# PERFORMANCE OF A COMPUTATIONAL COST REDUCTION TECHNIQUE IN LENGTHY TIME INTERVAL ANALYSES

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**Key Words:** *Time Integration, Accuracy, Computational Cost, Digitized Excitation, Numerical Stability.*

**Abstract.** Time integration is the most versatile tool to analyze semi-discretized equations of motion. However, the results are not exact and the computational costs are considerable. In 2008, a technique is proposed, for reducing the computational costs, for cases, when the excitations are available as digitized records, and the digitization step sizes are smaller than needed for accuracy. In view of practical cases in seismic analyses, many numerical tests are carried out, and it is observed that, by implementing the technique, in time integration analyses, the computational costs can be reduced significantly, in the price of small and in cases negligible loss of accuracies. In order to ensure that the additional errors do not accumulate, and can not lead to numerical instability, the performance of the technique is herein studied, in lengthy time interval analyses, theoretically and numerically. As the outcome, the responses are stable. Nevertheless, when assigning values to the technique's parameter close to its limiting values, the errors might be large and unacceptable. Still, the technique can be considered considerably successful, for the smaller values of the parameter.

## 1 INTRODUCTION

For analyzing structures' dynamic behaviours, in view of the semi-discretized models [1-3]

$$\mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{f}_{\text{int}}=\mathbf{f}(t), \qquad 0\le t\le t_{end} \tag{1}$$

$$\text{Initial Conditions}: \left| \begin{array}{l} \mathbf{u}(t=t_0)=\mathbf{u}_0 \\ \dot{\mathbf{u}}(t=t_0)=\dot{\mathbf{u}}_0 \\ \mathbf{f}_{\text{int}}(t=t_0)=\mathbf{f}_{\text{int}_0} \end{array} \right. ,$$

$$\text{Additional constraints}: \mathbf{Q}$$

time integration is the most versatile and broadly accepted approach. In Eqs. (1), $t$ and $t_{\text{end}}$ imply the time and the duration of the dynamic behavior; $\mathbf{M}$ is the mass matrix; $\mathrm{f}_{\text{int}}$ and $\mathbf{f}(t)$

stand for the vectors of internal force and excitation; $\mathbf{u}(t)$, $\dot{\mathbf{u}}(t)$, and $\ddot{\mathbf{u}}(t)$ denote the unknown vectors of displacement, velocity, and acceleration; $\mathbf{u}_0$, $\dot{\mathbf{u}}_0$, and $\mathbf{f}_{\mathbf{int}_0}$ define the initial status of the model [4,5]; and $\mathbf{Q}$ represents some restricting conditions, in nonlinear problems, e.g. additional constraints in problems involved in impact or elastic-plastic behavior [6,7]. Nevertheless, due to the step-by-step nature and simple formulation of time integration methods [8,9] (see Figure 1), the responses are inexact and the computational costs are

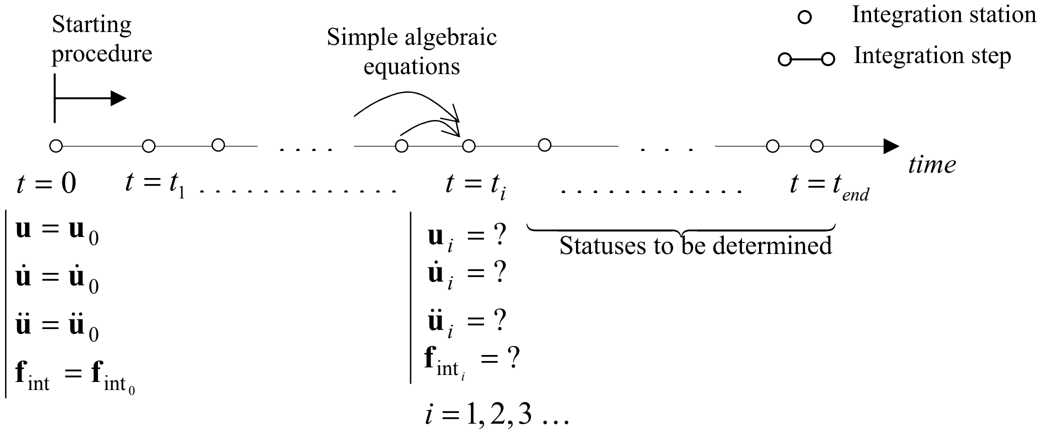


**Figure 1**: A pictorial representation of the time integration process

considerable. Considering the fact that generally more accuracy corresponds to more computational cost, both under the control of the integration step size, comments exist on the integration step size selection. A broadly accepted comment is

