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PLATE BENDING ANALYSIS BY A MULTI-SCALE MODEL COUPLING BEM AND FEM, CONSIDERING DIFFERENT BOUNDARY CONDITIONS FOR THE RVE

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Abstract. A multi-scale modelling for analysing the bending problem of plates composed by heterogeneous materials is presented. The macro-continuum is modelled by the non-linear formulation developed in [1] of the boundary element method (BEM) taking into account the consistent tangent operator (CTO) and based on Kirchhoff's theory. The micro-scale is represented by the RVE (representative volume element) being its equilibrium problem solved by the finite element formulation presented in [2, 3] that takes into account the Hill-Mandel Principle of Macro-Homogeneity while the volume averaging hypothesis of the strain and stress tensors is used to make the micro-to-macro transition. The microscopic equilibrium problem consists of, given the history of the macroscopic strain tensor, finding the field of displacement fluctuation such that, for each instant t, the RVE equilibrium equation is satisfied. In the numerical example a narrow plate subjected to simple bending is analysed where is adopted a RVE with a void defined in its central and different boundary conditions are imposed to the RVE.

1 INTRODUCTION

The pre-existence of initial defects in the material's micro-scale as microcracks and microvoids plays an important role in the stiffness of the structure or component. Moreover, in general, the materials, even the metallic, are heterogeneous at the micro and grain scale. The concrete, as example, has a very complex microstructure, since it is composed by different phases (or materials) that have different Young's moduli and present different non-linear behaviour. Besides, often the material microstructure is appropriately manipulated by adding certain constituents to a matrix phase, in order to change the material properties to attend specific applications. As any heterogeneity of the material as well as the microcracking initiation and propagation in the micro-scale affect directly the macro-continuum response, modelling heterogeneous material in different scales is very important to better represent the behaviour of such complex materials [2-5]. In many situations the traditional phenomenological approach for constitutive description does not provide a sufficiently general predictive modelling capability.

At micro-scale, the material behaviour is monitored individually to each RVE (representative volume element) that represents the microstructure, at grain level, of the macro-continuum at the infinitesimal material neighbourhood of a point (see [2, 3]). The strain related to the macro-continuum point is imposed to the cell (RVE) defined at micro-scale and the micro-to-macro transition is made by applying a homogenization process, after solving the equilibrium problem at micro-scale.

The boundary element method (BEM) has already proved to be a suitable numerical tool to deal with plate bending problems (see [1, 6]). The method is particularly recommended to evaluate internal force concentrations due to loads distributed over small regions that very often appear in practical problems. Moreover, the same order of errors is expected when computing deflections, slopes, moments and shear forces. Therefore, the use of BEM is very adequate to deal with dissipative phenomena in heterogeneous materials, such as strain localization, fracture process and plastic deformation on macro-scale level.

In the present work, the non-linear BEM formulation for plate bending presented in [1] is used to model the macro-continuum, while to solve the equilibrium problem at micro-scale a FEM formulation (see [2, 3]) has been adopted. The microscopic equilibrium problem consists of, given the history of the macroscopic strain tensor, finding the field of displacement fluctuation such that, for each instant t, the RVE equilibrium equation is satisfied. Depending on the boundary conditions adopted for this displacement fluctuation field in the RVE, different multi-scale models can be obtained, leading to different numerical responses. In the present work the following boundary conditions will be imposed to the RVE: (i) linear displacements, (ii) periodic displacement fluctuations and (iii) uniform boundary tractions. In general, the proposed modelling is an alternative tool to simulate the mechanical behaviour of the heterogeneous materials, like ductile and porous ductile materials. Besides, with the adoption of properly cohesive fracture model and plastic criterion, the proposed modelling will be able to deal with brittle materials, like concrete, in future works.

