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BEM-BASED FINITE ELEMENT METHOD WITH PROSPECTS TO TIME DEPENDENT PROBLEMS

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Abstract. The requirement of flexible mesh handling gains more and more importance in the numerical treatment of boundary value problems. The recent BEM-based Finite Element Method is applicable on general polygonal and polyhedral meshes and yields conforming approximations, due to the implicit definition of trial functions in a Trefftz-like manner. These functions are treated by means of local Boundary Element Methods (BEM) in the computational realization. The method is reviewed and applied to a time dependent model problem, where the integration in time is performed with the help of a singly diagonally implicit Runge-Kutta scheme.

1 INTRODUCTION

The BEM-based Finite Element Method was first introduced in 2009, see [1], and goes back to domain decomposition in Boundary Element Methods (BEM), see [2]. The decomposition of the domain is understood as a finite element mesh and the extensions of boundary element trial functions are interpreted as implicitly defined finite element functions fulfilling the underlying differential equation. This new approach was analysed in the following years, see [3, 4, 5, 6]. Since then it has undergone several developments. This includes residual error estimates for adaptive mesh refinement [7], the application for convection–diffusion problems [8], higher order approximations [9, 10] and mixed formulations with $H(\mathrm{div})$-conforming discretization [11] as well as improved generalizations to three dimensional problems with polyhedral elements [12]. The main results have been gathered in two doctoral theses [13, 14].

Due to the implicit definition of trial functions in a Trefftz-like manner, the BEM-based Finite Element Method yields conforming approximations on general polygonal and polyhedral meshes which have attracted more and more attention in the last years.

For example, there are recent developments in mimetic discretization techniques [15] and within the new methodology of Virtual Element Methods [16, 17]. But also within the Finite Element Method there are developments towards polygonal and polyhedral discretizations using generalized barycentric coordinates, see [18, 19, 20].

Till now, the BEM-based Finite Element Method has been studied for elliptic equations. In this presentation, however, the first numerical experiment is given for a parabolic equation, where the time dependent diffusion equation is considered in a model problem. The space variable is discretized in the usual way with Trefftz-like basis functions and an implicit Runge-Kutta scheme is used for the time integration.

The paper is organized as follows. In Section 2, the model problem is presented and some notation is fixed. In Section 3, the Trefftz-like basis functions are reviewed and a semi discretization is given. In Section 4, the fully discrete problem is obtained by a SDIRK method. Finally, a numerical experiment is presented, in Section 5.

2 PRELIMINARIES

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary $\Gamma = \Gamma_D \cup \Gamma_N$ and $|\Gamma_D| > 0$. Furthermore, let $T \in \mathbb{R}$ with $T > 0$. The time dependent diffusion problem is to find a function $u : \Omega \times (0,T) \to \mathbb{R}$ which fulfills

$$\begin{aligned} \frac{\partial u}{\partial t} - \operatorname{div}(k\nabla u) &= f \quad && \text{in } \Omega \times (0,T), \\ u &= g_D && \text{on } \Gamma_D \times (0,T), \\ \frac{\partial u}{\partial n_\Omega} &= g_N && \text{on } \Gamma_N \times (0,T), \\ u &= u_0 && \text{on } \Omega \times \{0\}, \end{aligned} \tag{1}$$

for some given Dirichlet datum g_D, some Neumann datum g_N, a right hand side f and initial value u_0. Here, n_Ω denotes the outer normal vector of Ω and the diffusion coefficient fulfills $k_{\min} < k < k_{\max}$. For simplicity, we assume that k, g_D and g_N do not depend on the time variable t and that u_0 and g_D are compatible. Without loss of generality, let $g_D \equiv 0$. Otherwise, an extension of the Dirichlet datum is utilized.

For fixed $t \in (0,T)$, the function $x \mapsto u(x,t)$ is interpreted as a parameter-dependent element $u(t)$ of the space

$$V = \{v \in H^1(\Omega) : \ v = 0 \text{ on } \Gamma_D\}.$$

With $H = L_2(\Omega)$, the well known variational formulation reads: Find $u \in L_2((0,T),V)$ with $\frac{\partial u}{\partial t} \in L_2((0,T),H)$ such that

$$\begin{aligned} \left(\frac{\partial}{\partial t}u(t), v\right)_{L_2(\Omega)} + a(u(t),v) &= \ell(t,v) \quad \text{for } v \in V \text{ and } t \in (0,T), \\ u(0) &= u_0, \end{aligned} \tag{2}$$

