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COMPUTATIONAL MODELING OF A MULTI-LAYERED PIEZO-COMPOSITE BEAM MADE UP OF MFC

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Abstract. The Variational Asymptotic Method (VAM) is used for modeling a coupled non-linear electromechanical problem finding applications in aircrafts and Micro Aerial Vehicle (MAV) development. VAM coupled with geometrically exact kinematics forms a powerful tool for analyzing a complex nonlinear phenomena as shown previously by many in the literature [3 - 7] for various challenging problems like modeling of an initially twisted helicopter rotor blades, matrix crack propagation in a composite, modeling of hyper elastic plates and various multi-physics problems. The problem consists of design and analysis of a piezocomposite laminate applied with electrical voltage(s) which can induce direct and planar distributed shear stresses and strains in the structure. The deformations are large and conventional beam theories are inappropriate for the analysis. The behavior of an elastic body is completely understood by its energy. This energy must be integrated over the cross-sectional area to obtain the 1-D behavior as is typical in a beam analysis. VAM can be used efficiently to approximate 3-D strain energy as closely as possible. To perform this simplification, VAM makes use of thickness to width, width to length, width multiplied by initial twist and strain as small parameters embedded in the problem definition and provides a way to approach the exact solution asymptotically. In this work, above mentioned electromechanical problem is modeled using VAM which breaks down the 3-D elasticity problem into two parts, namely a 2-D non-linear cross-sectional analysis and a 1-D non-linear analysis, along the reference curve. The recovery relations obtained as a by-product in the cross-sectional analysis earlier are used to obtain 3-D stresses, displacements and velocity contours. The piezo-composite laminate which is chosen for an initial phase of computational modeling is made up of commercially available Macro Fiber Composites (MFCs) stacked together in an arbitrary lay-up and applied with electrical voltages for actuation. The expressions of sectional forces and moments as obtained from cross-sectional analysis in closed-form show the electro-mechanical coupling and relative contribution of electric field in individual layers of the piezo-composite

laminate. The spatial and temporal constitutive law as obtained from the cross-sectional analysis are substituted into 1-D fully intrinsic, geometrically exact equilibrium equations of motion and 1-D intrinsic kinematical equations to solve for all 1-D generalized variables as function of time and an along the reference curve co-ordinate, x_1.

1 INTRODUCTION

After the invention of MFC by NASA in 1996 and commercialization by Smart Material Corp. in 2002, it has evolved a lot in terms of design, properties and effectiveness. MFC is popular for its great flexibility, surface conformability, excellent actuation properties (d33 = 460 pc/N) and are quoted as being capable of a maximum strain of 1800 $\mu\epsilon$. It can be used both as an actuator as well as a sensor. As an actuator it finds applications in numerous fields ranging from flapping wing or morphing wing MAVs, UAVs, aircraft control surface, vibration control of helicopter rotor blades or aircraft rudder, satellite booms and so on. And as a sensor it can be used for energy harvesting, structural health monitoring (SHM) etc. In the present study, MFC is being modeled only as an actuator. M8528-P1 and M8528-F1 are chosen as constituent lamina in the piezo-composite actuator.

Keeping in mind the actuation authority provided by MFCs, it is difficult to analyze such a complex coupled nonlinear behavior using conventional ad-hoc beam theories or other engineering beam theories which are based on some truncation schemes or small strain approximations. Hence VAM is used to model it in an efficient way. From the cross-sectional analysis, VAM gives closed-form solutions for simple geometries and hence certain parametric studies are carried out which make significant contribution in the design process. For the simplest case, stacked layers are made up of MFC in $[0^0/45^0/0^0]$ configuration, a linear 3-D electromechanical constitutive law is incorporated and the theoretical formulation is constructed considering the piezo-composite beam as a thin strip S-class beam.

Smart material Co. [1] provides following structural and piezoelectric properties of MFC

$$\begin{aligned} E_{11} &= 30.336GPa \quad E_{22} = 15.857GPa \quad G_{12} = 5.515GPa \quad \nu_{12} = 0.31 \quad \nu_{21} = 0.16 \\ & d_{15} = d_{24} = 400pCN^{-1} \quad d_{31} = -170pCN^{-1} \end{aligned}$$

The stiffness matrix, $[C]_{6\times6}$, for an orthotropic material in terms of the engineering constants, piezoelectric strain coefficient matrix, $[d]_{6\times3}$, and piezoelectric stress constants $e = [C] \cdot [d]$ can be obtained from the above data relevant to MFC.

