11th World Congress on Computational Mechanics (WCCM XI)
5th European Conference on Computational Mechanics (ECCM V)
6th European Conference on Computational Fluid Dynamics (ECFD VI)
E. Oñate, J. Oliver and A. Huerta (Eds)

MODAL DERIVATIVES BASED REDUCTION METHOD FOR FINITE DEFLECTIONS IN FLOATING FRAME

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Key words: Modal Derivatives, Floating Frame, Geometric Nonlinearities

Abstract. Model order reduction techniques are widely applied in the floating frame of reference. The use of linear vibration modes, however, is not applicable when the elastic deformations become finite. In this paper, the non-linear elastic formulation, where the higher-order terms will be included in the strain energy expression to consider the bending-stretching coupling effect, is applied in the floating frame of reference. In this case, the complexity of the formulation diminishes the advantages of the floating frame of reference formulation because of the relatively high computational cost. Therefore, the linear reduction basis of vibration modes is augmented with the relevant modal derivatives to accurately reproduce the nonlinear elastic deformation on the reduced basis. The numerical results presented in this paper demonstrate that the proposed approach can be applied to accurately investigate problems featuring arbitrary large rigid body rotations and finite elastic displacements.

1 INTRODUCTION

The floating frame approach, which follows a mean rigid body motion of an arbitrary flexible component, is widely applied in the flexible multibody dynamics [1]. The major advantage of floating reference, comparing with the corotational frame and inertia frame, is the ability to naturally use Model Order Reduction (MOR): the local generalized coordinates are expressed as a linear combination of a small number of modes. However, the traditional linear MOR is limited to small displacements applications. In order to accurately represent the behavior of flexible systems, a geometrical nonlinear model must be employed. This includes the bending-stretching coupling when the elastic deformation become finite.

The floating frame of reference (FFR) approach combined with linear elastic finite element model was illustrated by Shabana[2]. In a later contribution the classical geometric stiffness was added to the constant stiffness matrix in order to account for geometric elastic nonlinearity[3]. A similar formulation was extended to 3D FFR[4] recently. However, the formulation of strain energy developed by literature[3, 4] neglected some higher terms for strain tensor, which can be indispensable for some practical application. J. Mayo [5] extended the formulation in literature[3, 4], then obtained additional geometric stiffness matrix and two further nonlinear elastic elastic force vectors. Meanwhile, a new formulation[6, 7] which moves the geometric elastic nonlinearity terms from elastic forces to inertial forces was developed. Although this formulation can lead to a constant stiffness matrix, it will also generate more complicated inertia and constraint forces.

Although a MOR that reduces the elastic degrees of freedoms as a linear combination of vibration modes (VMs) is widely used in FFR, high frequency axial modes have to be included to consider the longitudinal displacement in the nonlinear formulation developed by [5]. In this paper, we propose a MOR technique for the nonlinear elastic component deformation based on the concept of Modal Derivatives (MDs) [8], which could be used to describe the second-order effects that occur when considering (geometrical) nonlinearities. By adding the MDs to the basis formed with VMs , the reduced system is able to capture the second-order nonlinear effects, and as a result accurately describe the nonlinear dynamic behaviour [9]. The proposed approach in this paper will be illustrated by several numerical examples and the reduced solution will be compared to the one obtained with corotational frame approach.

2 FINITE ELEMENT FORMULATION IN FLOATING FRAME

To illustrate the concept, we present in this paper the framework relative to two-dimensional Euler-Bernoulli beam, which can be generalized to slender beams and thin plates in three dimensions. In the FFR formulation, in addition to the global inertia coordinate XY, three extra coordinate systems are used to describe the global position vector r^{ij} of an arbitrary point on j^{th} element of i^{th} body, as shown in Fig.(1): 1. the body coordinates X^iY^i associated to each body i; 2. the element coordinates $X^{ij}Y^{ij}$ associated to each element j; 3. the intermediate coordinate system $\overline{X}^{ij}\overline{Y}^{ij}$ parallel to $X^{ij}Y^{ij}$ and whose origin is rigidly attached to the origin of X^iY^i. Therefore, r^{ij} can be expressed in the FFR formulation as:

$$\mathbf{r}^{ij} = \mathbf{R}^i + \mathbf{A}^i(\mathbf{e}_0^{ij} + \mathbf{N}^{ij}\mathbf{q}_f^i), \tag{1}$$

where $\mathbf{R}^i$ represents the translation orientation of X^iY^i with respect to XY ; $\mathbf{A}^i$ is the transformation matrix from body system X^iY^i to global system XY; $\mathbf{e}_0^{ij}$ is the undeformed position of point relative to X^iY^i system; $\mathbf{N}^{ij}$ related to the appropriated shape functions; $\mathbf{q}_f^i$ is the vector of elastic coordinates of body i.