$$\Delta t \cong\leq \mathrm{Min}\left(\frac{T}{10}, h_s\right) \tag{2}$$

where, $\Delta t$ implies the constant size of the integration steps, $T$ is the smallest period of the oscillations with considerable contribution in the response, and $h_s$ stands for the maximum step size preserving numerical stability. In earthquake engineering, however [2, 10, 11],

$$\Delta t \cong\leq \mathrm{Min}\left(\frac{T}{10}, h_s, {}_f\Delta t\right) \tag{3}$$

where, the new parameter, ${}_f\Delta t$, is the size of steps, by which, the strong ground motion is available in a digitized manner. When the values of $\Delta t$ obtained from Eqs. (2) and (3) are considerably different, significant computational cost is being spent for correctly modeling the excitation, not directly for the accuracy. In order to eliminate this additional computational cost, a technique is proposed in 2008 [12], that replaces the excitation with an excitation digitized at steps, $n$ times larger, materializing time integration with steps equal to

$$\Delta t = n_f \Delta t \cong\leq \text{Min}\left(\frac{T}{10}, h_s\right) \quad , \quad n \in Z^+ - \{1\} \tag{4}$$

Numerical tests have displayed the good performance of the proposed technique; see Table 1 [13-21]. Still, with attention to the rare, but existing, lengthy strong ground motions (see the example in Section 3), it is essential to also examine the performance of the technique, and specifically the probability of any numerical instability, when the analyses are carried out in lengthy intervals. This is discussed theoretically in Section 2, and backed numerically in Section 3, resulting in the conclusions presented in brief in Section 4.

**Table 1**: Experiences on the technique proposed in [12]

| Structural system | Reduction of computational cost in the price of trivial loss of accuracy (%) | Reference |
|---|---|---|
| Eight storey shear frame | 80 | [13] |
| Building in pounding | 12.7 | [14] |
| Thirty-storey building | 50 | [15] |
| Water tank | 66.7 | [16] |
| 3-component earthquakes | 66.7 | [17] |
| Silo | 77.65 | [18] |
| Bridges subjected to severe earthquakes | 45-80 | [19] |
| Residential buildings | 50-87 | [20,21] |

## 2 THEORY

Considering the independency of numerical stability from nonlinearity [4,22-25], and the fact that, in the study of numerical stability, it is conventional to consider the damping classical [8,9], the theoretical study of numerical stability can be limited to linear single degree of freedom system with $u$, $\dot{u}$, and $\ddot{u}$, as the variables defining the status of the system. With these considerations, for one step time integration methods [8,9],

$$\mathbf{Y}_i = \mathbf{A}\mathbf{Y}_{i-1} + \mathbf{L}\mathbf{F}_i \tag{5}$$

where,

$$\mathbf{Y}_i = \begin{bmatrix} u_i \\ \dot{u}_i \Delta t \\ \ddot{u}_i \Delta t^2 \end{bmatrix}$$
$$\mathbf{F}_i = \mathbf{F}(t_i) = \begin{bmatrix} f_{i-1} \\ f_i \end{bmatrix}$$
$$\mathbf{A} = [A(\omega \Delta t, \xi)] \quad \text{(amplification matrix)}$$
$$\mathbf{L} = [L(\omega \Delta t, \xi)] \quad \text{(load operator)} \tag{6}$$

$\omega$ and $\xi$ are parameters introducing the natural frequency and viscous damping of the single degree of freedom system, and the mass is implied in the definition of $f$; see the equation below:

$$\ddot{u} + 2\xi\omega\,\dot{u} + \omega^2 u = f(t) \tag{7}$$

In implementation of the technique proposed in 2008 [12], Eq. (5) is being replaced with

$$\begin{aligned} \mathbf{Y}_i &= \mathbf{A}\mathbf{Y}_{i-1} + \mathbf{L}\widetilde{\mathbf{F}}_i \\ \widetilde{\mathbf{F}}_i &\neq \mathbf{F}_i \ \ (= \mathbf{F}(t_i)) \end{aligned} \tag{8}$$

