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DIRECT EVALUATION OF THE LOAD-CARRYING CAPACITY OF STEEL-REINFORCED CONCRETE BEAMS BY LIMIT ANALYSIS

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Summary. *A numerical limit analysis methodology is adopted to predict the load-carrying capacity of steel reinforced concrete beams. The methodology, based on iterative elastic analyses, is here refined and implemented through a two-yield-criteria formulation where post-elastic behaviour of both concrete and reinforcement bars is taken into account. The effectiveness of the methodology is shown by the analysis of reinforced concrete beams.*

1 INTRODUCTION

Although plain concrete is not a ductile material, reinforced concrete (RC) can exhibit, when loaded up to failure, considerable ductility injected by the presence of reinforcement bars (re-bars). The typical failure mode of properly designed so-called under-reinforced elements subjected to bending, in fact, initiates by yielding of the steel re-bars followed, with the load increasing, by crushing of concrete in compression zone. This is, actually, a failure mode regarded as ductile and normally assured by a reinforcement ratio below the balanced ratio. Obviously, in these cases steel re-bars play a significant role in determining the post-elastic behaviour of the RC element.

In this contribution, a numerical methodology, founded on the limit analysis theory and aimed at the evaluation of the load-carrying capacity of steel-reinforced concrete elements, is applied. This finite element-based (FE-based) methodology, promoted by the authors to predict the limit state solution of RC elements (see [1], [2]), allows the determination of an upper and a lower bound to the peak load of the RC structural elements and gives some useful indications on the expected failure mechanism. In the quoted papers the methodology has been performed by adopting a Menétrey–Willam-type (M–W-type) yield criterion [3] endowed with cap in compression for concrete, and by postulating an indefinitely elastic behaviour of steel re-bars, which is acceptable in over-reinforced concrete elements. However, a more consistent approach, surely more suitable for under-reinforced beams, should simultaneously take into account both the post-elastic behaviour of concrete and the actual contribution of yielded re-bars to the post-elastic behaviour of the structural element as a whole. To this aim, the above numerical methodology is here refined and implemented within a *two-yield-criteria* formulation, where concrete is governed by the *Menétrey–Willam-type yield criterion* and steel bars are handled by a *von Mises criterion*, so enabling the prediction of possible steel bars yielding at incipient collapse. In order to validate the effectiveness and reliability of the promoted approach, a comparison is made between numerical predictions and experimental findings on large-scale beams tested in laboratory [4]–[6].

2 FUNDAMENTALS OF THE NUMERICAL LIMIT ANALYSIS METHODOLOGY

The constitutive assumptions together with the key ideas of the numerical methodology are here outlined. For the sake of brevity, only a few basic concepts are given; more detailed information can be found in [1], [2].

As said, the limit analysis methodology gives the peak load multiplier of the analyzed RC beams by detecting an upper and a lower bound to it. Since concrete is viewed as a nonstandard material, a *nonstandard limit analysis approach* is adopted. According to this approach, the yield surface of the nonstandard material is encircled with two surfaces, precisely an outer and an inner surface, and two bounds are then computed with reference to such surfaces (or materials). Concrete is assumed to comply with a M–W-type yield criterion [3] with cap in compression, which plays itself the double role of inner and outer surface in the static and kinematic nonstandard limit analysis approach [7]. A traditional (standard) limit analysis approach with von Mises yield surface is instead adopted for steel re-bars.

The methodology consists of two limit analysis numerical procedures, namely the Linear Matching Method (LMM) and the Elastic Compensation Method (ECM). Both the procedures, originally conceived for von Mises type materials (see [8], [9]), have also been successfully applied by the authors, within a wider research program, in the context of composite laminates ([10]–[14]). The application to RC elements with fiber reinforced polymer (FRP) bars has also been investigated in [15]. Following a kinematic approach of limit analysis, the LMM is aimed at constructing a collapse mechanism for the evaluation of an upper bound P_{UB} to the collapse load of the analyzed elements. It is an iterative procedure involving one sequence of linear FE-based analyses in which the studied structure is assumed made of a *fictitious* material with spatially varying moduli. At each iteration the fictitious moduli are adjusted so that the computed fictitious stresses are brought onto the yield surface at a fixed strain rate distribution so defining a collapse mechanism, i.e. strain and displacement rates together with the associated stresses at yield. By applying the upper bound theorem of the limit analysis theory a P_{UB} load multiplier can then be computed. Following a static approach of limit analysis, the ECM attempts to construct an admissible stress field suitable for the evaluation of a lower bound P_{LB} to the collapse load. It is an iterative procedure involving many sequences of linear FE-based analyses in which highly loaded regions of the structure are systematically weakened by the reduction of the local modulus of elasticity ("*redistribution procedure*") in order to simulate the effects of inelasticity.

