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GEOMETRIC ADAPTIVE FUNCTIONALS FOR STRUCTURED GRID GENERATION

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Abstract. In this work we present a method for structured adaptive grid generation on geometric objects over plane irregular regions. We show that we can accumulate the cells around an object or the boundary. We present some results using a reference meshes to define a control function in general cases. In order to hand, we present the new version of UNAMALLA system that uses the recently developed discrete functionals.

1 Introducción

The adaptive generation of structured grids by adjustment or movement of nodes can be used to increase the accuracy of the numerical solution of partial differential equations and reduce the computational cost in the solution. It's very attractive that given a grid can be adapted without increasing its size or connectivity as it does not increase the computational cost required for use.

Numerical grid generation is a process which demands an important amount of computational resources in numerical modelling. Recent methods and theory show how to generate an ϵ-convex structured grid on very irregular domains in an efficient way [1]. Currently, we are interested on the extension of these methods for generating adaptive grids with additional properties such as smoothness and orthogonality. The key for achieving this goal is the use of geometrical functional that generate and preserve convexity [8, 1]. We are interested in extend these methods to generate adaptive grids with additional properties such as smoothness and orthogonality.

In some simulation problems we wish the grid nodes fitted around an internal object or in some particular places previously established. First, it is important to present the framework for the variational grid generation of structured grids on irregular planar domains. Second, we discuss the basic properties of the required functionals, with specific emphasis on the functionals that preserve convexity.

In this work we present some variational methods for adaptive geometric grid generation that we will present in the next sections.

2 Problem formulation

Problem 1 *Given a set* Γ *in* Ω*, we are interested in accumulate the cell around* Γ*.*

For example, in the figure 1 bellow, Γ is a thin set on the left that models a fissure on a metal, you can see the adapted grid.

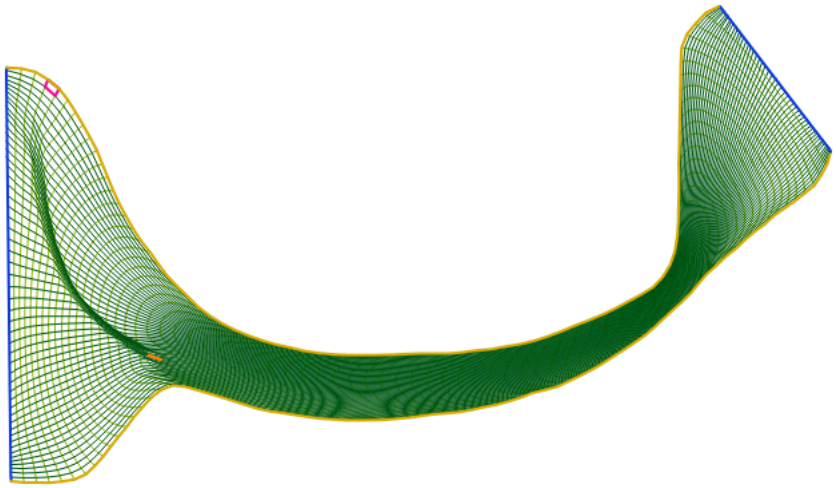


Figure 1: The grid was adapted following Γ.

3 Numerical structured grid generation: recently formulation

Definition of a discrete grid. Let B be the unit square and $U(m,n)$ the uniform mesh of size $m \times n$ on B given by

$$U(m,n) = \left\{ \left(\frac{i}{m}, \frac{j}{n} \right) | \quad 0 \leq i \leq m, 0 \leq j \leq n \right\}$$

