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LAMINATE ELEMENT METHOD FOR ELASTIC GUIDED WAVE DIFFRACTION SIMULATION

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Abstract. A meshless method suitable for investigation of wave propagation and diffraction problems for layered structures with local inhomogeneities is presented. The approach is based on the use of fundamental solutions for the layered structure as a whole. The corresponding basis functions, called laminate elements, satisfy identically the governing equations in the sublayers and all interface and boundary conditions on the plane-parallel surfaces. Thus, only conditions on the obstacle boundaries are to be approximated. With the approach presented, guided wave diffraction by plane and volumetric obstacles in laminate plates is investigated.

1 INTRODUCTION

Theoretical and experimental investigations on elastic guided wave (GW) interaction with various obstacles in lengthy laminate structures have wide applications in seismology, non-destructive evaluation and structural health monitoring. In engineering practice finite element (FEM) or finite difference (FDM) methods based on a mesh discretization of the waveguide structure are widely utilized for the numerical simulation of wave propagation phenomena. With increasing distances or frequency, these approaches require more and more elements and, as a consequence, become time and, especially, memory consuming. Additional computational costs are also connected with possible sharp difference among the elastic properties of sublayers as well as with a complex wave structure resulting from repeated reflections and refractions at the interfaces and defects. Therefore, efficient modifications of these techniques allowing significant reduction of computation costs have been recently developed and successfully implemented. For example, these are the spectral element method (SEM) [1], which is a high-order FEM, or the local interaction-simulation approach (LISA) [2] utilizing special finite-difference approximations.

As an alternative, various hybrid approaches such as a coupling of mesh-based approximations for areas adjacent to edges or inhomogeneities with the Lamb wave expansions are currently under intensive development [3]. Explicit analytical GW representations do not require any spatial discretization, and so they remain practically costless irrespective of the sample's size. Therefore, combining mesh or boundary-element discretization of local areas with an analytically-based continuation of incident and scattered wave fields among them may lead to sufficient reduction of computational costs.

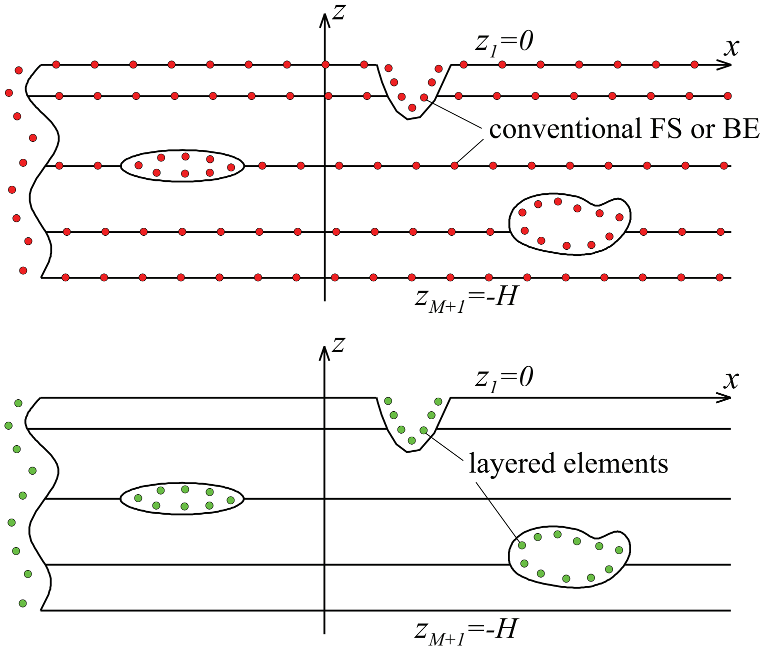

Figure 1: Comparison of the conventional MFS-BEM and LEM approaches to elastodynamic problems for layered waveguides with obstacles

For plate-like waveguides with localized obstacles there exists a way to avoid such a coupling. This approach called the laminate element method (LEM) [4] is a special case of the method of fundamental solutions (MFS). In contrast to the conventional MFS, which is based on classical fundamental solutions for a homogeneous elastic space, the LEM utilizes the fundamental solutions for the elastic layered structure under consideration as a whole. Such basis functions called laminate elements (LEs) identically satisfy the governing equations in the sub-layers and the homogeneous boundary conditions at all plane-parallel surfaces and interfaces. Therefore, only conditions on the remaining non-planar parts of the domain's boundary, e.g., at obstacle boundaries, surface irregularities or structural edges, are to be approximated by LEs (Fig. 1). This property considerably reduces the number of basis functions and thereby the computational expenses. It makes this method well-suitable for independent validation of various direct or hybrid numerical methods and enables efficient simulations of GW propagation and diffraction problems, especially in lengthy and expanded structures with different localized scatterers.