2 THEORETICAL FORMULATION

The beam kinematics is first drawn based on Jaumann Biot Cauchy's strain tensor [2] to get the expressions for the 3-D strains in terms of the 1-D generalized strains and

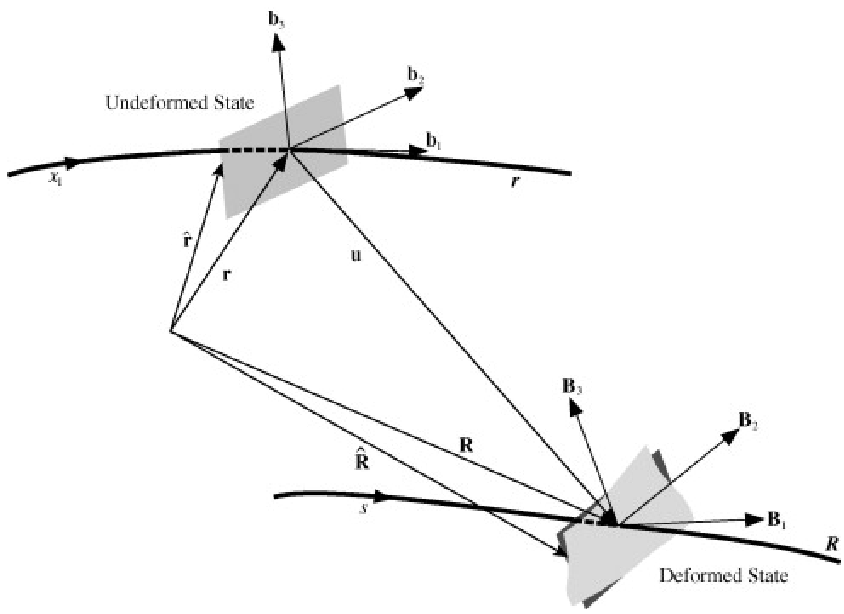

Figure 1: Schematic of beam deformation

warping measures. Next, the 2-D cross-sectional analysis is performed by minimising the electromechanical energy of the system to obtain recovery relations, warping expressions and 1-D constitutive law. The results of 2-D analysis are input to the geometrically-exact 1-D equilibrium equations, which are again obtained by an asymptotic approach as outlined in [2], and then solved following an appropriate discretization scheme in space and time for 1-D variables in terms of length coordinate.

2.1 2-D Analysis

Consider an initially twisted cantilever beam with an initial twist given by $k_1(x_1)$ in its undeformed state as shown in Fig 2.1, where x_1 is the running axial coordinate along the length of the beam. Let the wavelength of deformation be denoted by l. The width and thickness of the strip are denoted by b and h, respectively. As mentioned earlier also, the geometric small parameters are $\delta_h = h/b$, $\delta_b = b/l$ and $\delta_t = bk_1$. The Cartesian coordinate measures x_i are directed along the length, width and thickness of the strip for $i = 1, 2, 3$ respectively, parallel to corresponding unit vectors $\mathbf{b_i}$. Here and hereafter throughout the text, unless otherwise specified, Greek indices assume values 1 and 2, while Latin indices assume values 1, 2, and 3. Repeated indices are summed over their range unless indicated otherwise. Note that all $\mathbf{b_i}$ are functions of x_1. The domain of the strip is such that $0 \leq x_1 \leq L$, $-b/2 \leq x_2 \leq b/2$ and $-h/2 \leq x_3 \leq h/2$.

2.1.1 Undeformed Beam Geometry

In the first step, we choose the reference curve $\mathbf{r}(x_1)$ of the beam along its length which is a locus of uniquely identifiable cross-sectional material points, for e.g. a locus of centroids of the cross-section or a locus of the left or right corner points of the cross-section. In the present work, we have chosen our reference curve to be a locus of centroids of the cross-section. At each point along $\mathbf{r}$, define a frame $\mathbf{b}$ in which are fixed mutually orthogonal dextral set of unit vectors $\mathbf{b_i}$ such that $\mathbf{b_2}(x_1)$ and $\mathbf{b_3}(x_1)$ are tangent to the coordinate curves x_2 and x_3 respectively and $\mathbf{b_1}$ is tangent to $\mathbf{r}$. The position vector from a point fixed in an inertial reference frame $\mathbf{a}$ to a generic point on the middle surface of the strip is $\mathbf{r_0}(x_1, x_2) = x_1\mathbf{b_1} + x_2\mathbf{b_2}(x_1)$. The position vector of an arbitrary material point in the strip is then

$$\hat{\mathbf{r}}(x_1, x_2, x_3) = \mathbf{r_0} + x_3\mathbf{b_3}(x_1) = x_i\mathbf{b_i} \tag{1}$$