where

$$\partial U(m,n) = \partial B \cap U(m,n)$$

A discrete grid G of size $m \times n$ on Ω is a mapping

$$G : U(m,n) \mapsto \mathbf{R}^2$$

such that

$$G(\partial U) \subset \partial \Omega$$

and

$$\partial G = G(\partial U) = \Gamma_{mn} \subset \partial \Omega$$

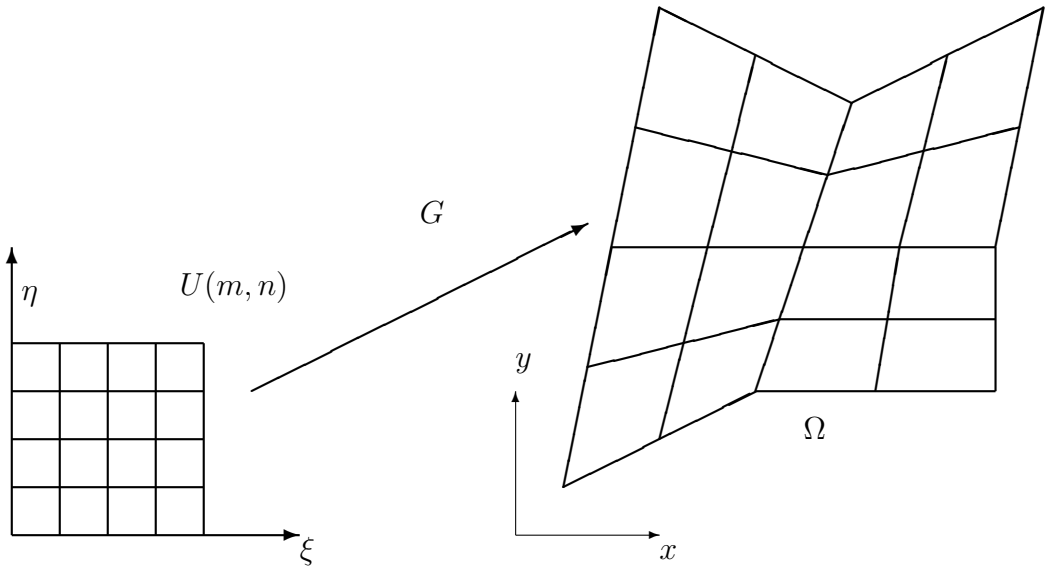


Figure 2: A discrete grid.

In the follow figure 2 we can see this idea.

If the bondary of Ω is positively oriented, an orientation of the boundary of the grid cell $c_{ij} = G(B_{ij})$ is induced in a natural way and in an analogous way the orientation of the boundary of the four triangles defined by the grid vertices (see figure 3).

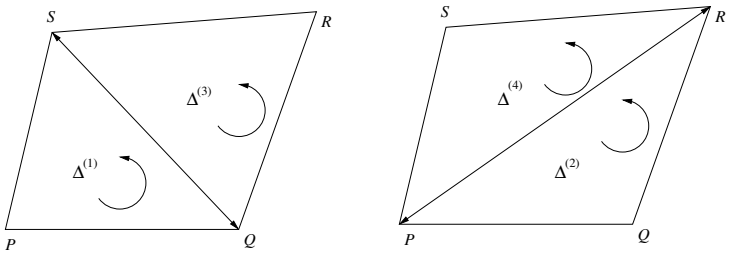


Figure 3: Four triangles defined by the vertices of a quadrilateral cell.

The problem of interest here is to generate a grid G with the property that the minimum signed area of the four triangles in every grid cell is greater than a given positive number ϵ, which is usually small. If this condition is fulfilled, then every grid cell is convex and its area is greater or equal than 2ϵ. A grid that satisfies this property is an ϵ-convex grid.

In order to generate convex grids for Ω, a well-posed definition of cell convexity is required, and it must be scale independent: given a grid G with $m \times n$ cells, let $\overline{\alpha}(G)$ is the average of the signed areas of all the grid cells triangles. Then, a G is ϵ-convex if for $\epsilon > 0$, every triangle area is greater than $\epsilon \cdot \overline{\alpha}(G)$.