This technique as well as other LEM peculiarities are to be discussed and illustrated by the application to structural health monitoring (SHM) problems. In particular, guided wave diffraction by plane and volumetric obstacles such as rigid inclusions, holes and surface irregularities has been investigated. Examples of FEM and experimental verification of the proposed approaches are presented as well.

2 LAMINATE ELEMENT METHOD

Let us consider time-harmonic oscillations $\mathbf{u}e^{-i\omega t}$, $\mathbf{u} = \{u_x, u_y, u_z\}$ of an M-layered linear elastic waveguide of thickness H with surface or internal localized inhomogeneities, governed by the elastodynamic equations in displacements with different elastic parameters for each layer. In the Cartesian coordinate system $\mathbf{x} = (x, y, z)$ the structure occupies the domain $D = \bigcup_{m=1}^{M} D_m$, where $D_m = \{|x, y| < \infty,\ z_{m+1} \leq z \leq z_m\}$ are sublayers, $z_1 = 0$, $z_{M+1} = -H$. Below, the harmonic multiplier $e^{-i\omega t}$ is conventionally omitted. The exterior plane-parallel surfaces $z = 0$ and $z = -H$ are stress free except, possibly, at a local source zone S_0 to which load $\mathbf{q}$ is applied; displacement and traction continuity is assumed at all internal plane-parallel boundaries. Obstacle boundaries are further denoted by S. Schematic geometry of the problem in 3D and 2D cases is shown in Fig. 2.

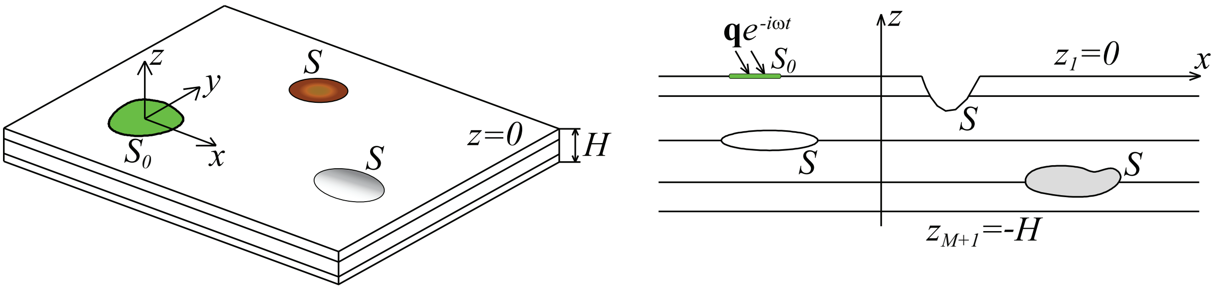

Figure 2: Geometry of the problem in 3D (left) and 2D (right) cases

In contrast to the classical fundamental solutions or boundary elements, which utilize the matrix of fundamental solutions (Green's matrix) $g(\mathbf{x}, \boldsymbol{\xi})$ for an infinite homogeneous elastic space [5], laminate elements are based on the fundamental matrices $l(\mathbf{x}, \boldsymbol{\xi})$ derived for an infinite laminate plate or half-space as a whole. The columns of the matrix are displacement vectors $\mathbf{u}_j$ associated with point sources $\delta(\mathbf{x} - \boldsymbol{\xi})\mathbf{i}_j$ directed along the coordinate unit vectors $\mathbf{i}_j$, $j = 1, 2, 3$. Here $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ is a point of source location. The basis vectors $\mathbf{i}_1$ and $\mathbf{i}_2$ are taken to be parallel to the laminate's boundary planes, while $\mathbf{i}_3$ is orthogonal to its surface.