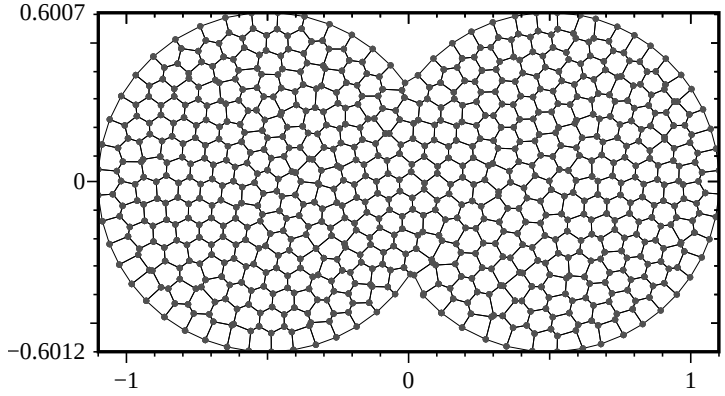

Figure 1: Polygonal mesh.

where

$$a(u,v) = \int_\Omega k\nabla u \cdot \nabla v \qquad \text{and} \qquad \ell(t,v) = (f(t),v)_{L_2(\Omega)} + (g_N,v)_{L_2(\Gamma_N)}$$

for $u_0 \in H$, $g_N \in L_2(\Gamma_N)$ and $f \in L_2((0,T),H)$.

For the numerical treatment of (2), the domain Ω has to be discretized. Therefore, we allow the decomposition of Ω into open and star-shaped polygonal elements $K \in \mathcal{K}_h$ with respect to a circle of radius ρ_K such that

$$\overline{\Omega} = \bigcup_{K\in\mathcal{K}_h} \overline{K} \qquad \text{and} \qquad K \cap K' = \varnothing \quad \text{for } K,K' \in \mathcal{K}_h : K \neq K'.$$

Furthermore, we assume that the aspect ratio of the element diameter h_K and the radius ρ_K is uniformly bounded from above, i.e. $h_K/\rho_K < \sigma$. An example of such a mesh is shown in Figure 1, which has been generated with the help of [21]. An edge $E \in \mathcal{E}_h$ in the mesh is a straight line which is always situated between two nodes. The set of all nodes is denoted by $\mathcal{N}_h$. We assume that the edge length h_E of each edge E of an element K is uniformly bounded from below by a constant times the element diameter, i.e. $ch_K < h_E$. Thus, we deal with quite general meshes, where the assumption on the aspect ratio ensures that the elements do not degenerate and the assumption on the edge length ensures that the number of nodes per element is uniformly bounded.

3 SEMI-DISCRETIZATION USING BEM-BASED SPACES

In a first step, we discretize in space, and thus follow the vertical method of lines. To handle the polygonal meshes, let $V_h^q \subset V$ be the BEM-based approximation space of order q introduced in [10]. It is defined as

$$V_h^q = \operatorname{span} \Psi_h^q \cap V$$

for same basis Ψ_h^q. The basis functions are defined in a Trefftz-like manner, such that they fulfill the underlying differential equation locally. We distinguish between nodal, edge and

element basis functions. For a first order approximation space, it is sufficient to use the nodal basis functions which are defined for each node $z \in \mathcal{N}_h$ as unique solution of

$$-\Delta\psi_z = 0 \quad \text{in } K, \quad \psi_z(x) = \begin{cases} 1, & x = z, \\ 0, & x \in \mathcal{N}_h \setminus \{z\}, \end{cases} \qquad \psi_z \text{ linear on each } E \in \mathcal{E}_h.$$

To obtain an approximation of order q, one additionally needs edge and element basis functions. The first ones are defined for each edge $E \in \mathcal{E}_h$ and $i = 2, \dots, q$ as unique solution of

$$-\Delta\psi_{E,i} = 0 \quad \text{in } K, \qquad \psi_{E,i} = \begin{cases} p_{E,i}, & \text{on } E, \\ 0, & \text{on } E' \in \mathcal{E}_h \setminus \{E\}, \end{cases}$$

where $p_{E,i}$ form a basis of the polynomial space $\mathcal{P}^q(E)$. The element basis functions $\psi_{K,i,j}$ are defined for each element $K \in \mathcal{K}_h$ and $i = 0, \dots, q-2$ and $j = 0, \dots, i$ as unique solution of

$$-\Delta\psi_{K,i,j} = p_{K,i,j} \quad \text{in } K, \qquad \psi_{K,i,j} = 0 \quad \text{else},$$

where $p_{K,i,j}$ form a basis of the polynomial space $\mathcal{P}^{q-2}(K)$.

