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AN IMPROVED WAVE / FINITE ELEMENT FORMULATION FOR STUDYING HIGH-ORDER WAVE PROPAGATION IN LARGE-SCALED WAVEGUIDES.

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Abstract. In this paper, a reduced formulation for the Wave Finite Element Method is used to study wave dispersion characteristics of a composite beam on a broadband frequency range. The proposed method efficiency is discussed and high-order propagating waves behaviour is described.

1 INTRODUCTION

The problem considered in this paper concerns the broadband analysis of wave propagation in large-scaled or thick laminated composite waveguides. Guided waves in such structures are increasingly encountered in automotive and aerospace industries or in the field of structural health monitoring. As wave propagation is studied in structurally advanced waveguides, there is an increasing need for numerical methods to be compatible with finite element modelling. The semi-analytical finite element (SAFE) and wave and finite element (WFE) methods are, among others, very efficient tools for this purpose. In the SAFE method, sinusoidal functions are employed to formulate the displacement field in the direction of propagation. Nevertheless, it is necessary to develop specific semi-analytical elements for each application, which can severely limit its interest for industrial purposes. The wave finite element method, combining periodic structure theory (PST) introduced by Mead [1] with a finite element method (FEM), uses commercial finite element packages and can be an effective numerical tool for such purposes. However, broadband analysis of composite waveguides suffers sizeable computational costs, due to the requirements for a high spatial discretization. In this paper, numerical analysis of propagating waves dispersion characteristics if provided on a refined multi-layered beam (see Figure 1) using a reduced computational strategy.

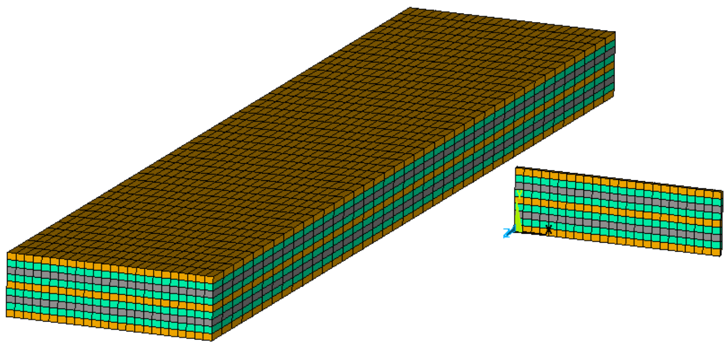

Figure 1: Cross-sectional discretization of a thick laminated composite beam.

2 Wave analysis of the composite waveguide

2.1 Classical Wave Finite Element Method

First, a formulation of the WFE method is given for free wave propagation in a one-dimensional waveguide. The structure is considered as a superposition of N identical subsystems of length d connected along the main direction x. Displacements and forces are written as $\mathbf{q}$ and $\mathbf{f}$, and subscripts 'L' and 'R' denote the left and right edges of a cell. The number of degrees of freedom is assumed to be the same on both edges and mesh compatibility is assumed between the N subsystems. The discrete dynamic equation of a cell at frequency ω is given by:

$$(-\omega^2\mathbf{M} + j\omega\mathbf{C} + \mathbf{K})\mathbf{q} = \mathbf{f} \tag{1}$$

where $\mathbf{M}$,$\mathbf{C}$,$\mathbf{K}$ are the mass, damping and stiffness matrices, respectively. By introducing the condensed dynamic stiffness operator $\mathbf{D} = -\omega^2\mathbf{M} + j\omega\mathbf{C} + \mathbf{K}$ and reordering degrees of freedom, equation of motion can be stated as follows:

$$\begin{bmatrix} \mathbf{D}_{LL} & \mathbf{D}_{LR} \\ \mathbf{D}_{RL} & \mathbf{D}_{RR} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_L \\ \mathbf{q}_R \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_L \\ \mathbf{f}_R \end{Bmatrix} \tag{2}$$

We denote $\lambda = e^{-j\kappa d}$ the propagation constant describing wave propagation over the cell length d and κ associated wavenumber. $\mathbf{D}_{LL}$ and $\mathbf{D}_{RR}$ are symmetric and $\mathbf{D}^t_{LR} = \mathbf{D}_{RL}$. Considering force equilibrium $\lambda\mathbf{f}_L + \mathbf{f}_R = 0$ in a cell and Bloch's theorem, $\mathbf{q}_R = \lambda\mathbf{q}_L$, Eq. (2) leads the following eigenproblem [2]:

$$\begin{bmatrix} -\mathbf{D}_{RL} & -(\mathbf{D}_{LL} + \mathbf{D}_{RR}) \\ \mathbf{0} & -\mathbf{D}_{RL} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_L \\ \lambda\mathbf{q}_L \end{Bmatrix} = \lambda \begin{bmatrix} \mathbf{0} & \mathbf{D}_{LR} \\ -\mathbf{D}_{RL} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_L \\ \lambda\mathbf{q}_L \end{Bmatrix} \tag{3}$$

Here, the eigenvectors represents both nodal displacements and forces associated with a wave.

