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A TWO-YIELD-CRITERIA LIMIT ANALYSIS APPROACH FOR STEEL-REINFORCED CONCRETE SLABS

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Summary. *A FE-based limit analysis methodology is presented for the peak load evaluation of steel reinforced concrete slabs. The numerical methodology, already applied to reinforced concrete elements, is here presented in an enhanced form to better take into account the post-elastic behaviour of concrete and steel reinforcement. The modification implies a two-yield-criteria formulation which enables the prediction of possible steel bars yielding at incipient collapse. The effectiveness of the methodology is shown through a few applications on large-scale reinforced concrete slabs.*

1 INTRODUCTION

For load-carrying capacity assessment of structures and within plasticity theory, if primary interest is in the final stage of the structural response, the two fundamental theorems of limit analysis theory allow the determination of the limit load at collapse in a direct manner. In the framework of reinforced concrete (RC) structures, ductile behaviour, which is an essential condition for applying limit analysis theory, is guaranteed by the confining effect of steel reinforcement bars (re-bars) which mitigate many complex post-elastic phenomena (such as fracturing or damaging mechanisms), thus making a limit analysis approach both applicable and effective.

In the present work a limit analysis methodology, recently proposed by the authors and successfully applied to RC elements (see [1], [2]), is further refined for a better modeling of the post-elastic behaviour of steel and concrete. The methodology employs conventional finite elements (FEs) in order to simulate a limit state solution of RC elements, searching for an upper and lower bound to the peak load multiplier. In [1] and [2] the methodology has been carried out by assuming a plasticity model for concrete, i.e. a Menétrey–Willam-type (M–W-type) yield criterion, [3], endowed with cap in compression, and by postulating an indefinitely elastic behaviour of steel re-bars. Such assumptions, which are acceptable in over-reinforced concrete elements, in which failure is mainly ascribed to crushing of concrete, may result not enough accurate at an ultimate limit state characterized by yielding of steel re-bars. Indeed, for under-reinforced RC elements which fail in a more ductile manner, despite the brittle nature of plain concrete, the yielding of steel re-bars is often exhibited and it plays a significant role in determining the plastic behaviour of the RC element as a whole. This is why the methodology is here extended to deal with possible steel bars yielding at incipient collapse through a *two-yield-criteria formulation*, where: concrete is described by the M–W-type yield criterion and steel re-bars are handled by a von Mises-type yield criterion. The reliability of the promoted methodology is verified by comparison between numerical results and experimental findings on large-scale RC slabs tested in laboratory [4]–[7].

2 THEORETICAL BACKGROUND AND FUNDAMENTALS

Some useful preliminary remarks concerning the numerical limit analysis methodology, along with the constitutive assumptions, are here outlined to pose the problem. For brevity, only a few basic concepts are given; further information and theoretical details can be found in [1], [2].

In the framework of the so-called *direct methods* (see [8]), the application of the two fundamental theorems of limit analysis theory, namely the kinematic and the static theorem, allows the determination of the limit load at collapse by detecting two limits, i.e. an upper and a lower bound to it. With regard to the analyzed RC slabs, the dilatancy of concrete implies the lack of associativity and underlies the adoption of a *nonstandard limit analysis approach*. The key concept of the nonstandard limit analysis approach is to “encircle” the yield surface of the nonstandard material with two surfaces, precisely an outer and an inner surface and to compute two bounds with reference to such surfaces (or materials). Precisely, a Menétrey–Willam-type yield criterion [3] with cap in compression is assumed for concrete. Such a strictly convex M–W-type yield surface plays itself the double role of inner and outer surface (see [9]). Instead, for what concern the steel re-bars post elastic behaviour, handled by a von Mises yield surface, the above nonstandard approach becomes meaningless dealing with a classical standard material. The whole RC structural element has obviously to be treated as made by a nonstandard material.