A noteworthy refinement of the methodology compared to the one proposed in [1], [2] is that the above mentioned concepts are taken into consideration for both concrete and steel re-bars with reference to the two adopted yield criteria. As shown next, this significantly improves the effectiveness of the procedures and the accuracy of the obtained numerical results.

3 NUMERICAL RESULTS

The proposed approach was tested on 7 RC beams that were numerically simulated to predict peak load and collapse mechanism. Details concerning laboratory test equipment, reinforcement arrangement and experimental results for the analyzed RC beams can be found in [4]–[6].

The elastic analyses have been carried out using the FE-code ADINA [16] with meshes of isoparametric and displacement-based 3D-solid 8-nodes elements with 2x2x2 GPs per element for modeling concrete and 2-nodes 1-GP truss elements for steel re-bars and stirrups. A perfect bond between concrete and re-bars is assumed in the FE-analyses. An isotropic material formulation has been employed for concrete and re-bars, so only two material constants are used to define the constitutive relations, namely the Young modulus E and the Poisson ratio ν. To set both the Menétrey–Willam-type and the von Mises yield surfaces, the material strength properties (compressive and tensile strength for concrete, f_c', f_t', and yield strength for longitudinal steel re-bars, f_y) have been assumed as indicated in the experimental campaigns (see again [4]–[6]) and are not reported here for the sake of brevity.

In particular, among the 7 beams numerically analyzed, specimen labelled A3 was tested at the University of Toronto by Vecchio and Shim [6] and was here taken into consideration to highlight the better performance of the proposed two-yield-criterion approach compared to that presented in [1]. In the latter paper the same RC beams were, in fact, analyzed but re-bars were modelled as indefinitely elastic members. Among the 12 Toronto beams, the above beam A3 experienced in facts a pronounced yielding of tension reinforcement unlike the other RC specimens in which yielding of re-bars was not detected at ultimate/failure conditions. By assuming re-bars to have an indefinitely elastic behaviour, the corresponding numerical result concerning beam A3 was rather poor in terms of peak load prediction (see [1]). As shown below, a significantly better performance of the proposed (refined) approach has been attained. The second group of specimens analyzed was that presented by Lau and Pam [4]. 12 hybrid FRP RC beams (with both FRP and steel re-bars) were tested in three point bending (simply supported). Three out of the twelve tested beams (namely specimens labelled MD1.3-A90, MD2.1-A90, T0.2-A135) were reinforced with steel re-bars only and failed by crushing of concrete in compression and yielding of steel re-bars in tension occurring almost simultaneously. Moreover, these three beams were designed as under-reinforced specimens so as to promote yielding of steel re-bars at ultimate states and increase flexural ductility. The ductility ratio of these three beams, computed as the ratio of midspan displacement at ultimate stage (Δ_u) and at yield stage (Δ_y), was in fact by far higher than that of the other hybrid FRP RC specimens. All the experimental conditions match quite well the assumptions of the proposed two-yield-criterion limit analysis approach and this is why these three beams have been numerically analyzed here. Finally, 16 RC continuous beams strengthened with different arrangement of internal steel re-bars and external CFRP laminates were tested by Ashour *et al.* [5]. The beams were classified into three groups according to the amount of internal steel reinforcement. Each group included one un-strengthened control beam designed to fail in flexure (namely specimens labelled H1, S1, E1). Conventional ductile flexural failure due to yielding of the internal tensile steel reinforcement followed by concrete crushing at both the central support and midspan sections occurred for the three control beams, with large ductility ratio. The remaining 13 strengthened RC beams with CFRP laminates presented by Ashour *et* *al.* [5] are outside the scope of this paper, therefore they are excluded in the present analysis.