(for the details of $\widetilde{\mathbf{F}}_i$, see [12]). Substituting the approximate values in Eq. (8) with the corresponding exact values, leads to

$$\mathbf{Y}(t_i) = \mathbf{A}\mathbf{Y}(t_{i-1}) + \mathbf{L}\mathbf{F}_i - \tau_i \tag{9}$$

where, $-\tau$ is added for the sake of the equality between the two sides of Eq. (9). Reducing Eq. (9) from Eq. (8), we obtain

$$\mathbf{E}_i = \mathbf{A}\mathbf{E}_{i-1} + \mathbf{L}\delta(\mathbf{F}_i) + \tau_i \tag{10}$$

where,

$$\begin{aligned} \mathbf{E}_i &= \mathbf{Y}_i - \mathbf{Y}(t_i) \\ \delta(\mathbf{F}_i) &= \widetilde{\mathbf{F}}_i - \mathbf{F}(t_i) \end{aligned} \tag{11}$$

Equation (10) results in

$$\begin{aligned} \mathbf{E}_i &= \mathbf{A}^2\mathbf{E}_{i-2} + \mathbf{A}\mathbf{L}\delta(\mathbf{F}_{i-1}) + \mathbf{A}\tau_{i-1} + \mathbf{L}\delta(\mathbf{F}_i) + \tau_i \\ &= \mathbf{A}^3\mathbf{E}_{i-3} + \mathbf{A}^2\mathbf{L}\delta(\mathbf{F}_{i-2}) + \mathbf{A}^2\tau_{i-2} + \mathbf{A}\mathbf{L}\delta(\mathbf{F}_{i-1}) + \mathbf{A}\tau_{i-1} + \mathbf{L}\delta(\mathbf{F}_i) + \tau_i \\ &= \mathbf{A}^4\mathbf{E}_{i-4} + \mathbf{A}^3\mathbf{L}\delta(\mathbf{F}_{i-3}) + \mathbf{A}^3\tau_{i-3} + \mathbf{A}^2\mathbf{L}\delta(\mathbf{F}_{i-2}) + \mathbf{A}^2\tau_{i-2} + \mathbf{A}\mathbf{L}\delta(\mathbf{F}_{i-1}) + \\ &\quad \mathbf{A}\tau_{i-1} + \mathbf{L}\delta(\mathbf{F}_i) + \tau_i \\ &\ \vdots \end{aligned} \tag{12}$$

from which, by numerical induction [26], and considering definite initial conditions (i.e. $\mathbf{E}_0 = \overline{\mathbf{O}}$),

$$\mathbf{E}_i = \sum_{j=1}^{i} \mathbf{A}^{j-1}\mathbf{L}\delta(\mathbf{F}_{j-1}) + \sum_{j=1}^{i} \mathbf{A}^{j-1}\,\tau_{i-j+1} \tag{13}$$

Equation (13) apparently implies that, regardless of the implementation/not implementation of the computational cost reduction technique [12], the numerical stability depends on the spectral radius of **A** (see [8,9]), and hence, the technique cannot induce numerical instability. Nevertheless, the percentage of the additional error, at the arbitrary time instant $t_i$, can be approximated by

$$100\frac{\left\|\sum_{j=1}^{i}\mathbf{A}^{j-1}\mathbf{L}\delta\left(\mathbf{F}_{j-1}\right)+\sum_{j=1}^{i}\mathbf{A}^{j-1}\tau_{i-j+1}\right\|}{\left\|\sum_{j=1}^{i}\mathbf{A}^{j-1}\ {}_{o}\tau_{i-j+1}\right\|} \tag{14}$$

where, $\| \ \|$ stands for an arbitrary norm [27], and o as a left subscript implies associating with ordinary analysis. Accordingly, without any knowledge about the dependence of $\delta(\mathbf{F})$, $\tau$ and ${}_{o}\tau$ (also see [12] and [28]), we can expect large errors, even though numerically stable.