The key-idea of the methodology is the combined use of two limit analysis numerical procedures, namely the Linear Matching Method (LMM) and the Elastic Compensation Method (ECM). The former, originally conceived in [10], is related to the kinematic approach of limit analysis and hence provides an upper bound to the peak load value, allowing also a prediction of the collapse mode. The latter, first theorized in [11], is based on the static approach of limit analysis and gives a lower bound to the peak load value. This methodology has already been used in a completely different context to predict collapse load and failure mechanism of pinned-joints in orthotropic composite laminates using a Tsai–Wu-type yield surface ([12]–[15]). The application to RC elements with fiber reinforced polymer (FRP) bars has also been investigated in [16].

The main purpose of both the LMM and the ECM is to simulate, or “to build”, limit-type distributions by carrying out *FE-based linear elastic analyses* during which the elastic moduli and the imposed initial stresses (for the LMM) are systematically adjusted within the FEs. In particular, the LMM is aimed at constructing a kinematically admissible distribution of strain and displacement rates, namely a collapse mechanism suitable for the evaluation of an upper bound P_{UB} to the collapse load. It is an iterative procedure involving one sequence of linear FE-based analyses in which the studied structure is assumed made of a *fictitious* material with spatially varying moduli. At each iteration the fictitious moduli are adjusted so that the computed fictitious stresses are brought onto the yield surface at a fixed strain rate distribution. The strain and displacement rates together with the associated stresses at yield can then be used for the evaluation of a P_{UB}. On the other hand, the ECM constructs an admissible stress field suitable for the evaluation of a lower bound P_{LB} to the collapse load according to the static approach. It is again an iterative procedure involving many sequences of linear FE-based analyses in which highly loaded regions of the structure are systematically weakened by the reduction of the local modulus of elasticity and this in order to simulate the effects of a "stress redistribution" arising within the structure before attaining its limit strength threshold.

Considering the FE model of a typical RC slab, unlike the previous formulation adopted in the quoted papers [1], [2] in which steel re-bars were assumed as indefinitely elastic members, the procedures outlined above have been applied for both concrete and steel re-bars, with reference to the M–W-type yield surface and the von Mises yield surface, respectively. As a result, the limit-state solution of the RC element as a whole is more faithfully described and numerically simulated especially when yielding of re-bars actually occurs. As shown next, in most cases this approach considerably improves the accuracy of the obtained numerical results.

3 NUMERICAL TESTS

Experimental tests on 7 RC slabs, carried out up to collapse, are numerically simulated to predict peak load and collapse mechanism of the specimens carrying on a comparison between experimental findings and corresponding numerical results. The experimental campaigns taken into consideration are those presented in [4]–[7], where further details on laboratory test equipment, reinforcement arrangement and experimental data can be found.

The elastic analyses, representing the iterations within both the LMM and the ECM, have been carried out using the FE-code ADINA [17] with meshes of 3D-solid 8-nodes elements with 2x2x2 GPs per element for modeling concrete and 2-nodes 1-GP truss elements for steel re-bars and stirrups. Each node has three degrees of freedom. Both the elements are isoparametric and displacement-based. The nodes of the truss elements are shared with those of the 3D-solid, therefore this compatibility condition reflects a perfect bond between concrete and re-bars assumed in the FE-analyses. An isotropic material formulation has been employed for concrete and re-bars. To set the Menétrey–Willam-type and the von Mises yield surfaces, the strength material properties (compressive and tensile strength for concrete, f_c', f_t', and yield strength for longitudinal steel re-bars, f_y) have been assumed as indicated in the experimental campaigns (see again [4]–[7]) and are not reported here for the sake of brevity.