The dynamic equations for flexible systems in a 2D plane are obtained with the Lagrange Multiplier for constraint bodies:

$$\frac{d}{dt}\Big(\frac{\partial T}{\partial \dot{\mathbf{q}}}\Big)^T - \Big(\frac{\partial T}{\partial \mathbf{q}}\Big)^T + \Big(\frac{\partial U}{\partial \mathbf{q}}\Big)^T + \mathbf{C}_q^T\boldsymbol{\lambda} = \mathbf{Q}_e, \tag{2}$$

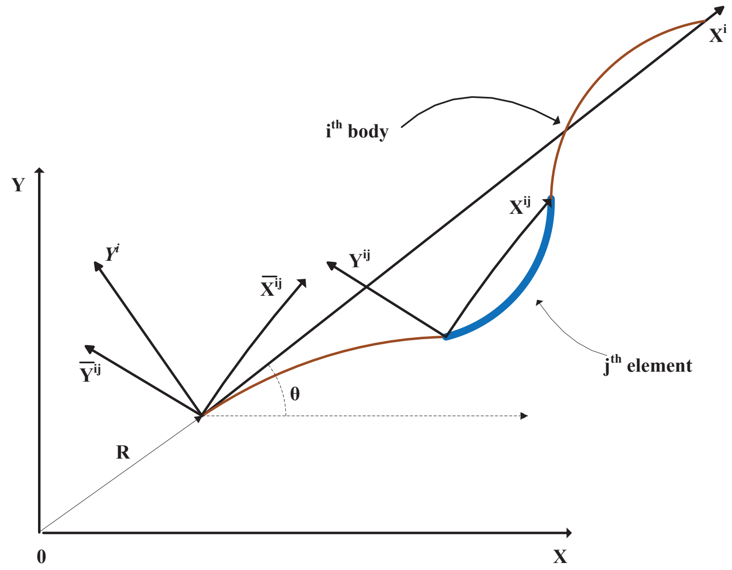


Figure 1: Generalized coordinates expression in FFR

where T and U are, respectively, the kinetic energy and strain energy; $\mathbf{C}_q$ is the constraint Jacobin matrix; $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers; $\mathbf{Q}_e$ is the vector of externally applied forces; t is the time; $\mathbf{q}$ is the total vector of the system generalized coordinates, which could be expressed as:

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_r \\ \mathbf{q}_f \end{bmatrix} = \begin{bmatrix} \mathbf{R} \\ \theta \\ \mathbf{q}_f \end{bmatrix}, \tag{3}$$

where subscripts $\mathbf{q}_r \in \mathbb{R}^3$ refers to the displacement and orientation of body coordinates and $\mathbf{q}_f \in \mathbb{R}^n$ refers to the elastic displacement in body coordinates, n is the number of elastic degree of freedoms.

The inertia coupling between the body coordinates system $\mathbf{q}_r$ and elastic coordinates system $\mathbf{q}_f$ leads to inertia terms which are configuration and velocity dependent[2]:

$$\frac{d}{dt}\Big(\frac{\partial T}{\partial \dot{\mathbf{q}}}\Big)^T - \Big(\frac{\partial T}{\partial \mathbf{q}}\Big)^T = \mathbf{M}(\mathbf{q}, \dot{\mathbf{q}})\ddot{\mathbf{q}} - \mathbf{Q}_v(\mathbf{q}, \dot{\mathbf{q}}), \tag{4}$$

where $\mathbf{M}$ is the system mass matrix, $\mathbf{Q}_v$ is the quadratic velocity vector.

In order to calculate the strain energy U, we hereby recall the strain expression for a 2D slender beam. An approximation of exact Lagrange strain expression is applied here, combined with the linearized form of displacement field, to investigate the model with small to moderate displacement[10]:

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} - y\frac{\partial^2 v_0}{\partial x^2} + \frac{1}{2}\Big(\frac{\partial v_0}{\partial x}\Big)^2, \tag{5}$$

where u_0 and v_0 denote the axial displacement u and vertical displacement v at $y = 0$.