2.2 Limitations for numerical applications

The computation of the eigenproblem Eq. (3) can lead to numerical errors when a large number of degrees of freedom are involved. However, when complex waveguides are considered, an insufficient discretization of the cross-section will produce significant errors, especially for eigensolutions associated to waves whose section shape have a short wavelength. On the other hand, refined meshes considerably increases computation time and worsen round-off errors due to the truncation of inertia terms. The layered beam considered in this paper requires a high degree of precision due its geometry, in order to determine high order propagating wave.

2.3 Reduced formulation

Classical techniques based on modal basis reduction are not available, since a cross-section boundary conditions are arbitrary for a uniform waveguide or subjected to structural periodicity otherwise. Hence, we propose a method, based on a cross-sectional transfer matrix projection on a reduced set of shape functions associated with propagating waves, to compute the dispersion curves of highly discretized waveguides. The reduced basis is build using propagating, positive-going wave solutions of a reduced number of generalized eigenvalue problems Eq. (3). Wave cut-on frequencies and associated wave shapes $\mathbf{\Phi}_A$ are solutions of the eigenproblem of size $n/2$:

$$(\mathbf{K}^*_{LL} + \mathbf{K}^*_{RR} + \mathbf{K}^*_{RL} + \mathbf{K}^*_{LR})\mathbf{\Phi}_A = \omega_A^2(\mathbf{M}_{LL} + \mathbf{M}_{RR} + \mathbf{M}_{RL} + \mathbf{M}_{LR})\mathbf{\Phi}_A \tag{4}$$

where n is the size of $\mathbf{K}$, $\mathbf{K}^* = (1 + j\eta)\mathbf{K}$ is the complex stiffness matrix and the solutions ω_A are the cut-on frequencies.

Eq. (3) are solved for a reduced number of cut-on frequencies. A solution subspace $\mathfrak{B}$ can be spanned using the normalized eigenvectors associated with propagating waves. Then denoting

$$\tilde{\mathbf{D}}_{ij} = \mathcal{B}^T \mathbf{D}_{ij} \mathcal{B} \tag{5}$$

the reduced dynamic stiffnesses and

$$\tilde{\mathbf{\Phi}} = \left\{ \begin{array}{c} \mu_L \\ \mu_R \end{array} \right\} \tag{6}$$

the associated eigenvectors, the generalized eigenproblem can be written :

$$\left[\begin{array}{cc} -\tilde{\mathbf{D}}_{RL} & -(\tilde{\mathbf{D}}_{LL} + \tilde{\mathbf{D}}_{RR}) \\ \mathbf{0} & -\tilde{\mathbf{D}}_{RL} \end{array} \right] \left\{ \begin{array}{c} \mu_L \\ \mu_R \end{array} \right\} = \lambda \left[\begin{array}{cc} \mathbf{0} & \tilde{\mathbf{D}}_{LR} \\ -\tilde{\mathbf{D}}_{RL} & \mathbf{0} \end{array} \right] \left\{ \begin{array}{c} \mu_L \\ \mu_R \end{array} \right\} \tag{7}$$

The method is summarized in Fig. 2.