## 3 NUMERICAL STUDY

The structural system under consideration here is the bridge model displayed in Figure 2(a) subjected to the lengthy strong ground motion record, addressed in Figure 2(b), where, with

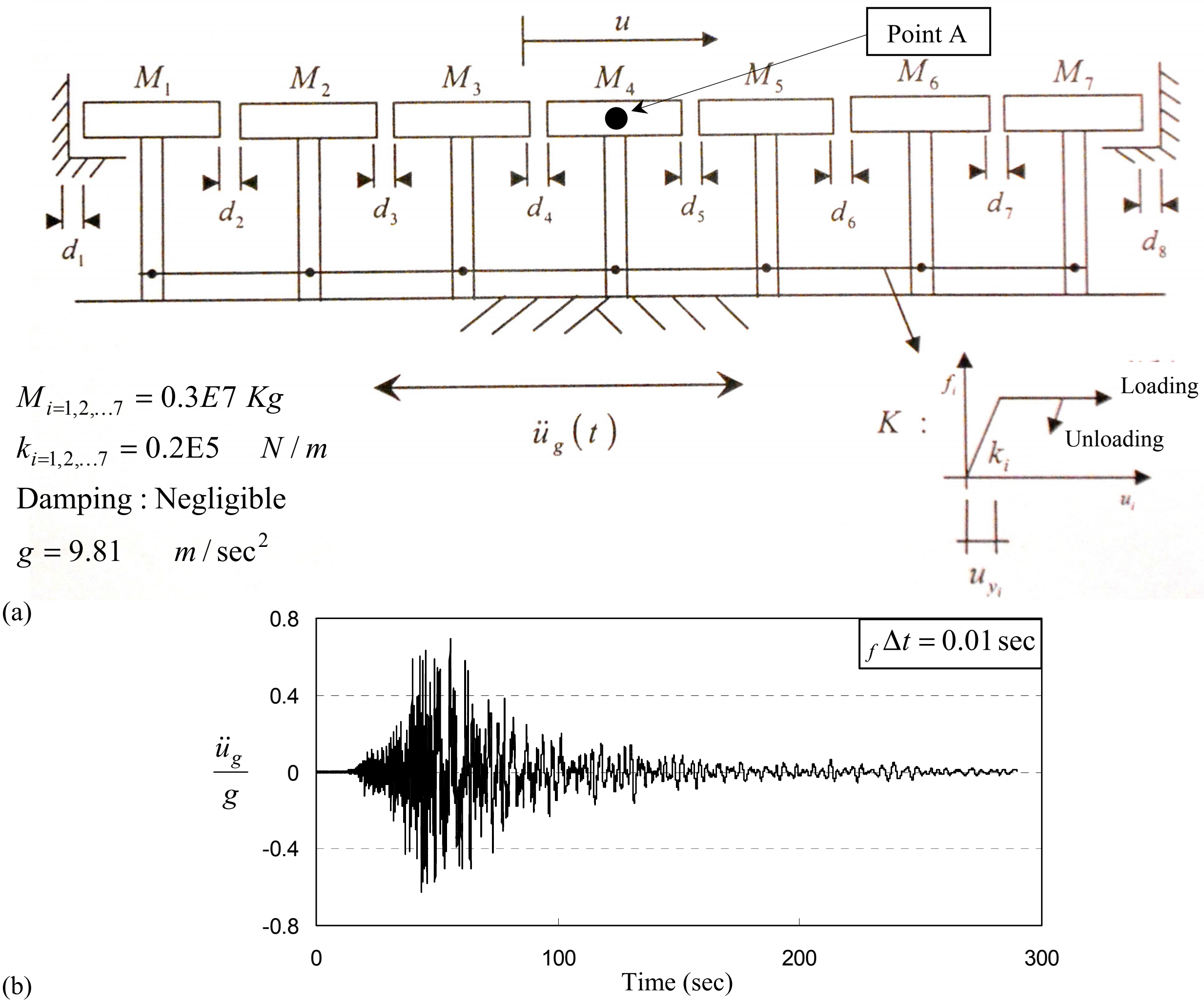


**Figure 2**: The structural system (a) the model [4] (b) the excitation (Tokashi oki)

attention to the objective of this paper, the sources of nonlinearities are inactivated (see also [22,25]). It is meanwhile, worth noting that compared to many strong ground motions (e.g. see [29], pp. 191), Figure 2(b) introduces a considerably lengthy earthquake record, appropriate for this study. The exact displacement history of the mid deck (point A in Figure 2(a)) is as displayed in Figure 3 (though the displacements are unrealistically large due to changes implemented in the model for another study, with attention to the linear behaviour and objectives of the study, the results can be considered acceptable here). In view of Figure 3