To come back to the variational formulation of the time dependent diffusion problem (2), let $u_{0h} \in V_h^q$ be an approximation to the initial data u_0. Then the semi-discrete problem reads: Find $u_h \in L_2((0,T), V_h^q)$ with $\frac{\partial u_h}{\partial t} \in L_2((0,T), H)$, $u_h(0) = u_{0h}$ such that

$$\left(\frac{\partial}{\partial t} u_h(t), v_h\right)_{L_2(\Omega)} + a(u_h(t), v_h) = \ell(t, v_h) \quad \text{for } v_h \in V_h^q \text{ and } t \in (0,T). \tag{3}$$

Testing only with basis functions and inserting the ansatz

$$u_h(t) = \sum_{\psi \in \Psi_h^q} u_\psi(t)\psi$$

for the approximation with time dependent coefficients, yields a system of linear ordinary differential equations. Namely,

$$\begin{aligned} M\mathbf{u}_h'(t) + A\mathbf{u}_h(t) &= r(t) \quad \text{for } t \in (0,T), \\ \mathbf{u}_h(0) &= \mathbf{u}_{0h}, \end{aligned} \tag{4}$$

where the bold letters refer to the vectors containing all coefficients and $r(t)$ is the vector obtained by testing $\ell(t, \cdot)$ with all basis functions. The entries of the matrix A are the values of the bilinear form $a(\cdot, \cdot)$ applied to basis functions and M is the mass matrix. For the computational evaluation of these values we refer to [9, 10].

4 TIME INTEGRATION

To obtain a fully discrete scheme, we have to discretize in time. Therefore, the interval $(0,T)$ is subdivided into $N \in \mathbb{N}$ equidistant intervals of length $\tau = T/N$. Furthermore, we set $t_n = n\tau$ for $n = 0, \dots, N$ and denote by $\mathbf{u}_h^n$ the approximation of $\mathbf{u}_h(t_n)$. To approximate the solution of the system of linear ordinary differential equations (4), a singly diagonally implicit Runge-Kutta scheme is chosen, because of computational advantages, see [22]. The coefficients of such a SDIRK method with s-stages is given by the corresponding Butcher scheme:

$$\begin{array}{c|cccc} c_1 & \gamma & & & \\ c_2 & a_{21} & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ c_s & a_{s1} & \dots & a_{s,s-1} & \gamma \\ \hline & b_1 & \dots & \dots & b_s \end{array}$$

Suitable coefficients can be found in the literature, see e.g. [22]. For a 2-stage method a possible choice is

$$\begin{aligned} c_1 &= 0.7886751345948129, & \gamma &= 0.7886751345948129, \\ c_2 &= 0.4227843350984671, & a_{21} &= -0.3658907994963458, \\ b_1 &= 0.2110347267759163, & b_2 &= 0.7889652732240837. \end{aligned}$$

Thus the system of linear ordinary differential equations (4) turns into

$$(M + \tau\gamma A)\kappa_i = r^n - A\left(\mathbf{u}_h^n + \tau \sum_{j=1}^{i-1} a_{ij}\kappa_j\right) \quad \text{for } i = 1, \dots s, \tag{5}$$

$$\mathbf{u}_h^{n+1} = \mathbf{u}_h^n + \tau \sum_{i=1}^{s} b_i \kappa_i, \tag{6}$$

where $r^n = r(t_n)$. In each time step, the systems of linear equations (5) have to be solved for the stages κ_i, $i = 1, \dots, s$ and afterwards, an approximation of the solution at the new time step is obtained by (6). Due to the choice of a SDIRK method, the system matrices in (5) are the same for all time steps and stages. Thus, this matrix can be factorized in a preprocessing step to speed up the simulation.