Within the geometry considered, the matrix $l(\mathbf{x}, \boldsymbol{\xi})$ can be represented in terms of the inverse Fourier transform $\mathcal{F}_{xy}^{-1}$ with respect to the horizontal coordinates x, y via the

Fourier symbol $L(\alpha_1, \alpha_2, z) = \mathcal{F}[l(\mathbf{x})]$:

$$l(\mathbf{x}) = \mathcal{F}_{xy}^{-1}[L] \equiv \frac{1}{4\pi^2} \int\limits_{\Gamma_1}\int\limits_{\Gamma_2} L(\alpha_1, \alpha_2, z) e^{-i(\alpha_1 x + \alpha_2 y)} d\alpha_1 d\alpha_2 \qquad (1)$$

The integration contours Γ_1 and Γ_2 go along the real axes α_1 and α_2 deviating into the complex planes for rounding real poles ζ_k of the integrand in accordance with the limiting absorption principle.

A key point of successful LEM implementation is the development of efficient algorithms for layered fundamental solution $l(\mathbf{x}, \boldsymbol{\xi})$ calculation. Two of them, based on fast and stable numerical schemes for Green's matrix $k(\mathbf{x}, \boldsymbol{\xi})$ calculation in multilayered isotropic or anisotropic waveguides have been developed by the authors [6, 7]. In the first algorithm, the matrix $l(\mathbf{x}, \boldsymbol{\xi})$ is constructed as a composition of the conventional fundamental solution matrix $g(\mathbf{x}, \boldsymbol{\xi})$ with the properties of the layer, in which the source is located, and non-singular matrix $v(\mathbf{x})$, accountig for the fields reflected from the interfaces and external sides and providing homogeneous boundary conditions at internal and exterior plane-parallel boundaries: $l = g + v$. The matrix v, in its turn, is further decomposed into four (or three, in case of the first or the last sublayers) terms; each of them, being derived in terms of Green's matrix for the considered layered waveguide with a point displacement or stress jump at the correspond internal boundary, stands for the displacements or tractions caused at the interlaminar boundaries by $g(\mathbf{x}, \boldsymbol{\xi})$ columns.

In the alternative scheme, an additional plane-parallel boundary passing through the vertical coordinate of the point source is introduced, increasing the number of layers by one. In this case, the columns of $l(\mathbf{x}, \boldsymbol{\xi})$ matrix are the wave fields generated by point stress jumps along $\mathbf{i}_j$, $j = 1, 2, 3$ directions at this fictitious boundary. It allows applying well-developed Green's matrix calculation algorithms without any modification.

A fast numerical integration of integrals (1) is a challenging problem as well. Even with contemporary powerful computers a straight numerical integration could hardly yield proper results. On the other hand, the level of computer expenses is radically reduced after certain analytical preprocessing using the complex variable theory, residual technique, special functions and asymptotics. Finally the integral representations for matrices $l(\mathbf{x})$ are brought to close analytical formulas involving cylindrical Bessel functions (for 3D cases) and residuals of the matrix L elements at the real and complex poles ζ_k. The latter have to be found and calculated beforehand. Thus the crucial points allowing one to handle efficiently integrals (1) are the efficient implementation of algorithms for matrix L computation together with fast and reliable methods of searching for complex poles and calculating residuals for functions given numerically.

As soon as layered fundamental solution $l(\mathbf{x}, \boldsymbol{\xi})$ is obtained, the displacement field $\mathbf{u}(\mathbf{x})$ caused in a laminate sample with possible local inhomogeneities (defects) by a given load $\mathbf{q}$ may be sought for in the form

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_{sc}, \quad \mathbf{u}_{sc} \approx \sum_{j=1}^{N} l_j(\mathbf{x})\mathbf{c}_j, \tag{2}$$