In case of isotropic linear material, the strain energy could be expressed as:

$$\begin{aligned} U &= \frac{1}{2}\int_0^L EA\varepsilon_{xx}^2 \,\mathrm{d}x \\ &= \frac{1}{2}\int_0^L EA\Big(\frac{\partial u_0}{\partial x}\Big)^2 \mathrm{d}x + \frac{1}{2}\int_0^L EI\Big(\frac{\partial^2 v_0}{\partial x^2}\Big)^2 \mathrm{d}x \\ &+ \frac{1}{2}\int_0^L EA\frac{\partial u_0}{\partial x}\Big(\frac{\partial v_0}{\partial x}\Big)^2 \mathrm{d}x + \frac{1}{2}\int_0^L \frac{EA}{4}\Big(\frac{\partial v_0}{\partial x}\Big)^4 \mathrm{d}x, \end{aligned} \tag{6}$$

where E is Young's modulus; A is the cross sectional area of beam; L is the length; I is the second moment of area.

The first and second order integrals in the equation will generate the linear stiffness terms; the third and last integrals are cubic and quartic function of displacements, respectively; therefore, the elastic forces $\mathbf{Q}_{nl}$ generated from the differentiation of strain energy is a cubic function of the elastic displacement $\mathbf{q}_f$. The equation of motion could be written as:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{Q}_{nl} + \mathbf{C}_q^T\boldsymbol{\lambda} = \mathbf{Q}_e + \mathbf{Q}_v. \tag{7}$$

Equation (7) can be conveniently written in a partitioned form in terms of a coupled set of reference and elastic coordinates:

$$\begin{bmatrix} \mathbf{m}_{rr} & \mathbf{m}_{rf} \\ \mathbf{m}_{fr} & \mathbf{m}_{ff} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_r \\ \ddot{\mathbf{q}}_f \end{bmatrix} + \begin{bmatrix} (\mathbf{Q}_{nl})_r \\ (\mathbf{Q}_{nl})_f \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{qr}^T \\ \mathbf{C}_{qf}^T \end{bmatrix} \boldsymbol{\lambda} = \begin{bmatrix} (\mathbf{Q}_e)_r \\ (\mathbf{Q}_e)_f \end{bmatrix} + \begin{bmatrix} (\mathbf{Q}_v)_r \\ (\mathbf{Q}_v)_f \end{bmatrix}, \tag{8}$$

where subscripts r and f refer to the reference and elastic coordinates, respectively.

3 MODAL DERIVATIVES BASED REDUCTION METHOD

The MOR approach aims at projecting the elastic displacement $\mathbf{q}_f$ into a much smaller space. The linear MOR method is to to form the reduced basis with low frequency VMs. For geometrically nonlinear analysis, however, VMs are not enough to describe the dynamic behavior since they are computed by solving the eigenvalue problem associated with the linearized equations of motion. The frequencies of fundamental axial modes are normally much higher than those of bending modes [5]: one has in principle to calculate several modes to find the proper axial contribution. This can be overcome by properly contrain the model. However, for more complex structures, this is extremely difficult. To overcome this difficulty, a set of MDs will be added to the existing linear reduction basis formed with VMs. Including MDs in nonlinear MOR will lead to a reduced systems that are able to accurately describe the geometric nonlinearities [8, 9, 11].

3.1 Computation of Modal Derivatives

The VMs are the solution of the eigenvalue problem corresponding to free dynamic equations around a certain point u_eq. The free vibrations eigenvalue problem is be written as:

$$(\mathbf{K}_{eq} - \omega_i^2 \mathbf{M})\boldsymbol{\phi}_i = 0, \tag{9}$$

where $\boldsymbol{\phi}_i$ represents the corresponding i^{th} VM, which is associated to the modal amplitude η_i.

Then, MDs can be computed analytically by differentiating the eigenvalue problem with respect to the modal amplitudes:

$$(\frac{\partial \mathbf{K}_{eq}}{\partial \eta_j} - \frac{\partial \omega_i^2}{\partial \eta_j}\mathbf{M})\boldsymbol{\phi}_i + (\mathbf{K}_{eq} - \omega_i^2 \mathbf{M})\boldsymbol{\theta}_{ij} = 0, \tag{10}$$

where $\boldsymbol{\theta}_{ij}$ represents the MDs, which could be expressed as:

$$\boldsymbol{\theta}_{ij} = \frac{\partial \boldsymbol{\phi}_i}{\partial \eta_j}. \tag{11}$$