2 THE NON-LINEAR PLATE PROBLEM

The non-linear plate bending analysis, that represents the macro-continuum problem in the present work, is modelled by a BEM non-linear formulation discussed in details in [1] and based on Kirchhoff's hypothesis. To define the plate bending problem, let us consider a flat plate of thickness t, external boundary Γ and domain Ω referred to a Cartesian system of co-ordinates with x_1 and x_2 axes laying on its middle surface and x_3 being the axis perpendicular to that plane. It is assumed that the plate supports only distributed load *g* acting on the plate middle plane, in the x_3 direction. The variables related to the plate bending problem are the following ones: $\dot{V}_n$ (effective shear force rate); $\dot{M}_n$ (bending moment rate); $\dot{w}$ (deflection rate); $\dot{w}_{,n}$ (rotation rate), being (n, s) the local co-ordinate system, with *n* and *s* referring to the boundary normal and tangential directions, respectively. As the present work deals with non-linear analysis, all variables are expressed in rates, i.e., $(\dot{x}) = dx/dt$, their time derivatives. The basic equilibrium equations for the plate problem will be omitted here, but they can be found in several works ([31]-[35])

The bending and twisting moment rates $\dot{m}_{ij}$ in the plate are obtained by integrating the Cauchy stresses $\dot{\sigma}_{ij}$ across the plate thickness *t*, as follows:

$$\dot{m}_{ij} = \int_{-t/2}^{t/2} z\dot{\sigma}_{ij} dz \tag{1}$$

Note that in a multi-scale analysis $\dot{\sigma}_{ij}$ is obtained after solving the RVE equilibrium problem. As this work only deals with small strain problems, the total strain will be split into its elastic and inelastic parts, $\dot{\varepsilon}_{ij}^{e}$ and $\dot{\varepsilon}_{ij}^{p}$ respectively, as follows:

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{e} + \dot{\varepsilon}_{ij}^{p} \tag{2}$$

By assuming the Kirchhoff´s hypothesis, the total strain component for the bending problem is given by: $\dot{\varepsilon}_{ij} = -x_3 \dot{w},_{ij}$, being $\dot{w},_{ij}$ the plate surface curvature. The moment rate predictor $\dot{m}_{ij}^{e}$, often defined as elastic trial used in non-linear algorithms, can be written in terms of the total curvatures $\dot{w},_{ij}$ as follows:

$$\dot{m}_{ij}^{e} = -D\left[\nu \dot{w},_{kk} \delta_{ij} + (1-\nu)\dot{w},_{ij}\right] \tag{3}$$

where δ_{ij} is the Kronecker delta, $D = Et^3 / 12\left(1-\nu^2\right)$ is the plate flexural rigidity and ν is the Poisson's ratio.

Thus, the inelastic moment rate $\dot{m}_{ij}^{p}$ is defined as:

$$\dot{m}_{ij}^{p} = \dot{m}_{ij}^{e} - \dot{m}_{ij} \tag{4}$$

3 BEM ALGEBRAIC EQUATIONS

Let us consider $\Delta t = t_{n+1} - t_n$, a typical time step in the non-linear solution. The finite step boundary value problem consists of searching the solution at the time step end t_{n+1} when this solution is known at the time step beginning t_n. From Betti's reciprocal theorem (see more details in [36]) we can obtain the following representation of deflections written for internal and boundary collocation points which is an exact integral representation:

$$K(q)\Delta w(q) = -\int_{\Gamma_m}\left(V_n^{*}\dot{w} - M_n^{*}\frac{\partial \dot{w}}{\partial n}\right)d\Gamma - \sum_{j=1}^{N_o} R_{cj}^{*} w_{cj} + \sum_{j=1}^{N_c} \dot{R}_{cj} w_{cj}^{*} + \int_{\Gamma}\left(\dot{V}_n w^{*} - \dot{M}_n \frac{\partial w^{*}}{\partial n}\right)d\Gamma$$
$$+ \int_{\Omega_g} \Delta g w^{*} d\Omega - \int_{\Omega} w,_{jk}^{*} \Delta m_{jk}^{p} d\Omega \tag{5}$$

where *w**, *V** and *M** are plate bending fundamental values of deflections, effective shear forces and boundary moments, N_C are numbers of corners and Ω_g is the plate loaded area; for the free term K(q) values see [1].

The integral representation for the curvature increment $\Delta w,_{\ell i}$ can be obtained by differentiating equation (5) twice at an internal collocation point see more details in [1].