The first experimental study is that by Al-Rousan *et al.* [4]. In this study, 8 RC slabs strengthened with different types and configurations of CFRP sheets externally bonded to the soffit (tension face) of the slab were tested up to failure under simply supported condition. Among the 8 slabs two specimens (labelled C-1 and C-2) with conventional steel re-bars and without CFRP sheets were included in the experimental program and are here analyzed. These two slabs, with identical material properties and tension reinforcement, failed due to propagation and widening of flexural cracks; at failure, the tensile steel reinforcement was yielded and followed by concrete crushing at the top of the slab. As the ultimate load recorded for the two specimens was practically coincident one another, only one RC slab (labelled C-1 for simplicity) is here considered. As shown below, the assumption of indefinitely elastic behaviour of the steel re-bars (adopted in [1]) leads, in this case where yielding of re-bars is really marked, to rather incorrect conclusions in terms of peak load numerical prediction. The second group of specimens analyzed was that by Breveglieri *et al.* [5] (for more experimental details see Bonaldo [6]), in which the application of the near surface mounted (NSM) technique for the flexural strengthening of continuous RC slabs (i.e. with intermediate support) was investigated. The numerical analysis is here directed to the three reference slabs (without NSM CFRP laminates), namely specimens labelled SL15, SL30 and SL45, which experienced a ductile failure with yielding of re-bars at both central support (in the hogging region) and at the two midspans (in the sagging regions) together with concrete crushing at both central support and loaded sections. Finally, in the report by Gilbert and Nejadi [7] 6 simply supported slabs reinforced with large ductility longitudinal bars (N12) were tested. The 6 slabs have identical material properties and reinforcement arrangement in pairs (the so-called specimens “a” and “b”), therefore only three specimens (each representative of a couple of slabs with averaged material data) are here considered, namely specimens labelled S1, S2, S3. It is worth noting that the latter slabs failed due to crushing of concrete in the top of the compressive zone, i.e. the failure was not primarily ascribed to yielding of steel re-bars (which probably occurred but was not so pronounced). As shown next, the corresponding numerical predictions obtained keeping bars elastic are not so inaccurate as they are for the other specimens, although also for these cases an overall better performance of the proposed two-yield-criterion formulation is observed.

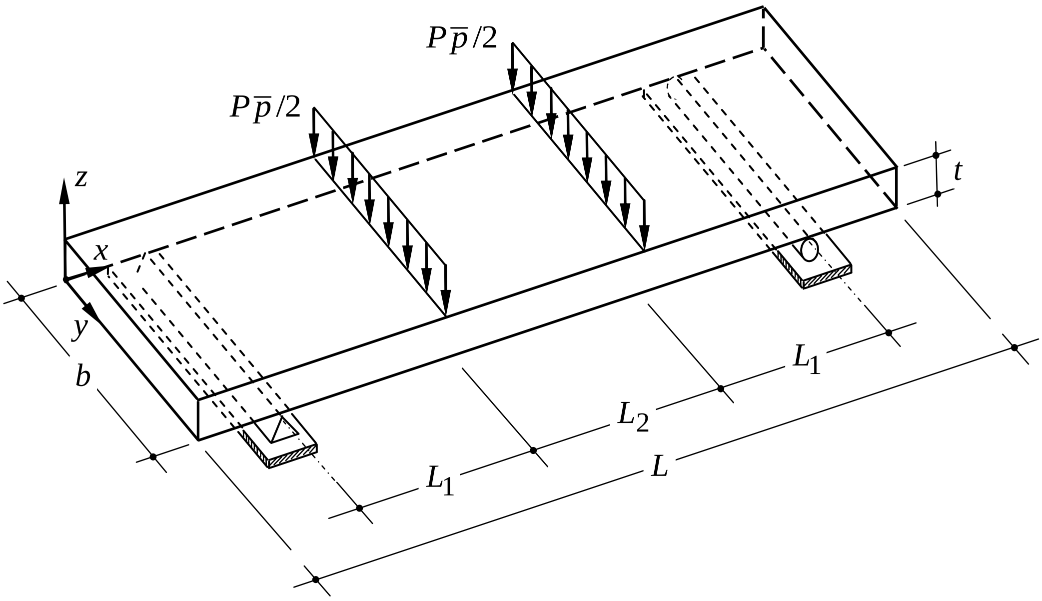

Figure 1: Mechanical model of simply-supported slabs C-1, S1, S2, S3

As shown in Figure 1, the simply-supported slabs (specimens C-1, S1, S2, S3) were subjected to two equal line loads symmetrically placed about midspan and denoted as $P\overline{p}/2$, with P representing the load multiplier and $\overline{p}$ the reference line load whose resultant is assumed equal to 100kN for all the tested slabs. The continuous-supported slabs (specimens

SL15, SL30, SL45), which comprised two equal spans, were similarly subjected to two equal line loads $P\bar{p}$ at the two midspans. In this case, only half of the slab has been modelled exploiting the symmetry in *x* direction, as shown in Figure 2, in order to guarantee an accurate FE elastic solution without increasing the computational effort. Details concerning geometrical data and reinforcement arrangement of all the 7 analyzed RC slabs are given in Table 1.