According to the literature [8, 9, 11], all the inertia terms in equation (10) could be neglected. So equation (10) could be simplified as:

$$\mathbf{K}_{eq}\boldsymbol{\theta}_{ij} = -\frac{\partial \mathbf{K}_{eq}}{\partial \eta_j}\boldsymbol{\phi}_i. \tag{12}$$

For a constrained problem, the stiffness matrix is invertible. The MDs are calculated as:

$$\boldsymbol{\theta}_{ij} = -\mathbf{K}_{eq}^{-1}\frac{\partial \mathbf{K}_{eq}}{\partial \eta_j}\boldsymbol{\phi}_i. \tag{13}$$

Given m selected VMs, only $p = m(m+1)/2$ modal derivatives exist due to the symmetry.

3.2 Modal order reduction in floating frame

Once the VMs $\boldsymbol{\Phi}$ and the MDs $\boldsymbol{\Theta}$ are calculated, the transformation for the elastic coordinates $\mathbf{q}_f$ can written as:

$$\mathbf{q}_f = \begin{bmatrix} \boldsymbol{\Phi} & \boldsymbol{\Theta} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{\xi} \end{bmatrix}, \tag{14}$$

where $\boldsymbol{\eta} \in \mathbb{R}^m$ and $\boldsymbol{\xi} \in \mathbb{R}^s,\ s \leq p$ are the modal coordinates vectors associated to VMs and MDs, respectively. The reference coordinates remain unchanged:

$$\begin{bmatrix} \mathbf{q}_r \\ \mathbf{q}_f \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & \boldsymbol{\Phi} & \boldsymbol{\Theta} \end{bmatrix} \begin{bmatrix} \mathbf{P}_r \\ \boldsymbol{\eta} \\ \boldsymbol{\xi} \end{bmatrix}. \tag{15}$$

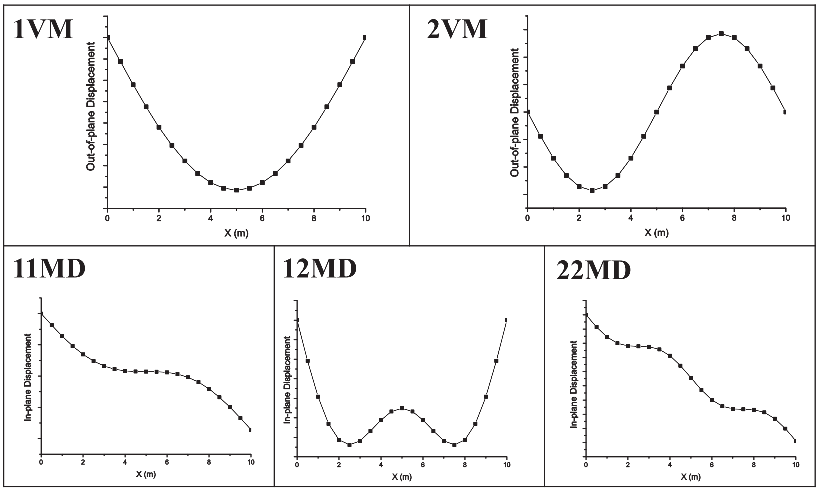


Figure 2: The shape modes for two-dimensional beam. Note that while the VMs feature out-of-plane displacements, the corresponding MDs contain in-plane motion only.

The system of equations of motion can be written as:

$$
\begin{bmatrix} \mathbf{m}_{rr} & \mathbf{m}_{rf}\begin{bmatrix}\boldsymbol{\Phi} & \boldsymbol{\Theta}\end{bmatrix} \\ \begin{bmatrix}\boldsymbol{\Phi}^T \\ \boldsymbol{\Theta}^T\end{bmatrix}\mathbf{m}_{fr} & \begin{bmatrix}\boldsymbol{\Phi}^T \\ \boldsymbol{\Theta}^T\end{bmatrix}\mathbf{m}_{ff}\begin{bmatrix}\boldsymbol{\Phi} & \boldsymbol{\Theta}\end{bmatrix} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{P}}_r \\ \ddot{\boldsymbol{\eta}} \\ \ddot{\boldsymbol{\xi}} \end{bmatrix} + \begin{bmatrix} (\mathbf{Q}_{nl})_r \\ \begin{bmatrix}\boldsymbol{\Phi}^T \\ \boldsymbol{\Theta}^T\end{bmatrix}(\mathbf{Q}_{nl})_f \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{qr}^T \\ \begin{bmatrix}\boldsymbol{\Phi}^T \\ \boldsymbol{\Theta}^T\end{bmatrix}\mathbf{C}_{qf}^T \end{bmatrix} \boldsymbol{\lambda} = \begin{bmatrix} (\mathbf{Q}_e)_r \\ \begin{bmatrix}\boldsymbol{\Phi}^T \\ \boldsymbol{\Theta}^T\end{bmatrix}(\mathbf{Q}_e)_f \end{bmatrix} + \begin{bmatrix} (\mathbf{Q}_v)_r \\ \begin{bmatrix}\boldsymbol{\Phi}^T \\ \boldsymbol{\Theta}^T\end{bmatrix}(\mathbf{Q}_v)_f \end{bmatrix}. \tag{16}
$$