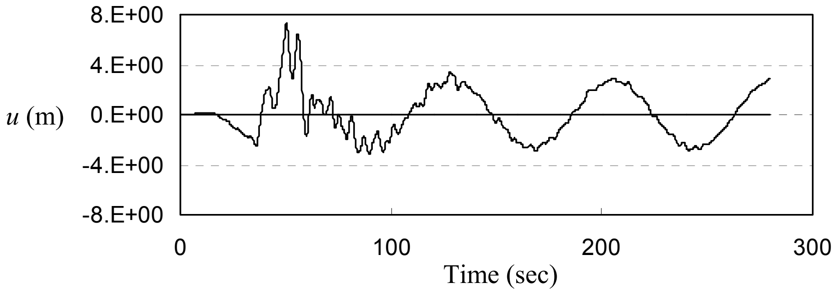


**Figure 3**: Displacement history for point A in Figure 2

and the formulation of the technique [12], the excitation can be replaced with an excitation digitized at steps at most thirty two times larger, i.e. in view of Eq. (4),

$$n = \{2, 3, ..., 32\} \tag{15}$$

Figure 4 displays the responses obtained from ordinary time integration (herein time integration with the Newmark average acceleration method [30]), without implementing the computational cost reduction technique ($n = 1$), and while implementing the technique with the values of $n$, below:

$$n = 2, 4, 10, 30 \tag{16}$$

In agreement with the explanations in Section 2, neither of the responses in Figure 4 implies numerical instability. For further reliability, the excitation in Figure 2(b) is repeated once and twice along the time axis and the resulting 560 and 840 seconds records are used in restudying the example. The outputs are reported in Figures 5 and 6 with vertical axes scaled similar to those in Figure 4. Obviously, Figures 4, 5, and 6 are conceptually identical and imply the numerical stability of the responses, in complete agreement with the theoretical discussion in Section 2. Still errors might be practically unacceptable, e.g. $n = 10$ and $n = 30$ (see Figures 4(c), 4(d), 5(c), 5(d), 6(c), and 6(d)). Considering these, besides the corresponding computational cost reduction, $A_C$, noted below ([12])

$$A_C = \frac{n-1}{n} \quad \% \tag{17}$$

(while, considering $n = 1,2,4,10,30$), in view of the cases $n = 2$ and $n = 4$ in Figures 4-6, where, $A_C$ respectively equals 50 and 75 percent, the technique [12] can be considered