The set of all the ϵ-convex grids for Ω will be denoted as $\mathcal{D}_\epsilon(\Omega)$. It is a subset of the euclidean space $\mathbf{R}^M$, where $M = 2(m-1)(n-1)$ is the number of degrees of freedom of the grid, twice the number of inner grid points. It is also a bounded set, and its boundary $\partial\mathcal{D}_\epsilon$ is the set of grids for which at least a signed triangle with area equals ϵ.

In addition, a G is ϵ-admissible is there exists an ϵ-convex grid $\tilde{G}$ such that $\partial\tilde{G} = \partial G$. We will denote as $\mathcal{A}_\epsilon(\Omega)$ the set of all the ϵ-admissible grids.

Problem 2 *Given an ϵ-admissible grid G_0, to find an ϵ-convex grid G^* such that $\partial G^* = \partial G_0$.*

We will say that a solution of this problem is a variational one if it is possible to get an ϵ-convex grid G^* that satisfies

$$G^* = \arg \min_{G \in \mathcal{A}_\epsilon(\Omega)} F(G).$$

3.1 Geometric classic functionals

On a triangle $A, B, C \in \mathbf{R}^2$ we define the length measure as

$$\lambda(\triangle(A, B, C)) = \|A - B\|^2 + \|C - B\|^2$$

and the measure area

$$\alpha(\triangle(A, B, C)) = (B - A)^t J_2 (B - C) = 2\text{área}(\triangle(A, B, C))$$

where

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Is easy to see that a grid G is ϵ-convex if and only if

$$\alpha_-(G) = \min \{\alpha(\triangle_q) |\, q = 1, \ldots, N\} > \epsilon \cdot \overline{\alpha}(G)$$

where $\epsilon > 0$, and $N = 4(m-1)(n-1)$ is the total number of triangles in G, 4 times the number of grid cells. Using these quantities, it is possible to define some other useful functions:

$$\alpha_+(G) = \min \{\alpha(\triangle_q) |\, q = 1, \ldots, N\}$$

and

$$\overline{\alpha}(G) = \frac{1}{N} \sum_{q=1}^{N} \alpha(\Delta_q)$$

the latter was described above. Since the triangle areas are signed we have

$$\sum_{q=1}^{N} \alpha(\Delta_q) = 4\text{área}(\Omega)$$

and therefore

$$\bar{\alpha}(G) = \frac{\text{área}(\Omega)}{(m-1)(n-1)}$$

which means that the average is independent of G, and depends only on Ω and the number of grid cells.This value is useful as a reference how spread out the area values are around the average.

Another important triangule measure is the orthogonality

$$o(\triangle(A,B,C)) = (B-C)^t(B-C)$$

These quantities are required to define the functionals over the grid G: the area functional used in [5]

$$F_A(G) = \sum_{q=1}^{N} \alpha(\triangle_q)^2$$

the classic length functional

$$F_L(G) = \sum_{q=1}^{N} \lambda(\triangle_q)$$

and the classic ortogonality functional [11]

$$F_O(G) = \sum_{q=1}^{N} o(\triangle_q)^2$$

There are also some useful linear combinations. For instance, the area-ortogonality functional

$$F_{AO}(G) = \sum_{q=1}^{N} \frac{\alpha(\triangle_q)^2}{2} + \frac{o(\triangle_q)^2}{2}$$

and the area-length functional

$$F_{AL}(G) = \sum_{q=1}^{N} [\sigma\alpha(\triangle_q)^2 + (1-\sigma)\lambda(\triangle_q)]$$

One must note that these functionals are positive grid functions.

The grid generation problem is closely related to the works develop by Kennon and Dulikravich[11], Castillo *et al.* [6] and Barrera [5], both of them requiring to minimize a suitable functional written as

$$F(G) = \sum_{q=1}^{N} f(\triangle_q), \tag{1}$$

all of then use suitable functionals that are variants of the area functional, but we found that on arbitrary irregular regions they fail to generate convex grids.