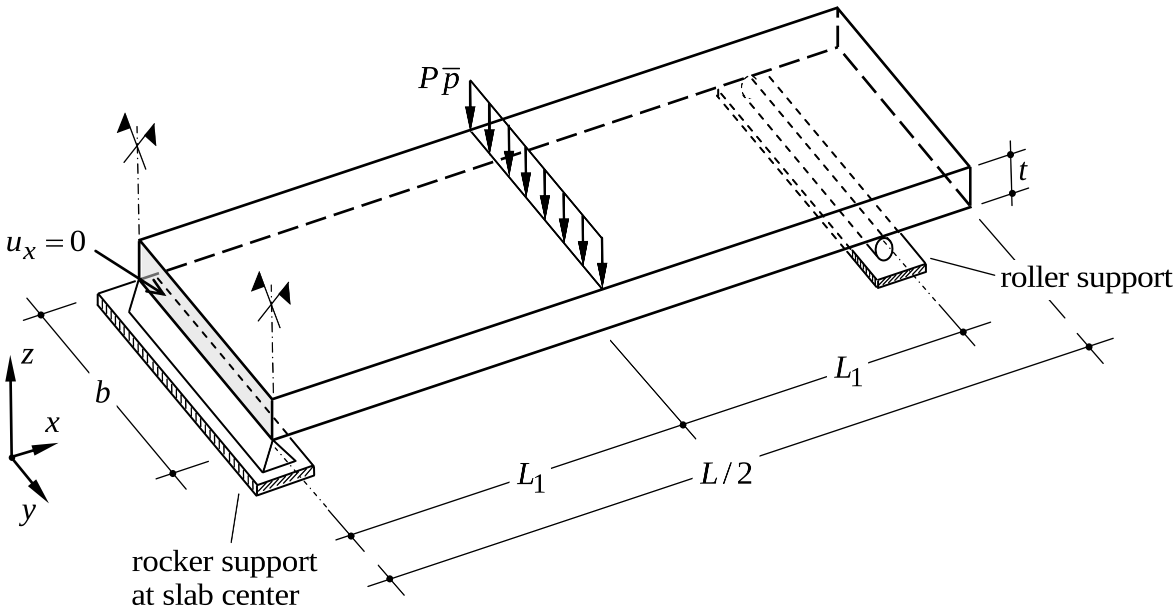

Figure 2: Mechanical model of continuous-supported slabs SL15, SL30, SL45

Table 1: Geometrical data and reinforcement arrangement of the tested RC slabs

Specimen label	*geometric properties*					*reinforcement arrangement*	
	b (mm)	*L (mm)*	*t (mm)*	*L_1 (mm)*	*L_2 (mm)*	*top re-bars*	*bottom re-bars*
C-1	600	2440	125	870	600	--	5 #4[a]
S1	400	3800	155	1167	1167	--	2 N12
S2	400	3800	155	1167	1167	--	3 N12
S3	400	3800	155	1167	1167	--	4 N12
SL15	375	5850	120	1400	--	5 Φ12	4 Φ12, 3 Φ8
SL30	375	5850	120	1400	--	4 Φ12	3 Φ12, 4 Φ10
SL45	375	5850	120	1400	--	3 Φ10, 2 Φ8	6 Φ12, 1 Φ8

[a] #3@300mm were placed along the short-span direction (see Al-Rousan *et al.* [4]).