Assuming there is no initial curvature in the beam, i.e. $k_2 = k_3 = 0$, derivatives of $\mathbf{b_i}$ can be expressed as follows:

$$\mathbf{b_i}' = k_j\mathbf{b_j} \times \mathbf{b_i} \tag{2}$$

Next, co-variant and contra-variant basis vectors for the undeformed beam geometry are obtained which are required for defining the Deformation Gradient Tensor (DGT)

2.1.2 Deformed Beam Geometry

After the application of loads, intially planar reference cross-section undergoes a rigid body translation and rotation as well as a warping displacement and hence the undeformed geometry $\mathbf{r}$ assumes a different configuration $\mathbf{R}$ with an arc-length parameter along the new reference curve denoted by s. At each point along $\mathbf{R}$, let us introduce an orthogonal, dextral triad $\mathbf{B_i}$. Different ways of introducing $\mathbf{B_i}$ makes it possible to express the one dimensional constitutive law obtained after cross-sectional energy minimization in the form of a classical theory where there are no generalized one dimensional shear strains ($2\gamma_{12}$ and $2\gamma_{13}$) or Timoshenko theory which does incorporate the effect of such transverse shear strains. Note that irrespective of whatever representation we choose for $\mathbf{B_i}$ the overall analysis neither introduces any additional approximation nor results in the loss of any information because as we shall see in the subsequent development, warping takes care of the effect of transverse shear strains in case it doesn't explicitly appear in the form of generalized one dimensional shear strains. In the present work, we develop a theory of the classical form and assume $\mathbf{B_1}$ to be tangent to $\mathbf{R}$ hence transverse shear strains ($2\gamma_{12}$ and $2\gamma_{13}$) would appear as a part of the warping fields.

The rotation of deformed state w.r.t. the undeformed state from $\mathbf{b_i}$ to $\mathbf{B_i}$ is accomplished by pre-dot multiplication with an orthogonal tensor that we call the global rotation tensor $\underline{\mathbf{C}}^{\mathbf{Bb}}$:

$$\mathbf{B_i} = \underline{\mathbf{C}}^{\mathbf{Bb}}.\mathbf{b_i} = C_{ij}^{Bb}\mathbf{b_j} \tag{3}$$

The specialized one-dimensional generalised strains also referred to as the generalised strains of classical theory can be expressed as:

$$\bar{\epsilon} = \left\{ \begin{array}{c} \overline{\gamma}_{11} \\ \overline{\kappa}_1 \\ \overline{\kappa}_2 \\ \overline{\kappa}_3 \end{array} \right\} \tag{4}$$

Next, we define the position vector of a material point in the deformed beam as follows:

$$\hat{\mathbf{R}}(x_1, x_2, x_3) = x_1\mathbf{b_1} + u_i\mathbf{b_i} + x_2\mathbf{B_2}(x_1) + x_3\mathbf{B_3}(x_1) + \overline{w}_i(x_1, x_2, x_3)\mathbf{B_i}(x_1) \tag{5}$$

where, $u_i(x_1)$ are the translational displacements of the material points lying on the reference curve and $\overline{w}_i(x_1, x_2, x_3)$ represent two inplane and one out of plane warping components which can be further broken down as follows:

$$\overline{w}_\alpha(x_1, x_2, x_3) = w_\alpha(x_1, x_2) + x_3\phi_\alpha(x_1, x_2) + \Delta_\alpha(x_1, x_2, x_3) \tag{6}$$

$$\overline{w}_3(x_1, x_2, x_3) = w_3(x_1, x_2) + \Delta_3(x_1, x_2, x_3) \tag{7}$$

Here, we have introduced a total of 14 variables namely; u_i, w_i, Δ_i, ϕ_α and $\mathbf{B_i}$; while an accurate description of the deformed configuration requires only three variables. Hence we constrain above unknown variables in the form of 11 constraining equations. There can be many choices of such constraints but it is important that those constraints render the displacement field unique. Two out of 11 constraints are implicitly obtained through the formulation of VAM, since we have $\gamma_{12}^{'} \approx u_2^{'}$ and $\gamma_{13}^{'} \approx u_3^{'}$. Remaining constraints are listed below:

$$\int_{-h/2}^{h/2} \Delta_i(x_1, x_2, x_3)dx_3 = 0 \qquad \int_{-h/2}^{h/2} \Delta_{\alpha,3}(x_1, x_2, x_3)dx_3 = 0$$
$$< w_i >= 0 \qquad < w_{3,2} >=< \phi_2 > \tag{8}$$

where the notation $< \bullet >= \int_{-b/2}^{b/2}(\bullet)dx_2$ and $(\bullet)_{i,j} = \frac{d}{dx_j}(\bullet)_i$

2.1.3 Three Dimensional Strains

Next, the covariant basis vectors $\mathbf{G_i}$ for the deformed geometry are determined by differentiating $\hat{\mathbf{R}}(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$ w.r.t. x_i. Now we can evaluate deformation gradient tensor $\underline{\chi} = \mathbf{G_i}\mathbf{g^i}$ and arrange the components of $\underline{\chi}$ in mixed bases into a matrix χ.