To obtain the algebraic equations linear elements with quadratic shape functions are adopted to approximate the four values defined along the plate boundary: Δw, $\Delta w,_n$, ΔM_n and ΔV_n. As two of these values are prescribed only two equations are required per node. One deflection equation is written for a collocation point defined along the boundary and another one is written for an external collocation very near the boundary. Besides, at boundary corners we may write extra equations as the reactions are preserved as unknowns.

The inelastic moment increments (Δm_{jk}^{p}) will be approximated over the domain by using triangular cells where continuous or discontinuous linear shape functions are considered. For the cells sides defined on the external boundary, the nodes and collocations are placed inside the cells because of the discontinuity. To perform the non-linear analysis of plate bending, three equations of elastic trial moment increments ΔM_{jk}^{e} at each node are required to evaluate the stress field over domain. These equations are obtained from the curvatures by applying the Hooke's law (equation (3)). After selecting conveniently the collocation points and performing the relevant integrals over boundary elements and over cells, one obtains a set of algebraic equations given in terms of boundary values and inelastic moment increments, which after applying the boundary conditions can be written as (see more details in [1]):

$$\Delta X = \Delta L + R_M \Delta m^P \tag{6}$$

where the vector ΔX contains the plate bending unknowns on the boundary and corners, ΔL represents the elastic parts of these unknowns, R_M expresses the corrections due to the inelastic moment increment.

We can also derive the following BEM algebraic equation for the actual moment increment ΔM (for more details see [1]):

$$\Delta M = C_M \Delta\chi - \Delta K_M - S_M\left(\Delta m^p\right) + \Delta m \tag{7}$$

where $\Delta\chi$ is the curvature increment in the plate, C_M a matrix that contains the elastic constant matrices (obtained from equation 3) of all nodes defined in the plate; the inelastic moment increment vector Δm^P is given by: $\Delta m^p = C_M \Delta\chi - \Delta m$; ΔK_M is the elastic solution given in terms of moment increments, S_M gives the bending moment effects due to the nodal inelastic increments, Δm^P.

Observe that $\Delta m^P, \Delta m^e$ and Δm defined, respectively in equations (4), (3) and (1) are computed locally for a particular point, i.e., they are obtained taking into account only the actual curvature increment $\Delta w,_{ij}$ and the actual stress tensor increment $\Delta\sigma_{ij}$ related to that point. To obtain the non-linear solution for an increment n, the following equilibrium equation has to be satisfied:

$$\Delta K_{M_n} - \Delta M_n = 0 \tag{8}$$

Replacing equation (7) into (8), one obtains the final algebraic relation to impose the equilibrium conditions over the time increment Δt_n:

$$R_M(\Delta\chi_n) = 2\Delta K_{M_n} - C_M\Delta\chi_n + S_M(C_M\Delta\chi_n - \Delta m_n) - \Delta m_n = 0 \tag{9}$$

After applying to the plate a curvature increment $\Delta\chi_n$ and obtaining the stress ($\Delta\sigma_{ij}$) distribution over the plate thickness for all nodes adopted in the domain discretization, if equation (9) is not satisfied, a residual moment R_M non null will be computed, i.e., the increment is not elastic. Then, equation (9) will be solved by applying Newton-Raphson's scheme, for which an iterative process may be required to achieve the macro-continuum equilibrium. Let us consider the iteration *i* where the curvature increment is known. The next trial increment, at (*i+1*), is obtained by finding the additive corrections $\delta\Delta\chi_n^{i+1}$:

$$\Delta\chi_n^{i+1} = \Delta\chi_n^i + \delta\Delta\chi_n^{i+1} \tag{10}$$

The corrections $\delta\Delta\chi_n^{i+1}$ are computed after the linearization of equation (9) to give:

$$\delta\Delta\chi_n^{i+1} = \left[-\frac{\partial R_M(\Delta\chi_n^i)}{\partial\Delta\chi_n^i}\right]^{-1} R_M(\Delta\chi_n^i) \tag{11}$$

where $-\dfrac{\partial R_M(\Delta\chi_n^i)}{\partial\{\Delta\chi_n^i\}} = S_M\left(C_{M_n}^{ep(i)} - C_M\right) + C_M + C_{M_n}^{ep(i)}$ is the Consistent Tangent Operator (CTO) obtained by differentiating equation (9); $C_{M_n}^{ep(i)}$ is a matrix that contains inelastic tangent modulus $\left[C_m^{ep}\right]_k^i$ (relating moments and curvatures) of all cell nodes.