With regard to the FE model adopted, the number of finite elements is different for each specimen and has been chosen after a preliminary mesh sensitivity study to assure an accurate FE elastic solution. A typical FE mesh is depicted in Figure 3. For the 7 analyzed RC slabs the number of 3D-solid elements ranges from 768 to 960, while that of truss elements from 44 to 278.

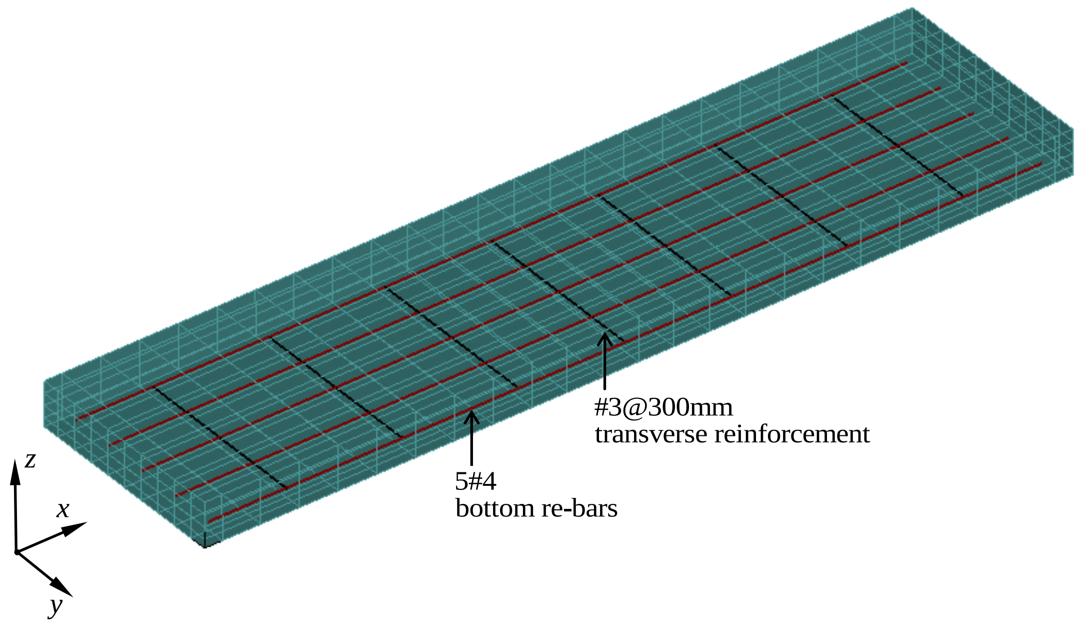

Figure 3: Typical FE mesh with 3D-solid elements modeling concrete and truss elements for re-bars (specimen C-1)

To point out the improvements achieved by using the proposed two-yield-criteria limit analysis approach compared to the one with indefinitely elastic behaviour of steel re-bars, the upper and lower bounds to the peak load multiplier were also computed keeping bars in the elastic field. These latter values are labelled P_{UB_eb} and P_{LB_eb} in Table 2 and 3, the subscript "eb" standing for "elastic bars". In Table 3 the relative errors are reported with sign comparing the numerical results with the experimental findings. Normally, the upper bound values are expected to have a positive relative error and the lower bound values a negative one.

By examining the numerical results reported in Tables 2 and 3, the predictive performance of the proposed limit analysis methodology is rather good for almost all the examined RC slabs, the upper and lower bounds detected in most cases "bracket" the experimental value quite closely. In detail, the upper bound values predicted by the LMM are always above the experimental ones (as it should be when searching for an upper bound), moreover a considerably better performance is observed using the proposed two-yield-criterioa approach (P_{UB} values) compared to the previous one (P_{UB_eb} values). The former formulation gives, in fact, numerical prediction having relative errors less than 10% in all the examined RC slabs. This improvement is even more marked in all those experimental cases where actually yielding of steel reinforcement is more pronounced (e.g. specimen C-1, SL45). Looking at the specimens C-1, SL45 also the lower bound values P_{LB_eb}, obtained by the ECM keeping steel re-bars elastic, are wrong as in these cases they exceed the value of the experimental load multiplier P_{EXP}, which is unacceptable for a lower bound value. On the other hand, when the failure is mainly ascribed to crushing of concrete in the compression zone with little yielding of steel re-bars occurring (e.g. in specimens S1, S2, S3), the hypothesis of indefinitely elastic behaviour of steel re-bars results in relatively satisfactory numerical predictions (with relative errors, on average, of approximately 15% for both the P_{UB_eb} and the P_{LB_eb} values), although also for these cases an overall better performance of the proposed two-yield-criteria formulation is achieved (relative errors less than 10% except for the P_{LB} value of RC slab S2). Moreover, only a few iterations/linear FE-elastic analyses (generally approximately fifteen) are sufficient to obtain a converged solution in terms of both upper and lower bounds.