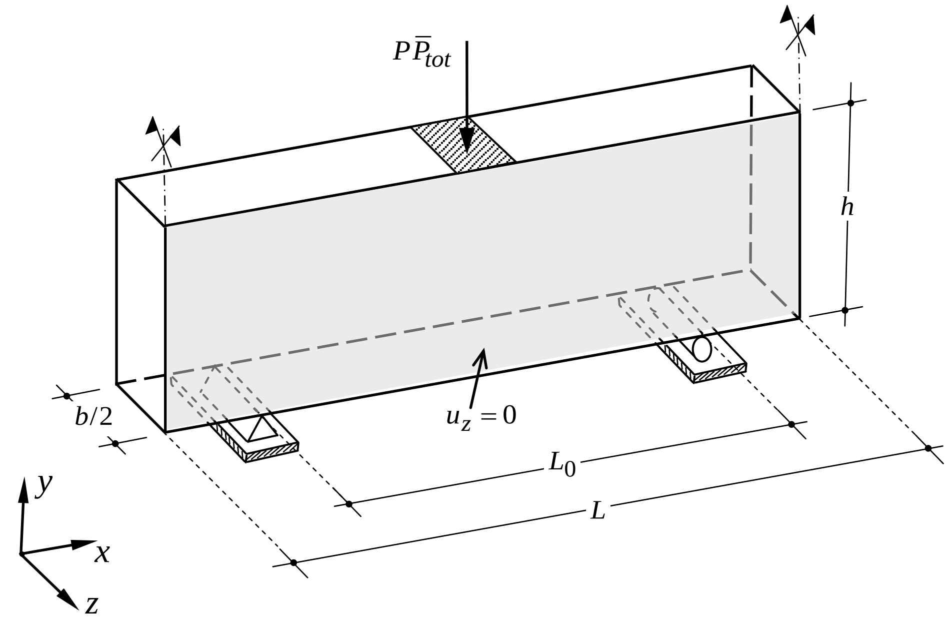


Figure 1: Mechanical model of simply-supported beams A3, MD1.3-A90, MD2.1-A90, T0.2-A135

