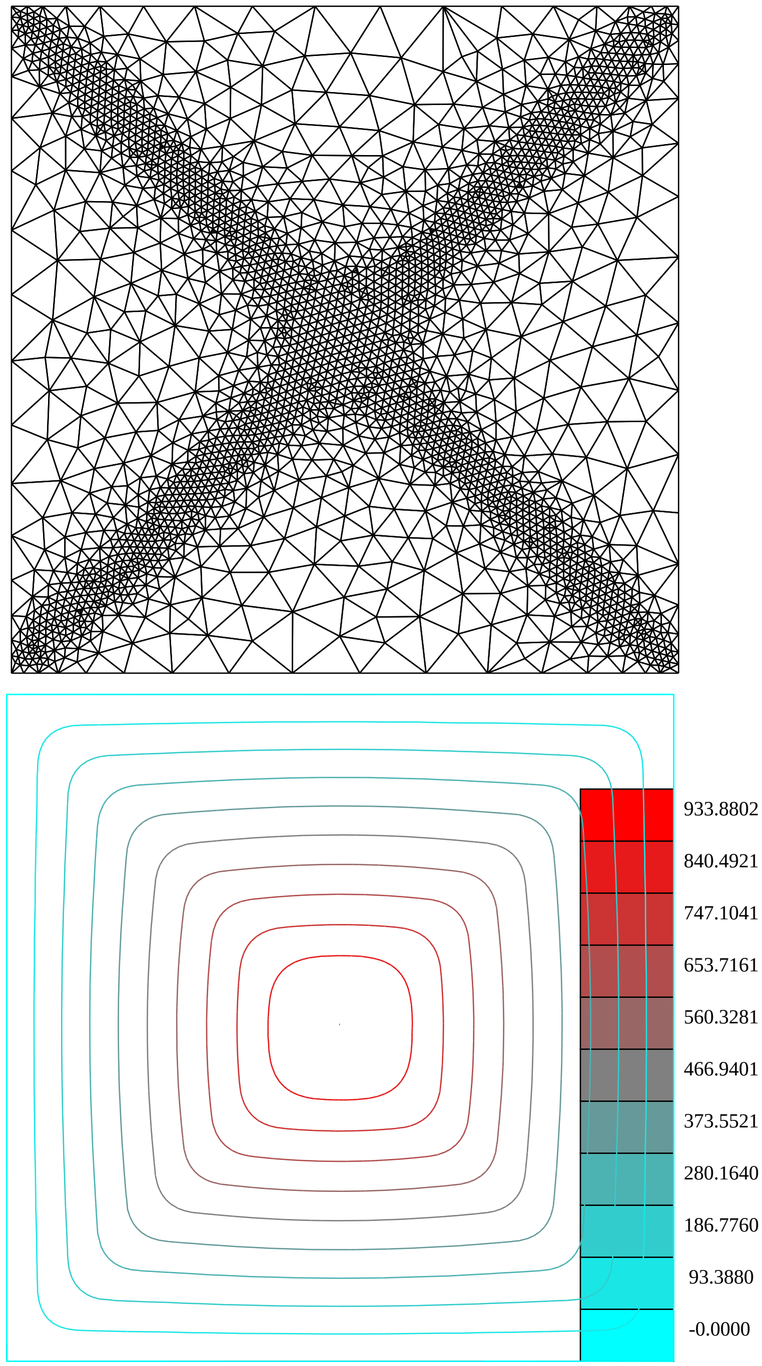

Figure 2: Mesh after 7 adaptation steps and Prandtl-Reuss function

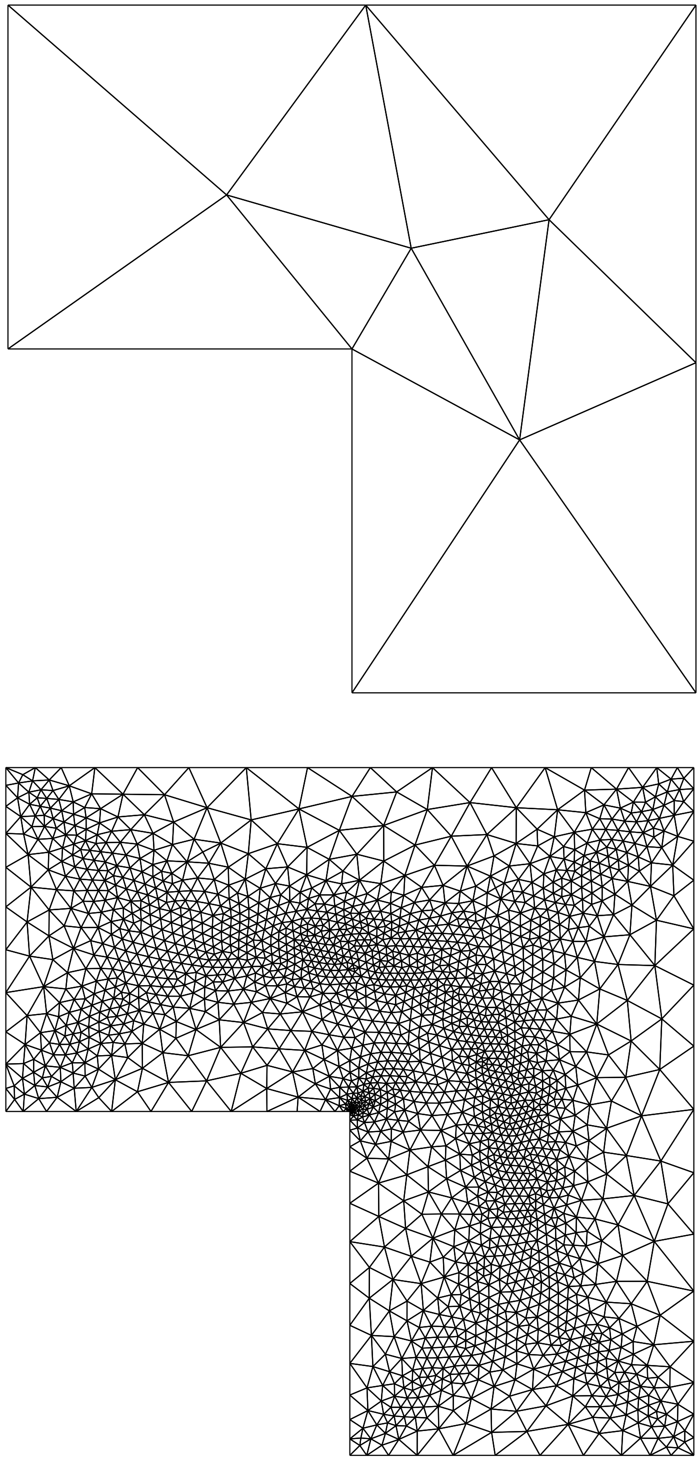

Figure 3: Initial and final mesh after 7 adaptation steps

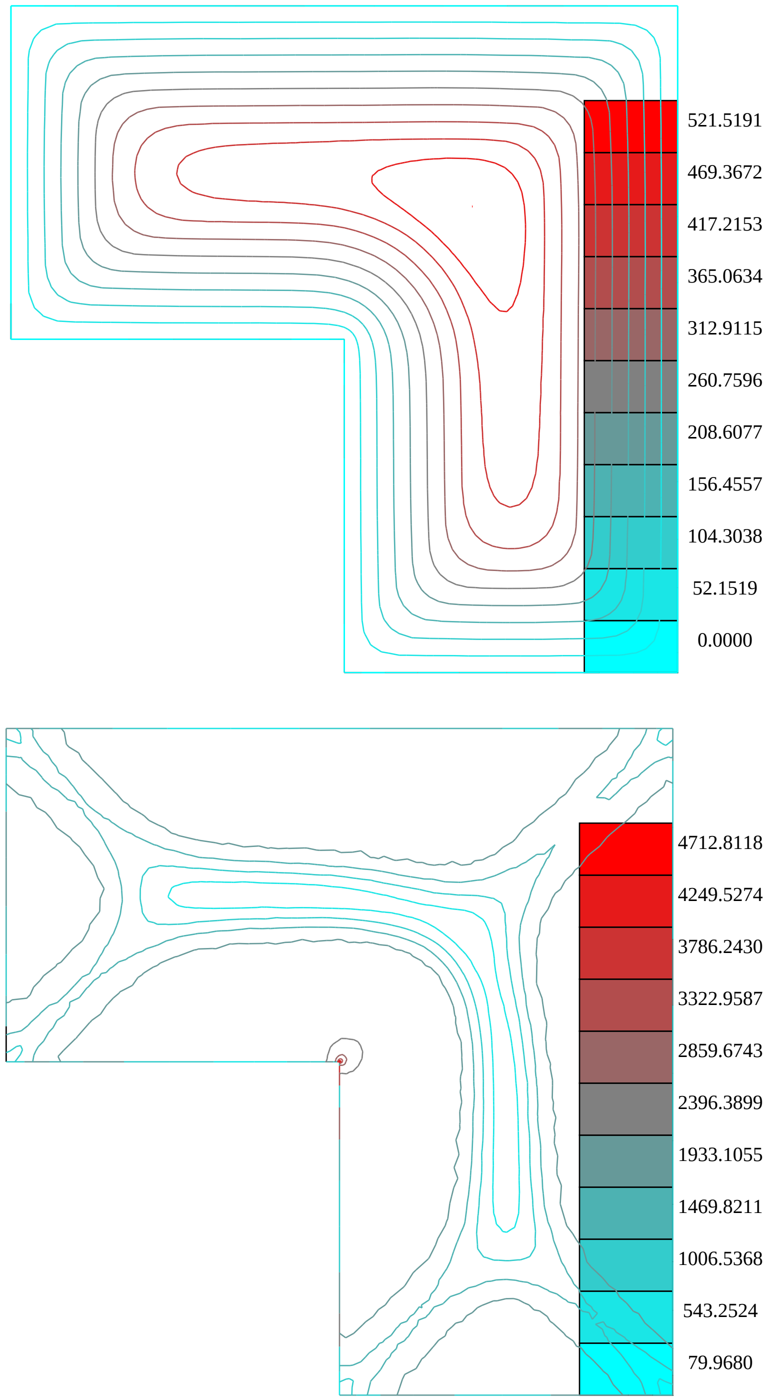

Figure 4: Prandtl-Reuss function

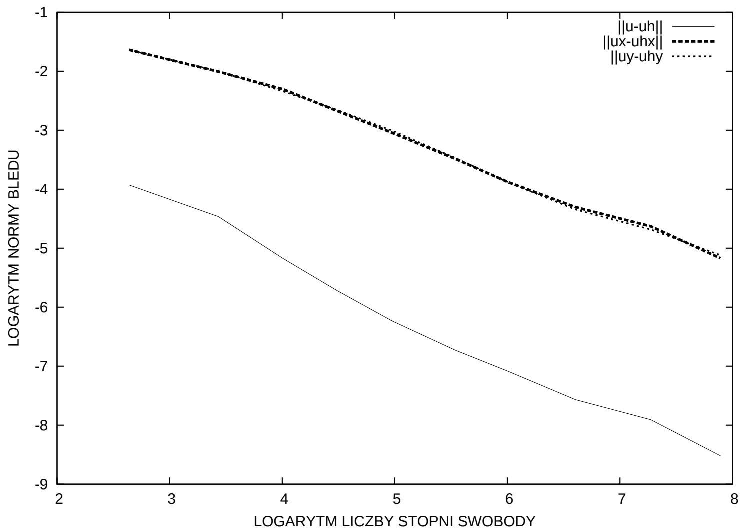

Figure 5: The convergence curve u, u_x, u_y with respect to the norms

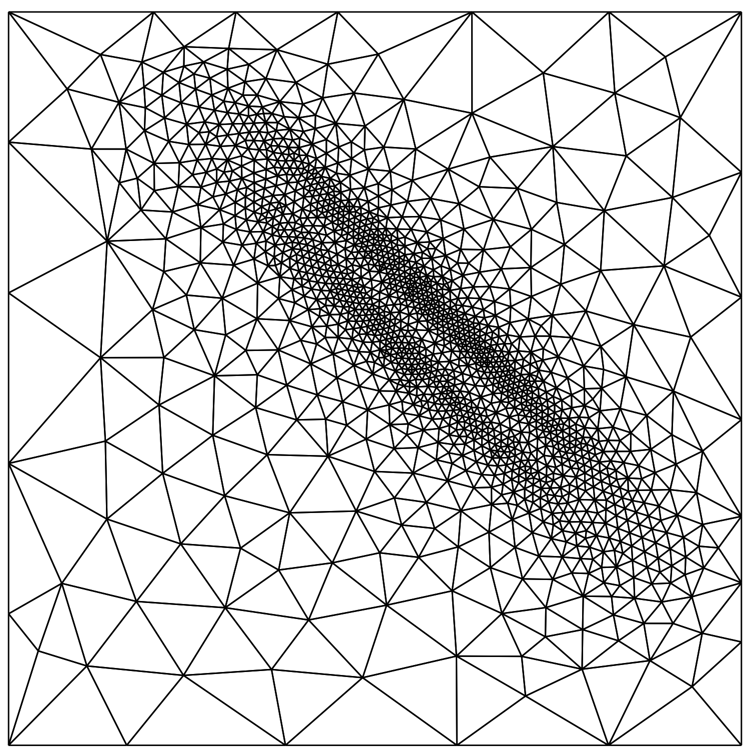

Figure 6: Adapted mesh for example problem from [14]

REFERENCES

[1] R. Bank, A. Sherman and A. Weiser Some Refine- ment Algorithms And Data Structures For Regular Local Mesh Refinement. Scientific Computing IMACS, 1983.