Table 2: Peak load multipliers for the analyzed RC slabs: values experimentally detected (P_{EXP}); values of upper and lower bounds (P_{UB} and P_{LB}, respectively); values P_{UB_eb} and P_{LB_eb} computed keeping bars elastic

	Peak load multipliers				
Specimen designation	P_{EXP}	P_{UB}	P_{LB}	P_{UB_eb}	P_{LB_eb}
C-1	0.765	0.826	0.712	1.042	0.876
S1	0.215	0.231	0.194	0.273	0.201
S2	0.365	0.382	0.322	0.434	0.309
S3	0.485	0.511	0.438	0.521	0.416
SL15	0.514	0.562	0.499	0.589	0.446
SL30	0.498	0.540	0.468	0.584	0.439
SL45	0.526	0.569	0.472	0.640	0.578

Table 3: Relative errors of the numerically predicted peak multipliers for the analyzed RC slabs

	Relative error (%)			
Specimen designation	P_{UB}	P_{LB}	P_{UB_eb}	P_{LB_eb}
C-1	7.99	-6.93	36.27	14.48[a]
S1	7.61	-9.95	26.76	-6.49
S2	4.72	-11.70	18.88	-15.31
S3	5.28	-9.72	7.47	-14.30
SL15	9.40	-2.87	14.75	-13.24
SL30	8.30	-6.14	17.14	-11.87
SL45	8.22	-10.09	21.87	9.97[a]

[a] wrong prediction ($P_{LB\_eb} > P_{EXP}$)

Some useful information on the mechanical behaviour of the RC slabs at collapse can be gained by the prediction of the failure modes, which can be obtained by identifying the plastic zones (collapse mechanism) at the last converged solution of the LMM. As said, the aim of the LMM is to construct a collapse mechanism on a *fictitious structure* (i.e. on the analyzed RC slab having a fictitious spatially varying distribution of elastic parameters) when loaded by $P_{UB}\,\bar{p}$. To this aim, the plots of the principal compressive strain rates in the deformed configuration are shown in Figure 4 with reference to the continuous slab SL15 at the final (converged) distribution of fictitious parameters. As declared in the experimental tests (see [6]) for this specimen, after excessive deformation three plastic hinges were formed, i.e. two in the sagging regions at the two midspans and one in the hogging region over the rocker support at slab center. The deformed shape shows how around such (confined) plasticized zones the remainder of the slab rotates rigidly (i.e. plastic rotations occur) as it should be in a flexural global collapse mechanism. Moreover, the stress-state numerically obtained in the steel reinforcement is such that the most critical re-bars (precisely, those in the sagging regions) are just yielded as observed in the experimental outcomes.