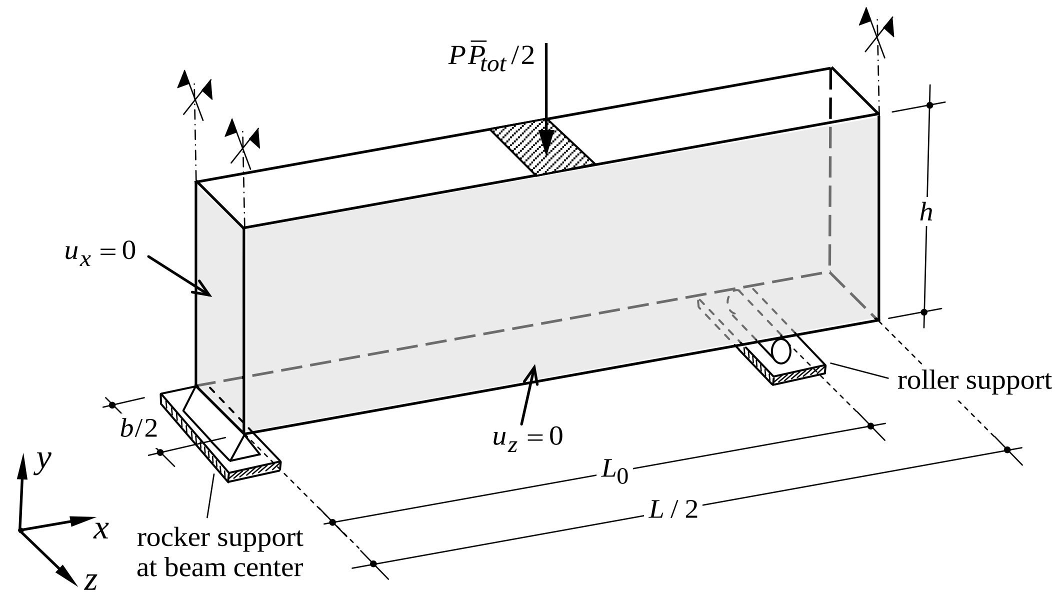


Figure 2: Mechanical model of continuous-supported beams H1, S1, E1

As shown in Figure 1, the simply-supported beams (specimens A3, MD1.3-A90, MD2.1-A90, T0.2-A135) were subjected to a concentrate load at midspan $P\overline{P}_{tot}$ with P denoting the load multiplier and $\overline{P}_{tot}$, assumed equal to 100kN for all the tested beams, the reference load. Due to the symmetry of the problem only half specimen has been analyzed and this by setting zero displacements in *z* direction to the FE-nodes lying on the shaded symmetry plane shown in Figure 1. The continuous-supported beams (specimens H1, S1, E1), which comprised two equal spans, were similarly subjected to two concentrate loads $P\bar{P}_{tot}/2$ at the two midspans. In this case, only a quarter of the beam has been modelled exploiting the symmetry in both *x* and *z* directions, as shown in Figure 2, in order to guarantee an accurate FE elastic solution without increasing the computational effort considerably. Details concerning geometrical data and reinforcement arrangement of all the 7 analyzed RC beams are given in Table 1.

Table 1: Geometrical data and reinforcement arrangement of the tested RC beams

	geometric properties				*reinforcement arrangement*		
Specimen label	*b (mm)*	*h(mm)*	*L (mm)*	*L_0 (mm)*	*top re-bars*	*bottom re-bars*	*stirrups*
A3	305	552	6840	6400	3 M10	4 M30, 2 M25	D4@168mm
MD1.3-A90	280	380	4600	4200	2 R6	4 MD20	R8@100mm[a]
MD2.1-A190	280	380	4600	4200	2 R6	4 MD25	R8@100mm[a]
T0.2-A135	280	380	4600	4200	2 R6	2 T12	R8@100mm[a]
H1	150	250	8500	3830	2 T8	2 T20	R6@100mm
S1	150	250	8500	3830	2 T20	2 T8	R6@100mm
E1	150	250	8500	3830	2 T16	2 T16	R6@100mm

[a] R8@50mm near the support (see Lau and Pam [4]).

With regard to the FE model adopted, the number of finite elements is different for each specimen and has been chosen after a preliminary mesh sensitivity study to assure an accurate FE elastic solution. For the 7 analyzed RC beams the number of 3D-solid elements ranges from 840 to 984, while that of truss elements from 376 to 738.

To point out the improvements achieved by using the proposed two-yield-criterion limit analysis approach compared to the one with indefinitely elastic behaviour of steel re-bars (i.e. the numerical formulation presented in [1] and [2]), the upper and lower bounds to the peak load multiplier obtained by means of the LMM and the ECM, respectively, were also computed keeping the bars in the elastic field (i.e. via the previous approach adopted in [1], [2]).

Figures 3a–g show, for the analyzed specimens, the plots of the upper and the lower bounds to the peak load multiplier versus the iteration number. As shown, only a few iterations/linear FE-elastic analyses (generally less than fifteen) are sufficient to obtain a converged solution in terms of both upper and lower bounds. The monotonic and rapid convergence is assured by a sufficient condition given by Ponter *et al.* [17] fulfilled by the assumed M–W-type and von Mises yield surfaces. In Figures 3a–g also the values of the upper and lower bounds obtained by using the previous approach (P_{UB_eb} and P_{LB_eb} values) are depicted for the sake of comparison.

In Table 2 the relative errors between all the computed bounds are reported with sign, comparing the numerical results with the experimental findings. Normally, the upper bound values are expected to have a positive relative error and the lower bound values a negative one.