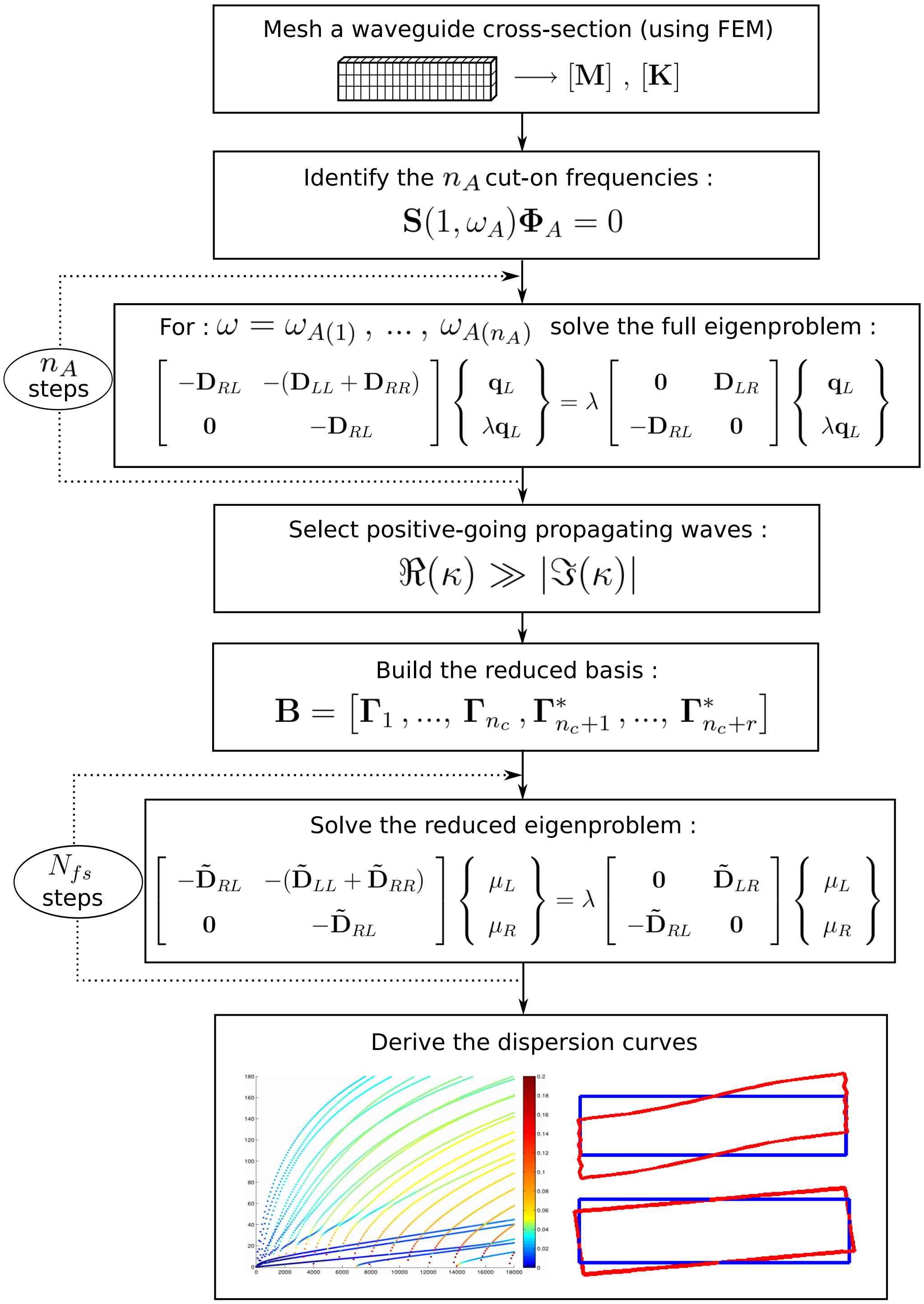

Figure 2: Procedure for the reduced wave finite element method.

This reduced formulation allows high-order wave analysis such as complex cross-sectional (Fig. 3) deformations or wave localization for thick laminated composite beams. A example is proposed next section.

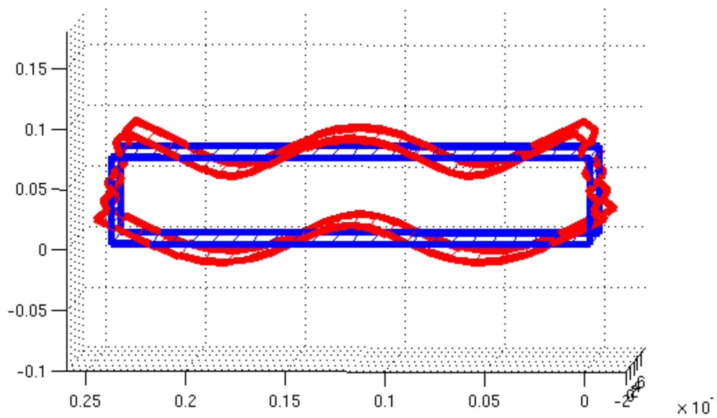

Figure 3: Deformed shape associated with a second-order propagating wave (red), undeformed shape (blue).

3 Application on a multi-layered composite beam

The aforementioned method is applied on the multi-layered beam presented Figure 1. The material properties of the 9 layers of the beam are defined Table 1.