Evaluating the deformation gradient tensor in this way, we can find out the three dimensional strain components Γ_{ij} corresponding to the Jaumann-Biot-Cauchy strain tensor resolved along purely $\mathbf{b_i}$ bases while using a moderate local rotation (ϕ) theory appropriate for strip kind of beams.

In the next step, a preliminary order of magnitude analysis is performed to arrive at leading order terms in the expressions of three dimensional strains so as to carry out zeroth order and higher order cross-sectional analysis through electromechanical energy minimization.

2.1.4 Energy Minimization using VAM

Consider the leading order terms in the expressions of three dimensional strains, we carry out energy minimization w.r.t. warping variables. For simplification, the three dimensional strains are broken into shell strains $\epsilon_{ij}$and curvatures ρ_{ij} and the electromechanical energy expression is written below:

$$U_{2D} = \frac{1}{2}\left\{\begin{array}{c}\epsilon_{11}\\ \epsilon_{22}\\ 2\epsilon_{12}\\ \rho_{11}\\ \rho_{22}\\ 2\rho_{12}\end{array}\right\}^T \left(\left[\begin{array}{cc}A & B\\ B & D\end{array}\right]\left\{\begin{array}{c}\epsilon_{11}\\ \epsilon_{22}\\ 2\epsilon_{12}\\ \rho_{11}\\ \rho_{22}\\ 2\rho_{12}\end{array}\right\} - \left[\begin{array}{c}N_a\\ M_a\end{array}\right]\right) \tag{9}$$

where the stiffness matrix $\left[\begin{array}{cc}A & B\\ B & D\end{array}\right]$ and actuation forces $\left[\begin{array}{c}N_a\\ M_a\end{array}\right]$, for zeroth order approximation are obtained through CLPT (classical laminated plate theory). We consider a laminate of 3 layers of MFC with fiber orientation $[0^0/45^0/0^0]$. The expressions of electric field applied to all the layers along the fiber direction is provided below:

$$\begin{aligned} E_0 &= A_0 sin(w_0 t + \phi_0) + 500 \quad V/m \\ E_1 &= A_1 sin(w_1 t + \phi_1) + 500 \quad V/m \\ E_2 &= A_2 sin(w_2 t + \phi_2) + 500 \quad V/m \end{aligned} \tag{10}$$

2.2 1-D Analysis & Sample Results

After performing cross-sectional analysis, 1-D constitutive law is obtained. Here, an expressions of axial sectional force F_1 and sectional twisting moment M_3 as obtained from the energy minimization are being provided for the case when $\delta_b = 0.339$, $\delta_h = 0.0257$ and $\delta_t = 0.177$, typical for M8528-P1 MFC laminate:

$$\begin{aligned} F_1 &= 8.05\times 10^5\gamma_{11} + 415.7\kappa_1 - 6.56\times 10^{-5}(E_0+E_2) - 1.37\times 10^{-5}E_1 \\ M_3 &= 82.22(\kappa_3 + 2\gamma_{12}^{'}) - 415.79(2\gamma_{13}) \end{aligned} \tag{11}$$

Next, the above static constitutive law and the generalized momentum-velocity relations as described by Hodges [2] are substituted into 1-D fully intrinsic equilibrium equations of motion and 1-D intrinsic kinematical equations as written below:

$$\begin{aligned} F^{'} + \widetilde{K}F + f &= \dot{P} + \widetilde{\Omega}P \\ M^{'} + \widetilde{K}M + (\widetilde{e_1} + \widetilde{\gamma})F + m &= \dot{H} + \widetilde{\Omega}H + \widetilde{V}P \\ \Omega^{'} + \widetilde{K}\Omega &= \dot{\kappa} \\ V^{'} + \widetilde{K}V + (\widetilde{e_1} + \widetilde{\gamma})\Omega &= \dot{\gamma} \end{aligned} \tag{12}$$

These are then solved in space and time using a piece-wise constant shape functions for all unknowns on the interior and discrete values of the unknowns at the ends. This time-marching algorithm is second order accurate along the beam and it satisfies the energy and momentum conservation laws.

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