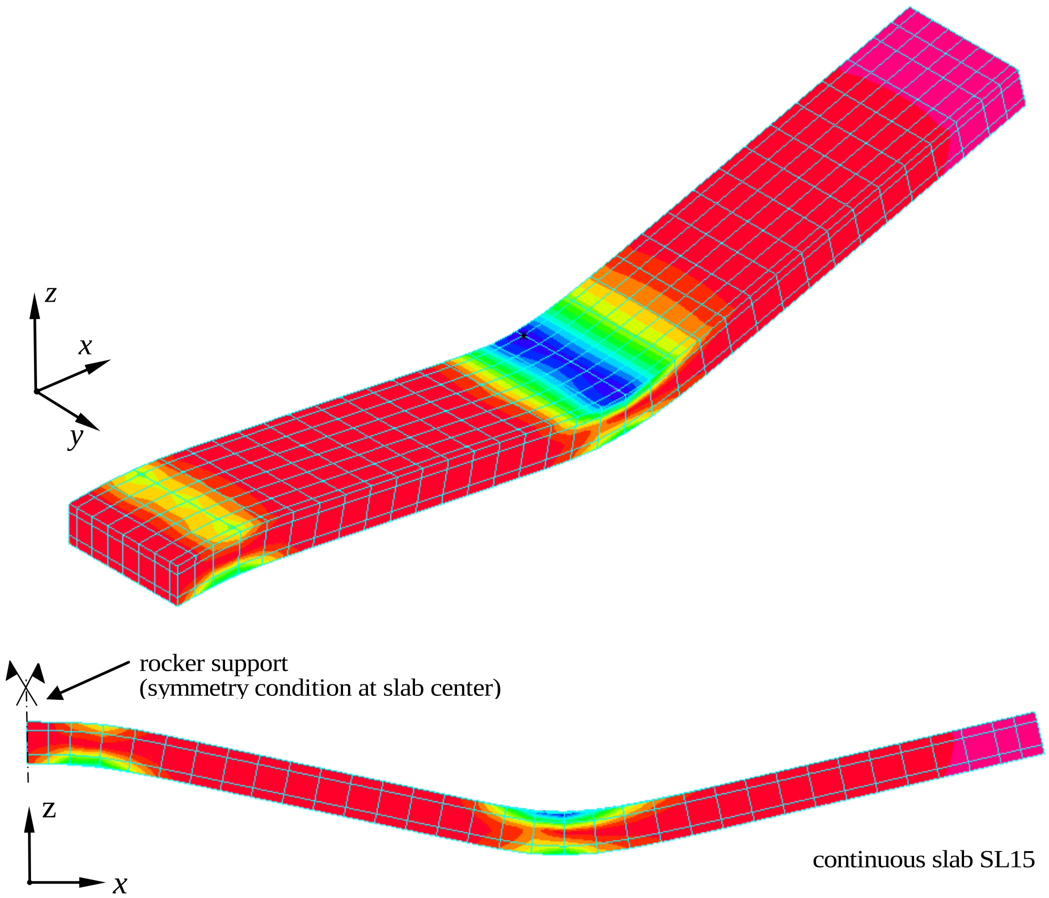

Figure 4: Prediction of the failure mechanism for the continuous slab SL15. Band plot of principal compressive strain rates in the deformed configuration at the ultimate value of the acting load obtained at last converged solution of the LMM

4 CONCLUDING REMARKS

Large-scale prototypes of steel-reinforced concrete slabs have been analyzed to show reliability and effectiveness of a limit analysis numerical methodology recently proposed by the authors. To this aim, experimental tests up to failure, available in the literature, have been taken into consideration and numerically simulated to predict peak load and collapse mechanism of the tested specimens.

The comparison between experimental findings and the corresponding numerical results proved to be quite satisfactory in terms of both peak load prediction, carried out by detecting an upper and a lower bound to the peak load multiplier and failure mechanism description.

Compared to a previous formulation the presented approach results to be more accurate and consistent accounting for the actual contribution of the yielded re-bars to the post-elastic behaviour of the structural element.

The improvements achieved are very encouraging especially when analysing RC slabs in which pronounced yielding of the steel re-bars experimentally occurs.

As the proposed methodology is a simple numerical tool based on conventional FE-based iterative analyses, it can easily be applied with any commercial FE-code without specialist programs and lends itself to applications on more complex RC structures to obtain useful (preliminary) information on peak load, failure modes, and critical zones.

The present approach may potentially be useful either for design purposes of new structures or for existing structures to locate weak links and vulnerable structural spots so as to plan structural interventions.

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