5 NUMERICAL EXPERIMENT

For a first numerical experiment, the 2-stage method of the previous section has been implemented. We choose

$$\Omega = \{x \in \mathbb{R}^2 : |x - x_0| < 0.6\} \cup \{x \in \mathbb{R}^2 : |x + x_0| < 0.6\},$$

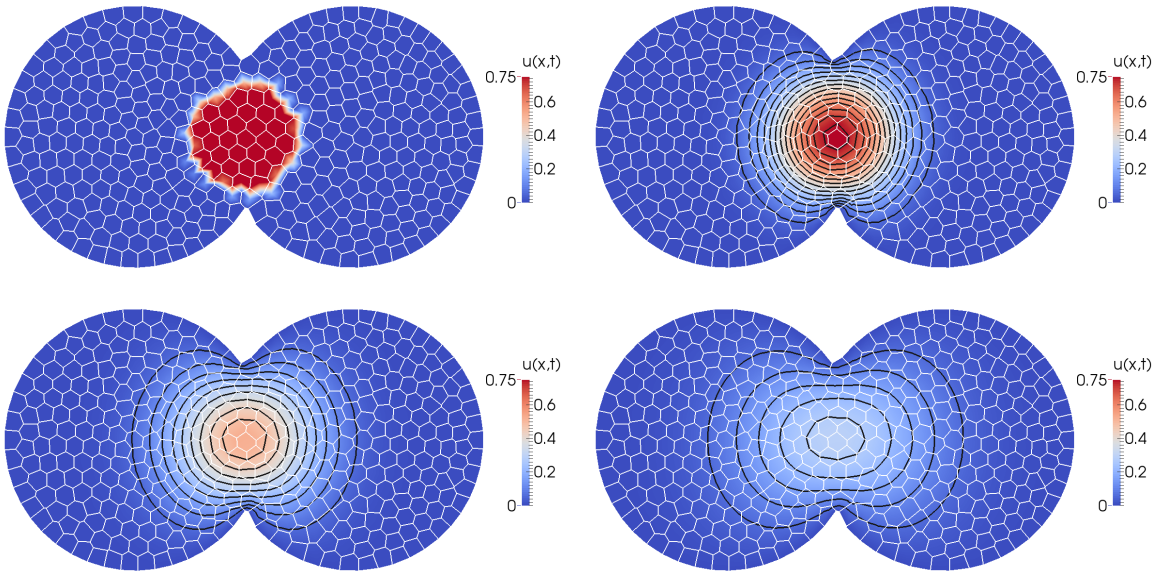

Figure 2: Approximation of initial data and the solution after 10, 20 and 40 time steps with isolines.

where $x_0 = (0.5, 0)^\top$, see Figure 1, with pure Dirichlet boundary conditions and $k \equiv 0.1$ as well as $f \equiv g_D \equiv 0$. The initial data u_0 is chosen to be one in $\{x \in \mathbb{R}^2 : |x| \leq 0.25\}$ and zero else.

In Figure 2, the approximation of the initial data as well as the approximation of the solution after 10, 20 and 40 time steps is visualized. Here, the time step size has been chosen to be $\tau = 0.01$. For the spacial discretization the trial space V_h^1 has been utilized. The expected behaviour of the diffusion process is observed on the polygonal mesh.

Furthermore, in Figure 3, the approximation of the solution is evaluated at the origin of the coordinate system and its value is plotted over time. It decreases monotonically and converges towards zero as expected from the theory of diffusion equations.

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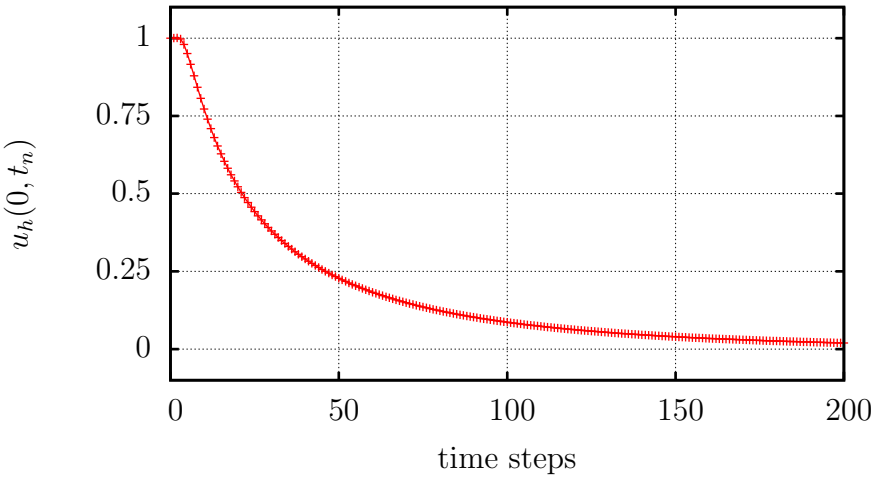

Figure 3: Approximation evaluated in point zero and plotted with respect to time.

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