Observe that the actual values of the internal forces increment Δm_{ij} defined in equation (1) as well as the tensor $\left[C_m^{ep}\right]_k^i$ are computed numerically adopting a Gauss scheme, for which a number of Gauss points has to be defined along the plate thickness.

4 EQUILIBRIUM PROBLEM DEFINED AT MICRO-SCALE

Let us initially consider the macro-continuum depicted in Figure (1), which in this work is represented by the plate, of characteristic length ℓ,being *x* an arbitrary material point of this continuum and y an arbitrary point of the microscopic cell, called RVE (Representative Volume Element) (see [2, 3]). For the RVE, the volume is denoted by V_μ, the domain by Ω_μ, the boundary by $\partial\Omega_\mu$ and the characteristic length by ℓ_μ. In order to perform the multi-scale analysis, one RVE must be associated with each point x of the macro-continuum where the stress vector computation is required. As in the present work, for a particular cell node of the macro-continuum, the stress σ has to be computed for a number of Gauss points defined over the plate cross-section one RVE has to be associated with each Gauss point defined over the plate thickness.

It is assumed that the strain tensor $\varepsilon(x,t)$ as well as the stress tensor $\sigma(x,t)$ at a point *x* of the macro-continuum is the volume average of their respective microscopic field ($\varepsilon_\mu=\varepsilon_\mu(y,t)$ or $\sigma_\mu=\sigma_\mu(y,t)$) over the RVE associated with *x*. That is, at an arbitrary instant t:

$$\varepsilon(x,t)=\frac{1}{V_\mu}\int_{\Omega_\mu}\varepsilon_\mu(y,t)dV \tag{12}$$

$$\sigma(x,t)=\frac{1}{V_\mu}\int_{\Omega_\mu}\sigma_\mu(y,t)dV \tag{13}$$

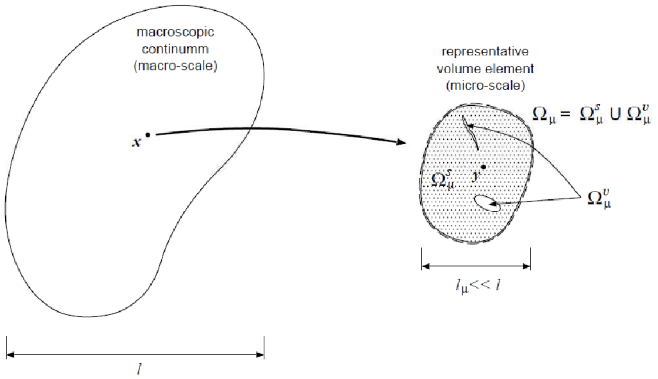

Figure 1 – Macro-continuum with a locally attached micro-structure

Note that in equations (12) and (13), a microscopic quantity (ε_μ or σ_μ) over the RVE is mapped into a macroscopic quantity (ε or σ) by means of a homogenization technique using the volume average, where the tensors ε and σ are referred to as the macroscopic or homogenised strain and stress, respectively. Besides, the microscopic stress can be written in terms of the microscopic strain as follows: $\sigma_\mu(y,t))= f_y(\varepsilon_\mu(y,t))$, being f_y the constitutive functional, which in this work can be defined by the Von Mises elasto-plastic criterion or by Hooke's law if an elastic behaviour is adopted. Moreover, the microscopic strain ε_μ can be written in terms of the microscopic displacement field u_μ of the RVE as follows: $\varepsilon_\mu(y,t)=\nabla^S u_\mu(y,t)$, where ∇^s denotes the symmetric gradient.