4 Convexity functional $S_{\omega,\epsilon}$

An efficient solution to problem 1 was proposed by Barrera and Domínguez-Mota [2], who posed theorem 1, which provides a characterization of the functionals whose minima are convex grids.

Theorem 1 *Let bet $0 < \epsilon \leq 1$. If f is a C^2 strictly decreasing convex and bounded below function such that $f(\alpha) \to 0$ as $\alpha \to \infty$, then*

$$S_{\omega,\epsilon}(G) = \sum_{q=1}^{N} f(\omega\alpha(\triangle_q) - \epsilon\overline{\alpha}(G)) \tag{2}$$

can be used as the objective function of the optimization problem 1, whose optimal grids are ϵ-convex is $\omega > 0$ is large enough.

One of the functions f we have used the most in practice with very satisfactory results is

$$f(\alpha) = \begin{cases} 1/\alpha, & \alpha \geq 1 \\ (\alpha - 1)(\alpha - 2) + 1, & \alpha < 1 \end{cases} \tag{3}$$

The ϵ-convex grids obtained by minimizing $S_{\omega,\epsilon}$ using (3) have been reported in [1, 4, 3].

4.1 Barrier property

The functionals $S_{\omega,\epsilon}$ have an infinite *barrier* at the boundary of the grid set ϵ-convex $\mathcal{D}_\epsilon \subset \mathcal{A}_\epsilon$. This means that at least one triangle's area is very close to ϵ when G is kept inside $\mathcal{D}_\epsilon$; then for ω large enough $S_{\omega,\epsilon} \longrightarrow \infty$ [8, 4].

Theorem 2 *Let $0 < \sigma \leq 1$. If there exists ϵ-convex G_0 for Ω and $S_{\omega,\epsilon}(G)$ is a functional that satisfies the hypotheses of theorem 1, then the functional*

$$F(G)) = \sigma S_{\omega,\epsilon}(G) + (1 - \sigma)F_c(G)$$

where $F_c(G)$ is a smooth positive grid functional, has an optimal ϵ-convex grid for $\omega > 0$ large enough.

In this sense the grid convexity can be guaranteed even when $S_{\omega,\epsilon}$ is combined with another functional related to a geometrical property sush as smoothness or ortogonality. Further details on this subject can be found in [4]

5 Geometric adaptive functionals

We are now to considere the problem of accumulate grids around objects Γ inside Ω and the question is how to build functionals to adapt our grids.

The main idea is that we will combinate the functional $S_{\omega,\epsilon}$ with another functional to move the cells of the grid towards the object Γ. Because our functional have the form

$$F_c(G) = \sum_{q=1}^{N} f(\triangle_q)$$

then we are going to use a weighted functional

$$F(G) = \sum_{q=1}^{N} w_q f(\triangle_q)$$

where the weights $w_q > 0$ are selected to have cell control. In consequence, it is convenient to rewrite the functionals in the form

$$F(G) = \sum_{i,j} w_{ij} f(c_{ij}).$$

A similar formulation has been studied by Kattri [14, 13], but is it useful only in simple regions Ω.

Problem 3 *To generate convex grids with preset cell size distribution using weighted functionals.*

This problem can be solved using the functionals with barriers like $S_{\omega,\epsilon}(G)$, as described in the following

Theorem 3 *Give an ϵ-convex grid G_0, there exists $\omega > 0$ such that the grid G^* satisfies*

$$G^* = \arg\min_{G\in\mathcal{A}_\epsilon} \sum_{q=1}^{N} [\sigma f(\omega\alpha(\triangle_q) - \epsilon\overline{\alpha}(G)) + (1-\sigma)w_q f_c(\triangle_q)]$$

Theorem 3 indicates that using positive weights it is osible to generate anϵ-convex grid for ω large enough.