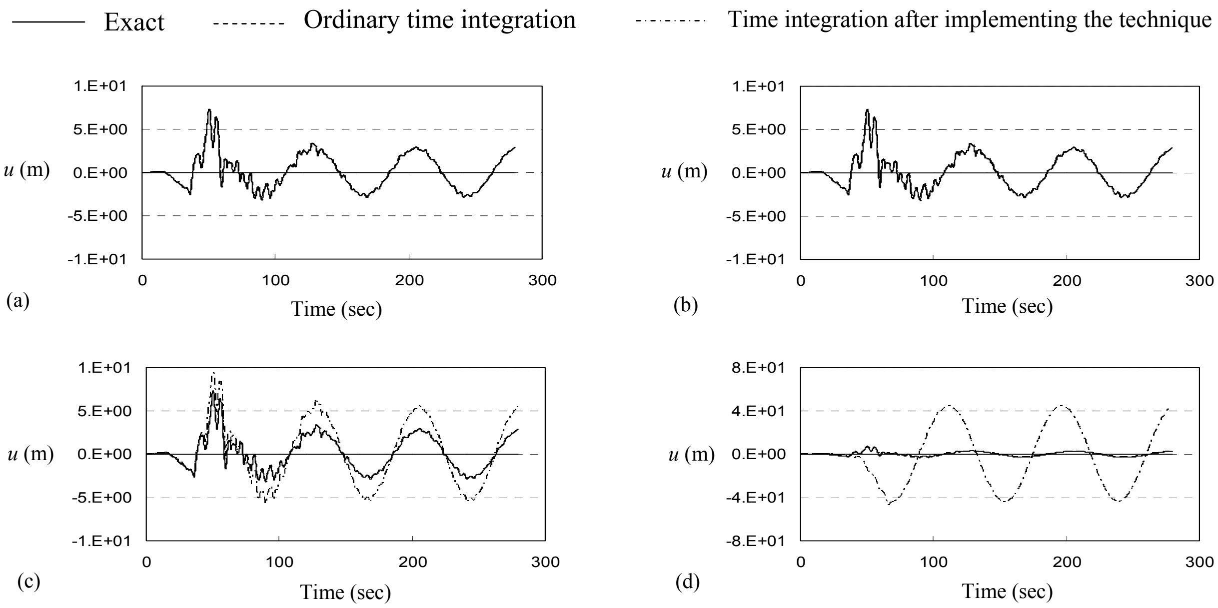


**Figure 4**: Displacements of point A in Figure 2(a) obtained from time integration, after implementations of the technique [12]: (a) $n = 2$, (b) $n = 4$, (c) $n = 10$, (d) $n = 30$

successful, i.e. no instability is probable and there exist values of $n$ leading to considerable computational cost reduction, in the price of negligible loss of accuracy. Merely, it should be stated that, in complete agreement with the discussion presented in Section 2, the limiting values of $n$ proposed by the technique [12], e.g. 30 in Eq. (15), should be restudied in detail. For instance replacing 32 with 7 would be appropriate for the above example (specifically, while the reductions of computational costs are not so different, i.e. 97 versus 86 percents). Finally, it is worth noting that, for more reliability regarding the above mentioned claims, especially that on stability, the example is restudied, by times, with larger values of $n$ (even out of the range of Eq. (15)) and in lengthier time intervals, and the responses were all stable; other structural systems subjected to other lengthy excitations are also studied leading to conceptually identical results, not reported here for the sake of brevity (also with attention to the presented theoretical discussions).

## 4 CONCLUSION

In time integration analyses with digitized records as the external excitations, e.g. seismic analyses, the computational cost reduction technique proposed in [12] provides the capability to reduce the computational cost with negligible loss of accuracy. This technique is examined in this paper for numerical stability, both theoretically and also via numerical examples. As the consequence,

1. Numerical instability because of the computational cost reduction technique proposed in [12] is not probable.
2. The technique can be successful, even in lengthy time integration analyses. However, the parameters of the computational cost reduction technique would rather be set more carefully in lengthy time integration analyses.