By the homogenization process we can also obtain the homogenised constitutive tangent modulus C^{ep}, as follows

$$C^{ep}(x,t)=\frac{\partial\sigma(x,t)}{\partial\varepsilon(x,t)}=\frac{\frac{1}{V_\mu}\int_{\Omega_{\mu S}}\partial\sigma_\mu(y,t)dV}{\partial\varepsilon(x,t)}=\frac{\frac{1}{V_\mu}\int_{\Omega_{\mu S}}\partial f_y(\varepsilon_\mu(y,t))dV}{\partial\varepsilon(x,t)} \tag{14}$$

Observe that after solving the RVE equilibrium problem, the microscopic fields of strain ε_μ and stress σ_μ are known and then the micro-to-macro transition can be made by using equations (13) and (14). Besides, any microscopic displacement field u_μ, may be split into the following sum:

$$u_\mu(y,t)=\varepsilon(x,t)y+\tilde{u}_\mu(y,t) \tag{15}$$

In equation (15) the portion εy varies linearly in y, and it is obtained by multiplying the macroscopic strain ε imposed to the RVE, which is constant, by the coordinates of point y. The portion $\tilde{u}_\mu$ is denoted displacement fluctuation and represents the strain variation in the RVE, i.e., in the case of having uniform microscopic strain ε_μ, the displacement fluctuation $\tilde{u}_\mu$ is null. Accordingly, the microscopic strain field is given by:

$$\varepsilon_\mu(y,t) = \varepsilon(x,t) + \tilde{\varepsilon}_\mu(y,t) \tag{16}$$

where ε is constant and represents the homogeneous strain imposed to the RVE by the macro-continuum and $\tilde{\varepsilon}_\mu(y,t) = \nabla^S \tilde{u}_\mu$ is the strains fluctuation field.

The necessary condition for a displacement fluctuation field $\tilde{u}_\mu$ to be kinematically admissible is that $\tilde{u}_\mu \in \tilde{K}^*_\mu$, being $\tilde{K}^*_\mu$ the minimally constrained vector space of kinematically admissible displacement fluctuation of the RVE defined as:

$$\tilde{K}^*_\mu \equiv \left\{ v,\ \ sufficiently\ \ regular\, /\ \int_{\partial\Omega_\mu} v \otimes_S n dA = 0 \right\} \tag{17}$$

where *n* denotes the outward unit normal field on $\partial\Omega_\mu$.

From the Principle of Virtual Work and considering the Hill-Mandel Principle of Macro-Homogeneity (see [2, 3]), which establishes that the macroscopic stress power must equal the volume average of the microscopic stress power over the RVE, we can obtain the following form of the equilibrium equation written in terms of displacement fluctuations:

$$\int_{\Omega^S_\mu} f_y\left(\varepsilon(x,t) + \nabla^S \tilde{u}_\mu(y,t)\right) : \nabla^S \eta dV = 0 \qquad \forall \eta \in \mathcal{V}_\mu \tag{18}$$

where $\mathcal{V}_\mu$ is a space of virtual displacements that satisfies $\mathcal{V}_\mu = \tilde{K}^*_\mu$.

Finally, the formulation is completed with the choice of an appropriate space $\mathcal{V}_\mu$, i.e., with the choice of kinematical constraints to be imposed on the RVE. Therefore, the microscopic equilibrium problem consists of, given the history of the macroscopic strain tensor ε, finding the field $\tilde{u}_\mu \in \mathcal{V}_\mu$ such that, for each instant t, the equilibrium equation (18) is satisfied. In view of the arbitrariness of *η*, after discretising the RVE domain into elements, the following incremental microscopic equilibrium equation must hold for a load increment of time $\Delta t_n = t_{n+1} - t_n$ and discretisation *h*, that allows one to find the fluctuation displacement $\tilde{u}_{\mu(n+1)} = \tilde{u}_{\mu(n)} + \Delta\tilde{u}_{\mu(n)}$:

$$G_h^{n+1} = \int_{\Omega^h_\mu} B^T f_y\left(\varepsilon_{n+1} + B\tilde{u}_{\mu(n+1)}\right) dV = 0 \tag{19}$$

where *B* is the global strain-displacement matrix, Ω^h_μ denotes the discretised RVE domain.