Unlike Kattri's functional, we do not take the weight w_q as a function of the grid nodes, is better as functions of the cells $g = g(c_{ij})$ to control the cell distribution. This is written as

$$w_q = g(c_{ij})$$

where $g(c) > 0$ is the function that control the grid shape. Function g is the adaptivity grid control, or density control, and depends on the problem to be solved.

Given $\Omega \subset \mathbf{R}^2$ and Γ in Ω such that

$$\Gamma \subset \Omega,$$

our interest is to adapt the grid to the shape of Γ. For this we calculate the distance of a point $P = (x, y)$ to Γ, which is defined as:

$$d(P, \Gamma) = \min_{Q\in\Gamma} d(P, Q)$$

in this problem we need an economic calculation of distances. Therefore, we proposed a control function like:

$$\varphi(P) = \frac{1}{1 + c_a d(P, \Gamma)^2}$$

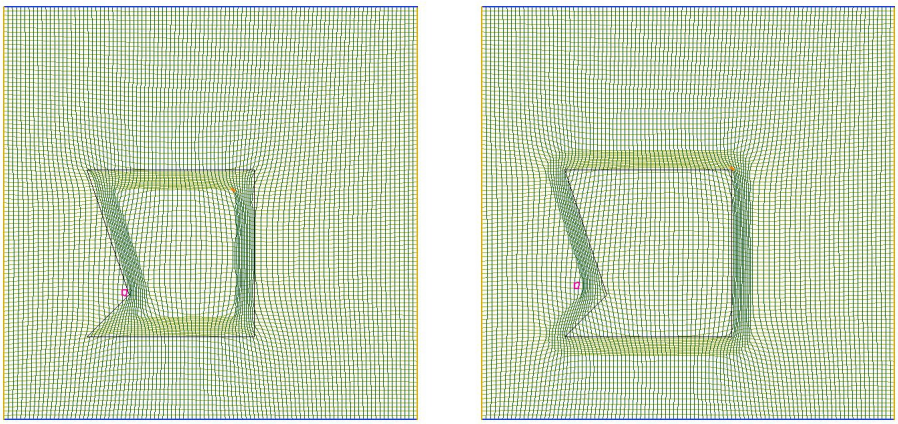
and using

$$w(c_{ij}) = \varphi(\tilde{P}_{ij})$$

where $\tilde{P}_{ij}$ is the center of mass of the cell c_{ij} and $c_a > 0$ is a parameter chosen to control the density.

Example 1 If what you want is to accumulate the cells around an internal region of the study region, like a hole or an island, we can use signed distances to identify whether we are inside or outside of the interest region, and by a density function as shown above can concentrate the cells around the inner region. Consider for Ω the unit square and Γ as an internal polygonal region, in the Figure 4 we can seen a cells concentration cells inside and outside the Γ region.

Example 2 Using this idea of geometric adaptivity, we can move the mesh with prescribed properties near of the boundary. In the figure 5 we can see an smothned adaptive grid over the Strait of Gibraltar, the grid lines control are near of the boundary.



(a) (b)

Figure 4: The cells have accumulated within the inner polygonal region, (b) the cells have accumulated outside the inner polygonal region.

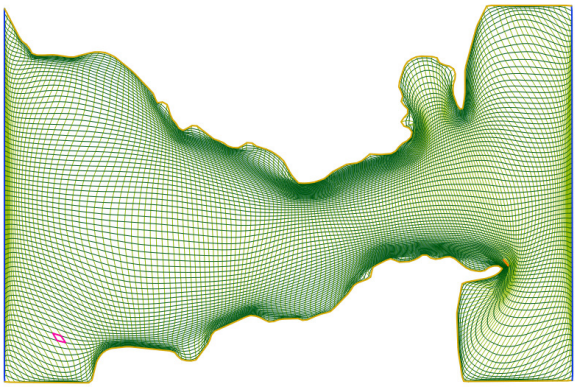


Figure 5: An adaptive grid on the Strait of Gibraltar

Example 3 In many applications it is necessary that the cells near of the boundary are as orthogonal such as possible. We can construct such grids by combining a functional that preserves convexity with the classic functional area-orthogonality or only orthogonality, and defining a properly weight function w_q which changes near of the boundary. This can be seen in the grid on the upper Arkansas subbasins, see the figure 6.