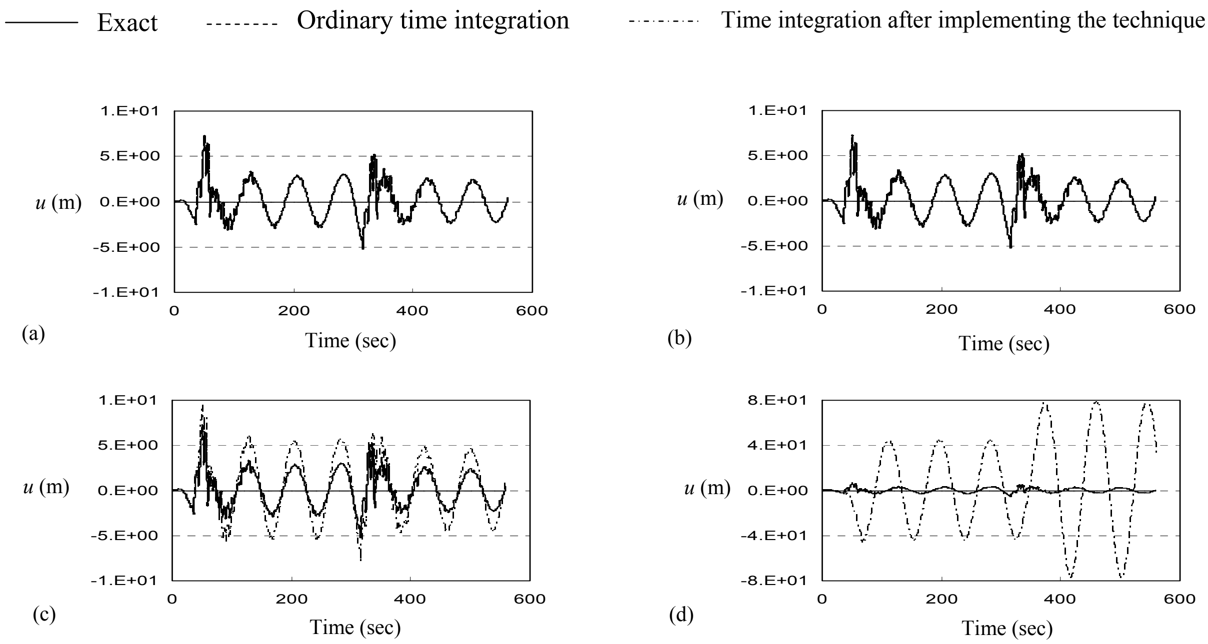


**Figure 5**: Displacement of point A in Figure 2(a) obtained from time integration after once repeating the excitation along the time axis: (a) $n = 2$, (b) $n = 4$, (c) $n = 10$, (d) $n = 30$

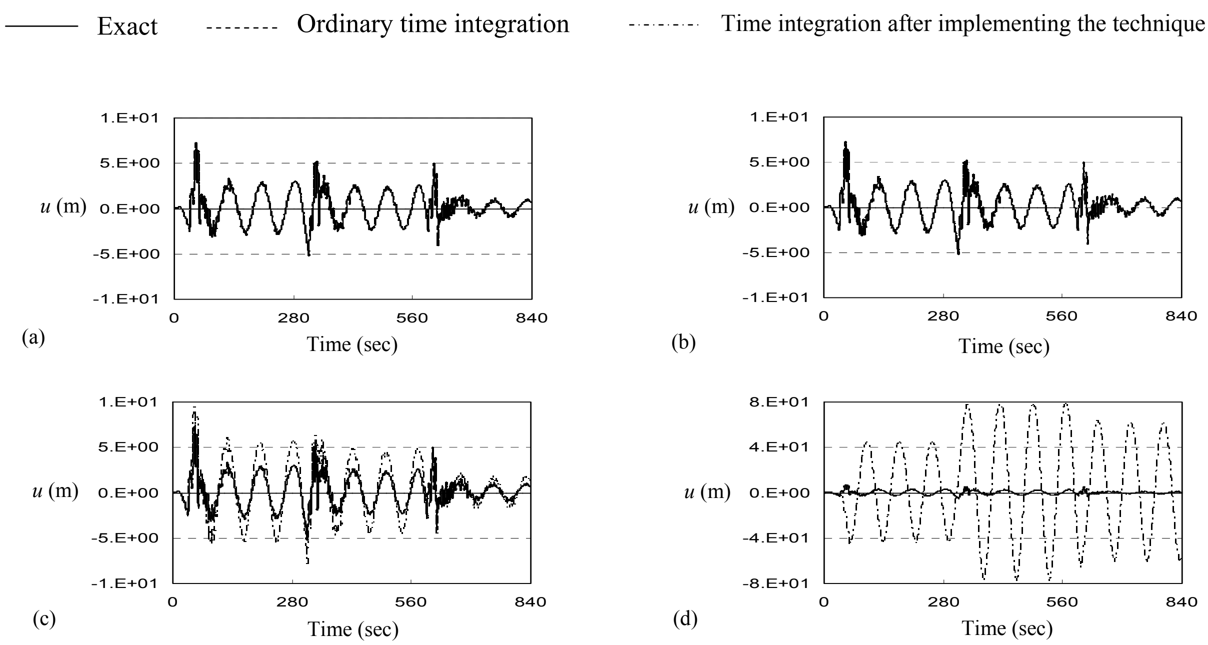


**Figure 6**: Displacement of point A in Figure 2(a) obtained from time integration after twice repeating the excitation along the time axis: (a) $n = 2$, (b) $n = 4$, (c) $n = 10$, (d) $n = 30$

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