If the load increment n is non-linear, equation (19) is solved by applying the Newton-Raphson Method which consists of finding the fluctuations corrections $\delta\tilde{u}_{\mu}^{i+1}$ for iteration $i+1$, such that:

$$F^{i} + K^{i}\delta\tilde{u}_{\mu}^{i+1} = 0 \tag{20}$$

where F is the tractions vector and K the tangent stiffness matrix of the RVE; denoting B_e the element strain-displacement matrix, N_e the number of finite elements and D_{μ}^{e} the constitutive tangent modulus of the element e, F and K are defined as:

$$F^{i} = \int_{\Omega_{\mu}^{h}} B^{T} f_{y}\left(\varepsilon_{n+1} + B\tilde{u}_{\mu}^{i}\right) dV = \sum_{e=1}^{N_e} B_{e}^{T}\sigma_{\mu}^{e(i)} V_{e} \tag{21}$$

$$K^{i} = \left[\int_{\Omega_{\mu}^{h}} B^{T} D_{\mu}^{i} B dV\right] = \sum_{e=1}^{N_e} B_{e}^{T} D_{\mu}^{e(i)} B_{e} V_{e}\ , \tag{22}$$

being $D_{\mu}^{i} = \left(\left.\dfrac{df_{y}}{d\varepsilon_{\mu}}\right|_{\varepsilon_{\mu}=\varepsilon^{n+1}+B\tilde{u}_{\mu}^{i+1}} \right)$ the consistent constitutive tangent matrix field over the RVE domain.

After computing the corrections $\delta\tilde{u}_{\mu}^{i+1}$ from equation (20), the next trial displacement fluctuation field to be considered in iteration $i+1$ related to the micro-continuum is given by: $\tilde{u}_{\mu}^{i+1} = \tilde{u}_{\mu}^{i} + \delta\tilde{u}_{\mu}^{i+1}$.

The homogenised stress is computed by equation (13) which after discretizing the RVE domain into elements, for a given increment of time $\Delta t_{n} = t_{n+1} - t_{n}$ used to model the loading incremental process, it can be written in the following incremental form:

$$\sigma_{n+1} = \frac{1}{V_{\mu}}\left[\int_{\partial\Omega_{\mu}^{h}} t_{n+1}^{e} \otimes_{s} y dA - \int_{\Omega_{\mu}^{S(h)}} b_{n+1} \otimes_{s} y dV\right] \tag{23}$$

where denoting u and v as arbitrary vectors, the following definition $u \otimes_{s} v = \frac{1}{2}\left(u \otimes v + v \otimes u\right)$ is used to compute the integrals defined in (23); t_{n+1}^{e} is the tractions across the RVE external boundary (equation 21) and y represents the coordinates vector of a point in the RVE.

The RVE formulation described so far is completed with the choice of an appropriate space $\mathcal{V}_{\mu}$, i.e., with the choice of kinematical constraints to be imposed on the RVE what leads to the following three different classes of multi-scale models:

(i) Linear boundary displacements

In this model the displacement fluctuations $\tilde{u}_\mu$ are assumed null along the external boundary $\partial\Omega_\mu$, i.e., the displacements are linear on the boundary $\partial\Omega_\mu$:

$$u_\mu(y,t) = \varepsilon(x,t)y \qquad \forall y \in \partial\Omega_\mu \tag{24}$$

and the space $\mathcal{V}_\mu$ is chosen as $\mathcal{V}_\mu = \left\{ \tilde{u}_\mu \in \tilde{K}^*_\mu \,/\, \tilde{u}_\mu(y,t) = 0 \quad \forall y \in \partial\Omega_\mu \right\}$.

(ii) Periodic boundary fluctuations

This model is usually adopted to represent a media with periodic microstructure. In fact, it can be shown that any material behaviour presents a periodic response if fine meshes are considered. In order to define its fluctuation displacements field, consider a square or a hexagonal RVE. For each side Γ_i^+ , whose normal direction is n_i^+, corresponds to a equally sized Γ_i^- with normal direction n_i^-, being $n_i^+ = -n_i^-$. Similarly, for each point y^+ defined over Γ_i^+ there is a point y^- over the side Γ_i^-. In this case, it is assumed that the displacement fluctuation must be periodic on the boundary of the RVE, i.e., for each pair { y^+, y^- } of points we must have:

$$\tilde{u}_\mu(y^+,t) = \tilde{u}_\mu(y^-,t) \qquad \forall \ \{y^+,y^-\} \in \partial\Omega_\mu \tag{25}$$