Material (x8mm)	Layers positions	Density ($kg.m^{-1}$)	Young Modulus (GPa)	Poisson Coefficient
QI GF	1 - 5 - 9	1500	75	0.22
Epoxy	2 - 4 - 6 - 8	1100	2	0.29
Aluminium	3 - 7	2700	63	0.35

Table 1: Material properties for the composite beam layers

The dispersion curves of the propagating waves are computed using classical WFEM and the reduced formulation described above. The real parts of wavenumbers $\kappa = j\ln(\lambda)/d$ are compared Fig.4 and a remarkable concordance is shown between these two solutions. Wave cut-on frequencies are successfully determined using Eq. (4) and the reduced basis is defined from a set of 9 eigensolutions. All the propagating waves are provided, and none of the highly decaying of evanescent waves appear in the reduced solution.

A summary of the CPU time needed is given table 2. Computations times are compared using the two processors AMD 64X2 dual core 6000+. A wave identification (or wave matching) based on eigenvectors comparison between two steps requires that the steps

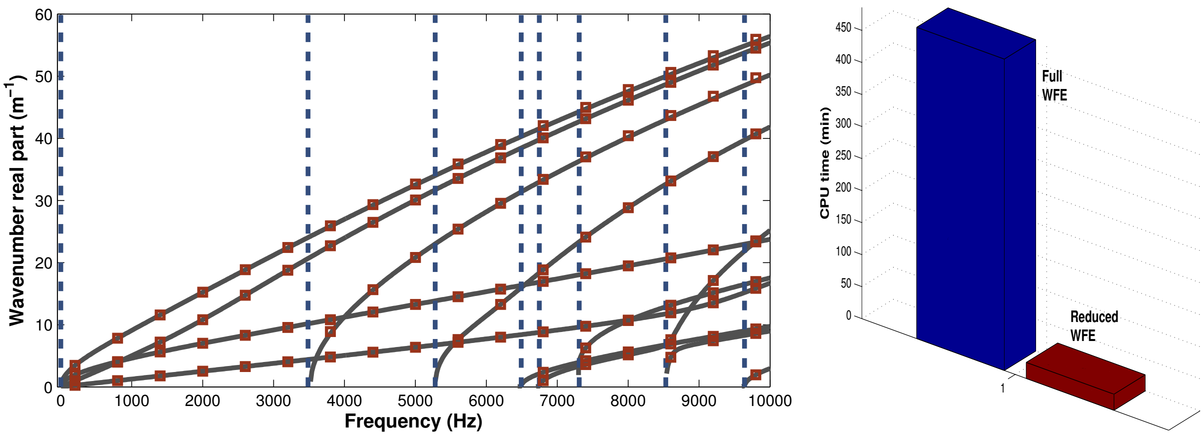

Figure 4: (a) : Comparison between full (red □) and reduced (black —) wavenumbers for the laminated waveguide. (b) : CPU time comparison between the classical WFE method (blue) and the proposed reduced formulation (red).

are smaller than 50 Hz. In the proposed example, the reduced WFE formulation saves 95% of computation time while the number of steps is increased from 200 to 500. It is also remarkable that most of the CPU time depends on the resolution of a fixed number of full eigenproblems, therefore the sampling has a minor influence on the computational cost. The reduced system to be solved is a 110×110 eigenproblem, instead of the 1500×1500 previous system. Note that this reduced system can be improved using GramSchmidt ortho-normalization algorithm and a more selective criteria for retaining eigenvectors in the reduced wave basis.

Classical WFE	Number of iterations	Time per iteration	Total CPU time
Resolution $[0, 10000Hz]$ per $50Hz$	×200	146s	**8h6m40s**
Reduced WFE	Number of iterations	Time per iteration	Cumulated CPU time
Appearance $\omega_{A(k)}$	×1	21s	21s
Full solution at $\omega_{A(k)}$	×9	146s	1335s
Building reduced basis	×1	153s	1488s
Resolution $[0, 10000Hz]$ per $20Hz$	×500	0.05	Total : **25m13s**

Table 2: Computation time associated with the different steps of the reduced WFE algorithm for a dispersion analysis between 0 Hz and 10.000 Hz

4 CONCLUSIONS

- A reduced formulation for the WFE method is described for one-dimensional waveguides. This numerical strategy is validated on a multi-layered elastic beam illustrated Fig. 1.
- The results (Fig. 3) are compared to the proposed formulation for high-order propagating waves. Computational issues due to the classical wave finite element method (WFE) are highlighted for elaborate or finely meshed waveguides.
- A CPU time needed for the computation is reduced by 95% while increasing the frequency sampling from 200 to 500 between 0 Hz and 10.000 Hz.
- Furthermore, this model order reduction strategy enables numerical analysis for structurally advanced waveguides on a broadband frequency range [3].

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