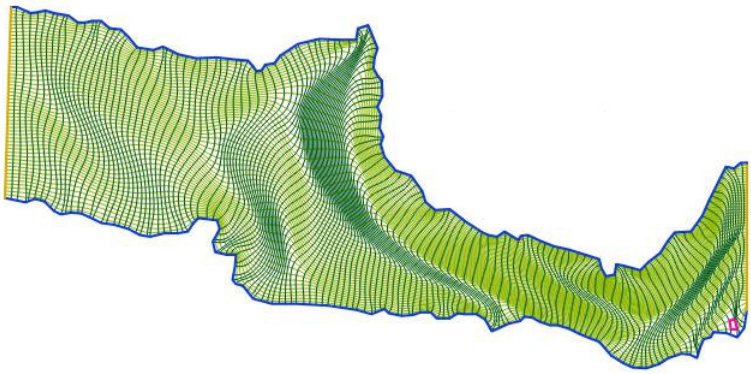


Figure 6: The upper Arkansas subbasins where the cells around the boundary are nearly orthogonal.

6 One way to construct control functions

The concept of reference grid generation was used by many people in the past where Nakamura [15, 16] was one of the first in use this concept to define control functions in parabolic grid generation. For us the the reference grid is very useful to define the weight functions for adaptive sense.

The functional that we working are

$$F_c(G) = \sum_{i,j} w_{ij}(B_{ij}) f_c(c_{ij})$$

where $w_{ij}(B_{ij}) > 0$ are values in function define over the parametric space. The reference grid (see figure 7) is construct *a priori* with the prescribed geometry properties over the physical grid.

Over the reference grid we can put some specific values to the cells to show a specific pattern, for example, if $w_q \geq 1$ then the cells are far form the object and $w_q < 1$ if we need concentrade the cell around the object. The formal description of this idea is work in progress [9].