[2] M.B. Bieterman, J.E. Busseletti, C.L. Hilmes, F.T. Johnson, R.G. Melvin, D.P. Young, An adaptive grid method for analysis of 3D aircraft configurations, *The Boeing Company Seattle*, Technical Report, Washington 1991.

[3] H. Borouchaki, F. Hecht, P.J Frey, Mesh gradation control, *Int. J. Num. Meth. in Engng*, **43**, 1143-1165, 1998.

[4] A. Huerta, P. Diez, Error estimation including pollution assessment for nonlinear finite element analysis, *Comp. Meth. Appl. Mech. Engng*, **181**, 21-41, 2000.

[5] L. M. Katchanov: Foundation of plasticity theory (in Russian), Ed. Nauka, Moscow 1968.

[6] Kucwaj J., The Algorithm of Adaptation by Using Graded Meshes Generator, *Computer Assisted Mechanics and Engineering Sciences*, *vol. 7*, 615-624, 2000.

[7] Thompson J.F., Soni B.K., Weatherwill N.P., *Handbook of Grid Generation,* CRC Press, Boca Raton, (1999).

[8] J. Kucwaj, The Algorithm of Adaptation by Using Graded Meshes Generator, *Computer Assisted Mechanics and Engineering Sciences*, **7**, 615-624, 2000.

[9] J. Kucwaj, Numerical Investigations of the covergence of a Remeshing Algorithm on an Example of Subsonic Flow, *Computer Assisted Mechanics and Engineering Sciences*, **17**, 147-160, 2010.

[10] S. H. Lo, Finite element mesh generation and adaptive meshing , *Progress in Structural Engineering and Materials*, **4**, 381-399, 2002.

[11] J.T. Oden, L. Demkowicz, W. Rachowicz, T.A. Westermann, Towards a universal h-p adaptive finite element strategy, part 2, Aposteriori error estimation, *Comp. Meth. Appl. Mech. Engng.*, **77**, 113-180, 1989.

[12] W. Rachowicz, An anisotropic h-type mesh-refinement strategy, *Comp. Meth. Appl. Mech. Engng*, **109**, 169-181, 1993.

[13] O.C. Zienkiewicz, R.L. Taylor, *The Finite Element Method*, 4-th edition, vol. 1, Basic Formulation and Linear Problems, McGraw-Hill Book Company, London, Washington 1989.

[14] O.C. Zienkiewicz, Achievements and some unsolved problems of the finite element method, *Int. J. Num. Meth. Engng*, **47**, 9-28, 2000.

[15] O.C. Zienkiewicz J.Z. Zhu, Adaptivity and mesh generation, *Int. J. Num. Meth. Engng.*, **32**, 783-810, 1991.

[16] MAdLib: an open source Mesh Adaptation Library, http://sites.uclouvain.be/madlib/, 2010.