Except for the inclusion of MDs, equation (16) is equivalent to the one botained via traditional MOR technique in FFR formulation. In the FFR formulation, a mixed set of global and body coordinates are used to define the configuration. The rigid body modes of shape function must be eliminated in order to define a unique displacement field. So a set of reference conditions which are consistent with the kinematic constraints should be imposed to boundary of deformable body. Therefore, the choice of deformable body coordinate system plays a fundamental role in the FFR equations. In this paper, the pinned frame, which has one of its axes passing through two nodes of the beam, is used as the body coordinates in FFR [12]. The first two VMs and corresponding MDs for the pinned frame are shown in Fig.(2). From Fig.(2) we can see that the first several VMs could only represent the bending displacement, while the corresponding MDs, which describe the second-order nonlinearities, can denote the longitudinal displacement due to the geometric nonlinearities.

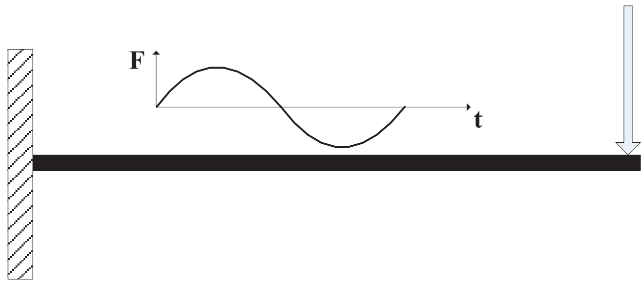


Figure 3: Cantilever beam example. An harmonic load is applied vertically at the free tip.

4 NUMERICAL EXAMPLES

In many references, the accuracy of nonlinear floating frame was only compared with traditional(linear [2] or classical nonlinear [3]) floating frame and inertia frame. In this paper, the obtained results are compared with a conceptually different formulation: the corotational frame of reference (CFR), which follows the mean rigid body motion of each element. The CFR could be treated as a specified FFR if the body coordinates are established at every element in FFR formulation.

The theory and formulation of the CFR was obtained from the literature [13], where cubic shape functions are used to derive both the inertia and elastic terms. Our code was validated against numerical examples found in literature [13]. Two numerical examples are analyzed in this section. The units are neglected here and may be assigned in any convenient system.

4.1 Cantilever beam

The first example is taken from [8] and it is shown in Fig.(3). It consists of a cantilever beam of length $L = 100$, with a rectangle cross-section, whose area A equals to 70. The clamped beam is subjected to a sinusoidal tip force $F = F_0 \sin(\Omega t)$ at the free end. The amplitude of the load F_0 is taken equal to 2500 and its frequency Ω is 9.5. The elastic modulus of the beam E is 16000 ,the mass per unit volume is $\rho = 0.0078$ and the moment of area I is 585.

We show in fig.(4) the results obtained with 4 formulations: 1.the corotational frame (denoted as CFR); 2.the linear floating frame with constant stiffness matrix (denoted as LFFR); 3.the nonlinear floating frame mentioned in this paper (denoted as NLFFR); 4.the modal derivatives based reduction methods in nonlinear floating frame (denoted as VMs+MDs). In order to investigate the effect of MDs, an extra MOR formulation with only VMs is also applied (denoted as VMs) as a reference.