Besides, in the corners the displacement fluctuations are assumed nulls. Therefore, in this model the space $\mathcal{V}_\mu$ is adopted as: $\mathcal{V}_\mu = \left\{ \tilde{u}_\mu \in \tilde{K}^*_\mu \,/\, \tilde{u}_\mu(y^+,t) = \tilde{u}_\mu(y^-,t) \quad \forall \ \{y^+,y^-\} \in \partial\Omega_\mu \right\}$.

(iii) Uniform boundary traction

This model is also denoted as the minimally constrained model, as the space $\mathcal{V}_\mu$ is adopted coincident to the space $\tilde{K}^*_\mu$ (the minimally constrained vector space of kinematically admissible displacement fluctuation of the RVE) defined in equation (17), i.e.: $\mathcal{V}_\mu = \tilde{K}^*_\mu$. Considering that the tractions t^e over $\partial\Omega_\mu$ can be written as $t^e = \sigma_\mu(y,t)n$, being *n* the normal direction to the boundary, in order to satisfy the Hill-Mandel Principle as well as equation (17) where the space $\tilde{K}^*_\mu$ is defined, we conclude that the distribution of stress on the RVE boundary must be constant, i.e. $\sigma_\mu(y,t) = \sigma_\mu(t)$, $\forall y \in \partial\Omega_\mu$.

Let us now to define the macroscopic or homogenised constitutive tangent modulus C^{ep} which is computed by equation (14). After the RVE discretisation into elements, for an iteration *i* of the increment *n* of time used to model the loading incremental process (being $\Delta t_n = t_{n+1} - t_n$), it can be written as

$$C_i^{ep} = C_i^{ep(Taylor)} + \tilde{C}_i^{ep} \tag{26}$$

where $C_i^{ep(Taylor)}$ is denoted the Taylor model tangent operator (obtained by assuming $\nabla^S \tilde{u}_{\mu(n+1)} = 0$) and is given by the volume average of the microscopic constitutive tangent:

$$C_i^{ep(Taylor)} = \frac{\frac{1}{V_\mu}\int_{\Omega_\mu^h} \partial f_y(\varepsilon_{n+1})dV}{\partial \varepsilon^i} = \frac{1}{V_\mu}\int_{\Omega_\mu^h} \left.\frac{\partial f_y(\varepsilon_{\mu\,n+1})}{\partial \varepsilon_\mu}\right|_i dV = \frac{1}{V_\mu}\int_{\Omega_\mu^h} D_\mu^i dV = \sum_{p=1}^{N_p} \frac{V_p}{V_\mu} D_\mu^{p(i)} \quad (27)$$

where D_μ is the microscopic constitutive tangent matrix and N_p the number of phases defined in the RVE.

The other part ($\tilde{C}_i^{ep}$) of equation (26) represents the influence of the displacement fluctuation into the homogenised tangent modulus (see more details in [2, 3]) and can be written in the following algebraic form:

$$\tilde{C}_i^{ep} = -\frac{1}{V_\mu} G_R^i K_R^{i\,-1} G_R^{i\,T} \quad (28)$$

where G_R and K_R are defined according to the multi-scale model and G is defined as $G = \sum_{e=1}^{Ne} D_\mu^e B_e V_e$.

6 NUMERICAL EXAMPLE

In the numerical example a narrow plate, see figure (2) subjected to simple bending is analysed where is adopted a RVE with a void defined in its central (see figure (3a)). The two small sides of the plate are adopted simply supported and the two others, in the span direction, assumed to be free. Its geometry is given by: thickness t=10.0cm, width b=50.0cm, length ℓ =200.0cm and an uniform distributed load g=1kN/cm^2 is applied over all plate domain. The plate boundary was discretized into 16 quadratic elements, while the domain moments are approached over 24 cells, for which the definition of 5 internal points is required as shown in Figure (2b). For the RVE, which is adopted square which length side equal to 1cm, 220 elements and 126 nodes have been used to discretize its domain (see Figure 3a). The material properties over the RVE are: Von Mises elasto-plastic criterion with isotropic hardening k=2000 kN/cm^2, Poisson's ratio ν=0.3; Young's modulus E=20000 kN/cm^2, yield stress σ_y=40.0kN/cm^2.