where $\mathbf{u}_0$ is an incident field generated by $\mathbf{q}$ in the infinite laminate plate without defects, and $\mathbf{u}_{sc}$ is an additional field arising due to $\mathbf{u}_0$ scattering by local defects (cracks, void, inclusions, surface irregularities, etc.) and/or the plate's edges (for finite samples). Whereas a close analytical representation for $\mathbf{u}_0$ is easily derived using the integral transform $\mathcal{F}_{xy}$ applied to the equations and boundary conditions for the infinite defectless structure, $\mathbf{u}_{sc}$ is approximated by a sum of LEs consisting of the fundamental matrices $l_j(\mathbf{x}) = l(\mathbf{x}, \boldsymbol{\xi}_j)$ and unknown vector coefficients $\mathbf{c}_j$. In accordance with the MFS general scheme the latter have to be chosen to minimize the discrepancy of the boundary conditions on the surface of scatterers S. For that the LE centers $\boldsymbol{\xi}_j$ are allocated along S at a certain distance d from this surface. Since $\mathbf{x} = \boldsymbol{\xi}_j$ are singular points of the matrices $l_j(\mathbf{x})$, one must put $\boldsymbol{\xi}_j$ outside the sample domain D, in which the field $\mathbf{u}_{sc}$ is approximated by LE sum (2).

It is worthy to note that since matrix l may be written as the sum $l = g + v$ its elements bear the same singularity at $\mathbf{x} = \boldsymbol{\xi}$ as that of the matrix of fundamental solutions used for classical BEs (matrix v has no singularities). In that way, the most of methods derived for operating with singular BEs may be used for LEM computations as well. In particular, $l(\mathbf{x}, \boldsymbol{\xi})$ may be used as a singular matrix kernel of the boundary integral representation in the context of indirect integral approach

$$\mathbf{u}_{sc}(\mathbf{x}) = \iint_S l(\mathbf{x}, \boldsymbol{\xi})\mathbf{c}(s)ds,$$

in which $\boldsymbol{\xi} = \boldsymbol{\xi}(s)$ ran over the surface S. Its discretization, for example in line with boundary element technique, lead to the following approximation of the reflected field

$$S = \bigcup_{j=1}^{N} S_j, \quad \mathbf{u}_{sc} \approx \sum_{j=1}^{N} \iint_{S_j} l(\mathbf{x}, \boldsymbol{\xi})\mathbf{c}_j(s)ds \tag{3}$$

Each term in the sum (3) may be treated as an "integral" laminate element. Though the use of LEs as boundary elements provides higher accuracy and better stability than approximation (2), it is more time consuming, especially in 3D, due to the necessity of multifold integration encountered in the course of calculation of the mutual influence between boundary elements lying on non-parallel planes. This task remains to be challenging even for contemporary powerful computers. In two-dimensional case, however, radical reduction of the computing expenses has been achieved utilizing certain analytical preprocessing using complex variable theory, asymptotic methods and residue technique. Among them, the key role plays the method of rotation in the Fourier transform domain firstly proposed for the modeling of 2D and 3D diffraction by inclined cracks [8].

The unknown coefficients $\mathbf{c}_j$ in representations (2) and (3) are obtained from a linear algebraic system to which the problem is reduced in the context of collocation or Galerkin scheme.

3 NUMERICAL EXAMPLES

Utilizing the efficient and well-developed algorithms for Green's matrix calculation, the method may be implemented for both isotropic and anisotropic laminates. Benchmark comparisons carried out with multiple-glass panes as an example have shown that a conventional FEM approach could not catch the peculiarities of stress concentration at contrast interlayers up to the absolute fail with too long samples [9], while the LEM results were stable disregard to the structure length and layer properties. As a further illustration, the results of LEM application to 2D and 3D Lamb wave propagation and diffraction by inner and surface obstacles are presented below.

In the 2D case diffraction of the fundamental Lamb mode S_0 by a surface-breaking rectangular notch simulating corrosion damage in an isotropic aluminium plate of thickness $H = 3$ mm has been investigated (Fig. 3, left). The dependence of the reflection coefficient $\mu^- = |u_{sc,x}(\mathbf{x}_1)|/|u_{0,x}(\mathbf{x}_1)|$ on the notch width - waveguide thickness ratio at a fixed frequency are shown in Fig. 3, right. The approximation (3) is used for scattered field calculation with $N = 64$. Unknown functions $\mathbf{c}_j$ are assumed to be constant within each boundary element and Galerkin scheme is used for obtaining unknown expansion coefficients. Good coincidence with the experimental and FEM results [10] are obtained.