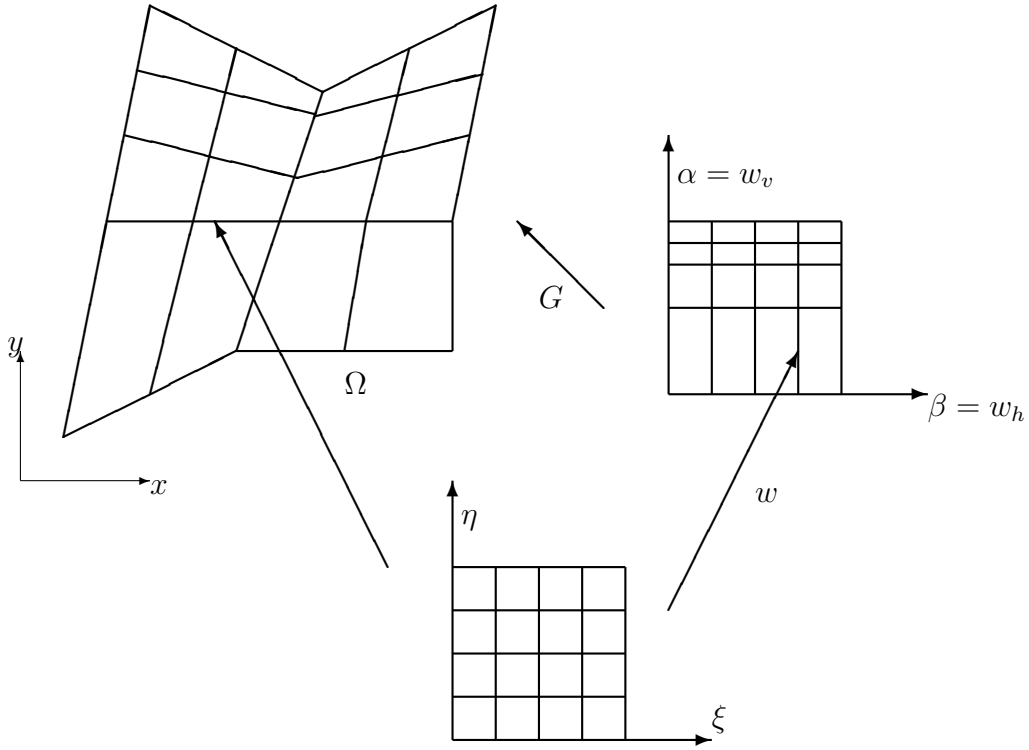


Figure 7: An squematic aproach to construt an adaptive grid using the reference grid.

Example 4 This adaptivity may be suitable geometry to achieve a grid that is adapted to the behavior we want to model, for example mesh formed by mosaic o tiles as shown in figure 8.

(a) (b)

Figure 8: Using a mosaic can build a floor or a carpet formed with a specific pattern.

Example 5 Using a weight function related to the intensity of grayscale of an image we can adapt the mesh to the shape of the image. In the figure we can see the image of Sergey Ivanenko a recognized specialist in grid generation who visited the UNAM months before his death, see the figure 9.

Figure 9: An image of Ivanenko (Sergey Ivanenko) in an adaptive grid.

Example 6 Another sample is whe ee can build an adaptive mesh to a physical phenomenon if known, for example, the behavior of the wind speed and using a suitable control function adapted mesh for later use in estimate speed storms and hurricanes as shown in the figure 10.

7 Conclusions

In this work we present a very simple description to generate adaptive grids around internal regions of the physical region using discrete classic functionals combinated with convex functionals that guarantee and preserve the convexity condition of the grid.

Using a collection of examples we show the potential of this technique for geometric adaptive grid using an adaptive function based on distances. The technique present can be used to specify prescribted properties mesh point near of the boundary like smooth or orthogonality lines.

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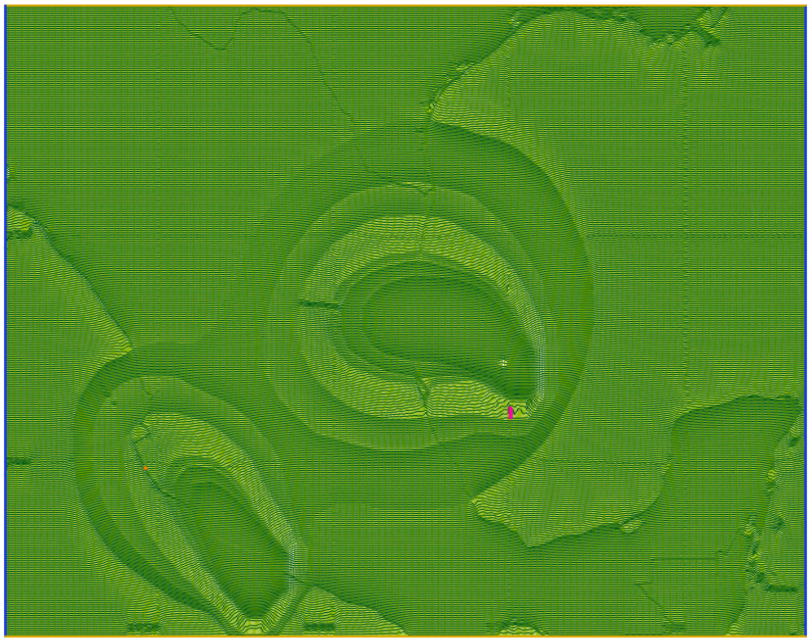

Figure 10: An adaptive grid over the map of Mexico and adapted to forecast the storm speeds Manuel and Hurricane Ingrid, September 15, 2013.

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