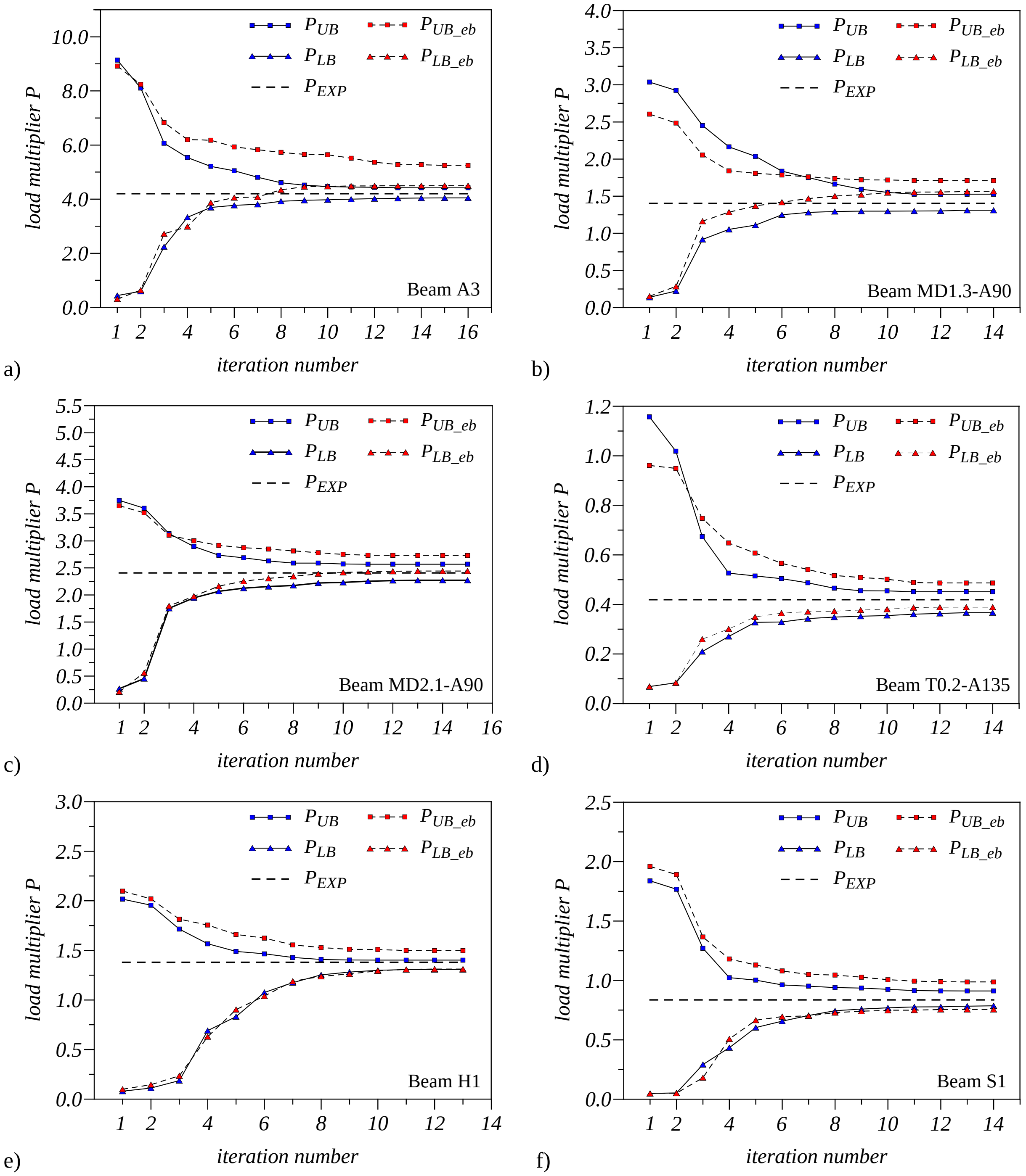
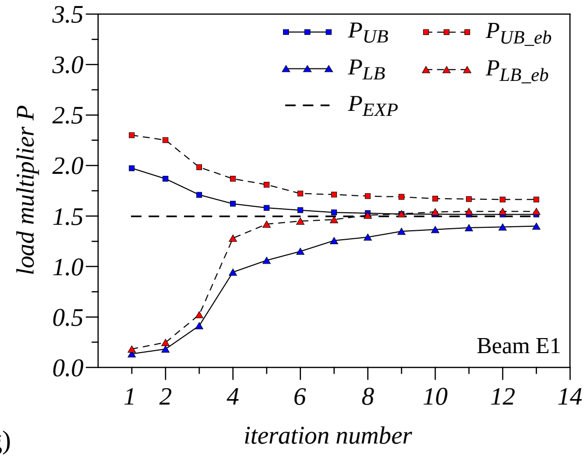