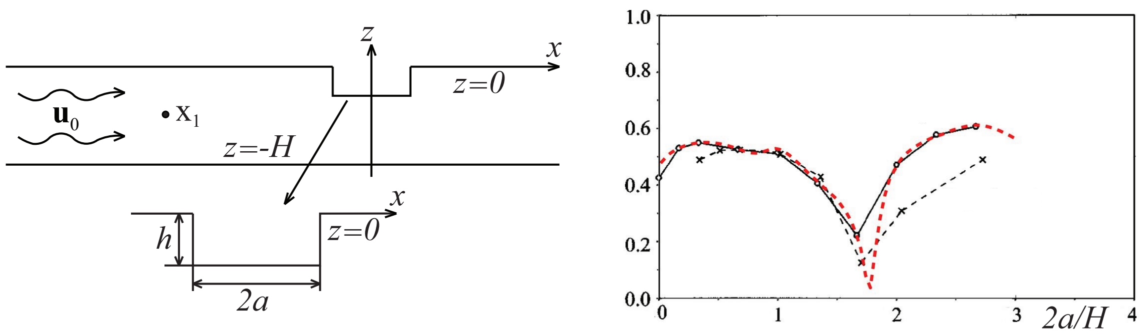

Figure 3: GW interaction with a rectangular notch. Geometry of the problem (left subplot); variation of S_0 reflection coefficient μ^- with notch width when notch depth h=0.5H; frequency-thickness product is 1.2 MHz-mm: solid line - FEM results [10], markers - experimental data [10], dashed line - LEM.

As an example of 3D diffraction problem, the incidence of plane S_0 and A_0 modes on through-thickness elliptical cavities in a 5 mm-thick aluminium plate at a frequency f=100 kHz is considered. Major and minor half axes of the ellipses are denoted by a and b respectively. The largest dimension of the ellipse, $2a$, is equal to the wavelength of the incident mode. Results for the scattering simulation are presented in Fig. 4, where thin solid lines stand for the data obtained using scattered field expansion in terms of cylindrical Bessel and Hankel function [11] and colored lines are LEM (2) results with $N = 256$ and collocation technique used for $\mathbf{c}_j$ evaluation. One can see very good coincidence of the results.

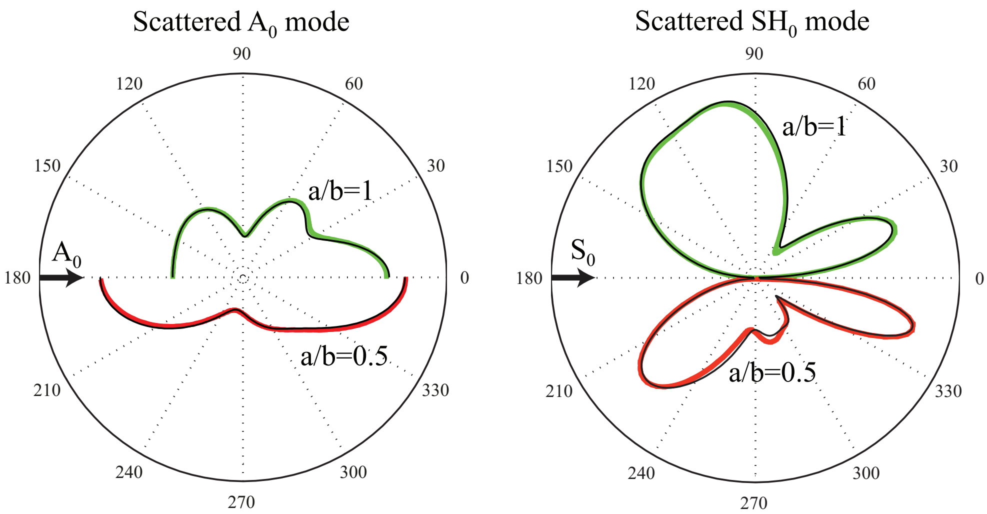

Figure 4: Directivity of scattered waves when A_0 (left subplot) or S_0 (right subplot) is incident on an elliptical through-thickness cavity in an aluminium plate at 500 kHz-mm. Thin black line - results of the work [11], coincided colored lines - LEM

4 CONCLUSIONS

The developed analytically-based LEM technique allows one to simulate guided wave propagation and diffraction in lengthy laminate structures with obstacles. The crucial point of LEM implementation is fast and stable algorithms of Green's matrix calculation which have been developed and computer implemented both for 2D and 3D problems. Applicability of LEM to wave diffraction problems has been validated over known independent theoretical and experimental results.

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