It is evaluated how the numerical response changes according to the adopted RVE boundary conditions which will be adopted as: (i) linear displacements, (ii) periodic displacement fluctuations and (iii) uniform boundary tractions. The results are shown in Figure (3b), where can be observed that the limit load obtained considering uniform boundary tractions is significantly smaller. The displacements related to the three different boundary conditions are very similar for $\beta \le 0.65$, when the dissipative processes in the microstructure due to presence of the voids and ductile behaviour are not so strong. After that the analysis considering uniform tractions along the RVE boundary did not converged, while for the other two boundary conditions the analysis continued further. For periodic fluctuations and linear displacements the limit load has been achieved, respectively for $\beta = 0.7$ and $\beta = 0.75$ .Thus the results confirm what had already been verified in other works (see [2, 3]): the linear boundary displacement gives the stiffest (most cinematically constrained) solution while the uniform boundary traction model produces the most compliant (least kinematically constrained).

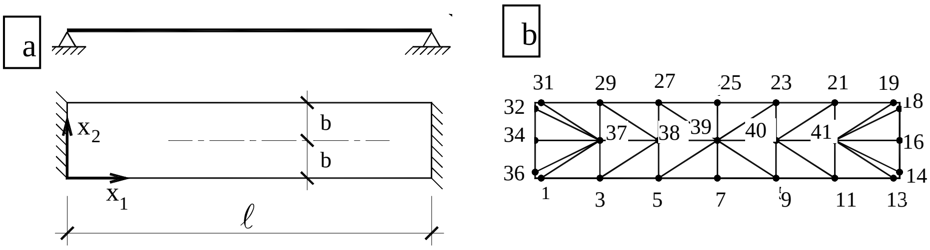

Figure 2 – Simply supported beam – geometry and discretization

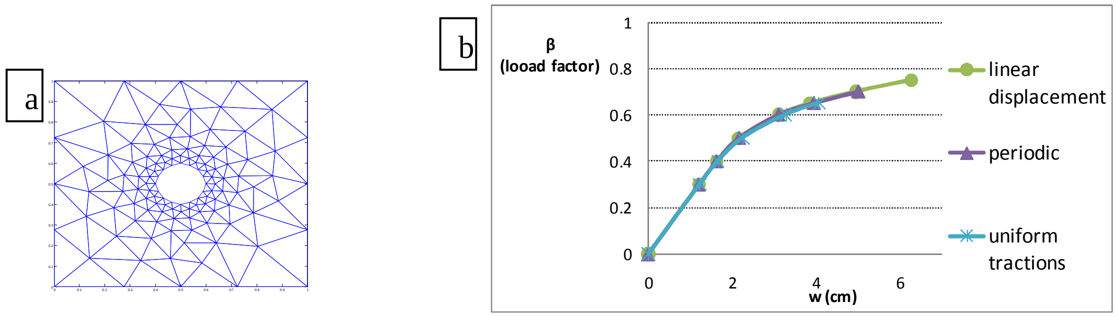

Figure 3: a)Discretization of a RVE with a void defined in the centre b)Deflection at the central point of the beam, using different boundary conditions in the RVE

7 CONCLUSIONS

A multi-scale modeling for plate bending analysis by coupling BEM and FEM has been presented. The macro-continuum has been modelled by a BEM non-linear formulation taking into account the consistent tangent operator while a FEM formulation is considered to solve the equilibrium problem defined at micro-scale in terms of displacement fluctuation. The Hill-Mandel Principle of Macro-Homogeneity as well as the volume averaging hypothesis of the strain and stress tensors has been used to make the micro-to-macro transition. In the numerical example three different boundary conditions for displacement fluctuation have been imposed over the RVE: (i) linear displacements (most cinematically constrained), (ii) periodic displacement fluctuation and (iii) uniform boundary tractions (least cinematically constrained).

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