Figure 3: Values of the upper and lower bounds to the peak load multiplier versus iteration number obtained by the proposed two-yield-criteria approach (solid lines with blue markers) and keeping bars elastic (dashed lines with red markers) compared to the collapse experimental threshold, dashed lines



g)

Figure 3 (continued)

Table 2: Relative errors of the numerically predicted peak multipliers for the analyzed RC beams

	Relative error (%)			
Specimen designation	P_{UB}	P_{LB}	P_{UB_eb}	P_{LB_eb}
A3	5.23	-3.62	24.92	7.10[a]
MD1.3-A90	8.76	-6.59	21.73	11.58[a]
MD2.1-A190	6.73	-5.57	13.45	1.47[a]
T0.2-A135	7.71	-12.46	16.11	-7.22
H1	1.60	-5.30	8.52	-4.93
S1	9.04	-5.92	18.09	-9.58
E1	1.22	-6.51	11.14	3.21[a]

[a] wrong prediction ($P_{LB\_eb} > P_{EXP}$)

By analyzing the numerical results, the proposed limit analysis procedure seems to be quite accurate in defining two close limits to the real—experimentally obtained—peak load value for almost all the examined RC beams. In detail, the upper bound values predicted by the LMM are always above the experimental ones as it should be when searching for an upper bound. As can be seen in Table 2, with regard to the upper bound values a significantly better performance is obtained by the proposed two-yield-criteria approach (P_{UB} values) compared to the previous one (P_{UB_eb} values). It is worth noting that also the relative errors concerning the upper bounds reduced significantly by using the proposed refined formulation (relative errors always less than 10%) and are more than halved in all the analyzed RC beams so witnessing a noteworthy improvement. Also with regard to the lower bound values obtained by the ECM a significant better performance of the refined formulation (P_{LB} values) compared to the previous one (P_{LB_eb} values) is observed. The relative errors concerning the P_{LB} values are less than 10% in all but one specimens (RC beam T0.2-A135). Moreover, it should be pointed out that in three out of seven specimens the lower bound predictions of the previous approach (P_{LB_eb} values) were incorrect as they exceeded the value of the experimental load multiplier P_{EXP}, which is unacceptable for a reliable lower bound value.

It is worth mentioning that some useful indications on the expected failure mode of the analyzed RC beams could also be obtained by identifying the plastic zones (collapse mechanism) at last converged solution of the LMM and this simply by plotting the strain rates in the deformed configuration on the beam loaded by $P_{UB}\overline{P}_{tot}$ at last iteration of the LMM. Such plots are not reported for sake of brevity.

4 CONCLUSIONS

A limit analysis methodology, already presented in [1] and [2] and successfully applied to RC elements, has been here modified and refined to include the possibility of modeling yielding of steel re-bars at incipient collapse that often occurs especially in under-reinforced RC beams. This more consistent approach results in a better predictive performance especially for those RC beams in which yielding of re-bars is more marked in the experimental tests. Operationally, the use of the two-yield-criteria formulation compared to that presented in [1] does not entail any significant increase in computational cost. The main difference is that the iterative updating of the elastic moduli, carried out within both the LMM and the ECM, has to be referred to two distinct materials, i.e. concrete and steel, by considering two distinct yield surfaces. In this regard, it is worth noting that accounting for the yielding in steel re-bars with reference to the von Mises yield surface reduces to an (iterative) updating of the elastic moduli of the bars depending on the ratio of the yield strength and the elastic stress in the truss bars elements. The obtained results, at least for the examined cases, seem to prove the effectiveness of the presented enhanced limit analysis approach.

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