By comparing the results with the referenced literature [8], we could observe that both the CFR and NLFFR are in good agreement with the reference solution, while the linear floating frame is not a proper approach in this example owing to the poor approximation of

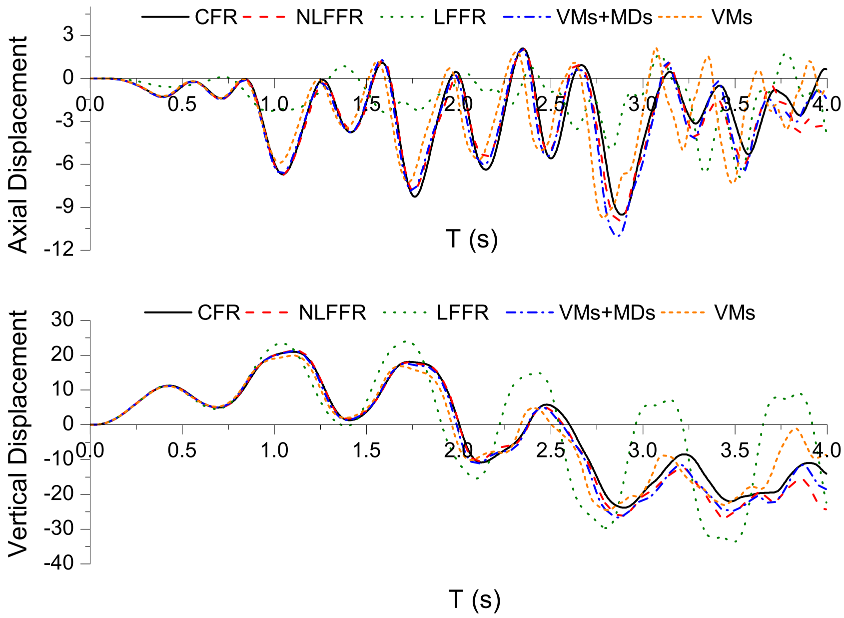
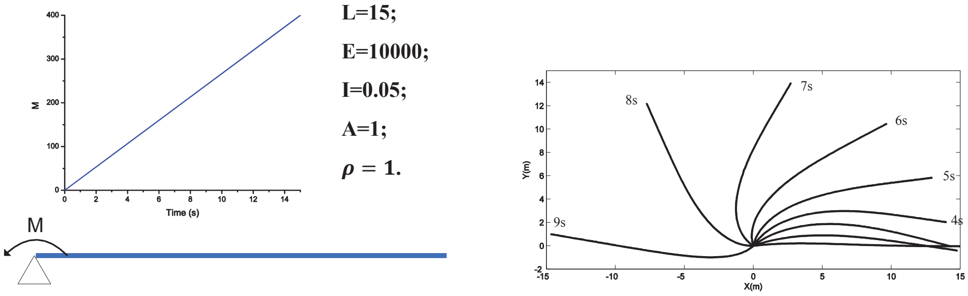


Figure 4: Displacement history at the free end of cantilever beam.

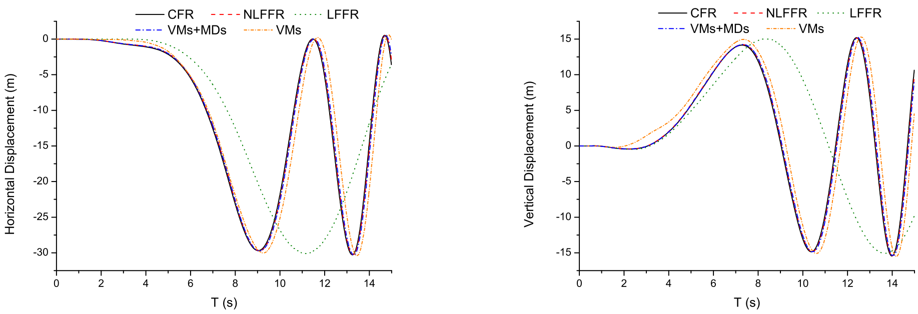
the elastic displacements. As for the proposed modal derivatives based reduction method, the first 12 VMs and 5 MDs are applied for the reduced basis. This formulation is in a good agreement with the two other formulations. But if only the first 17 VMs (to keep the same number of mode shapes) are applied in the reduced basis, the obviously difference with the accurate result will be observed even though 7 axial modes have already been included. Therefore, the modal derivatives is not the same thing as the axial modes in the VMs. The combination of VMs and MDs yields a better result since the MDs could correctly reproduce the in-plane/out-of-plane coupling in the nonlinear cases.

In this case, 20 elements are used in the NLFFR formulation while 17 mode shapes are applied in our proposed MOR method. So the degrees of freedoms are reduced from $3 + n = 66$ $(3 + 3 \times 21)$ to $k = 3 + m + s = 3 + 12 + 5$ in this example. The CPU time reduction, however, was not relevant in this case, as the advantage of the reduced model lays in the solution of a smaller $(k \times k)$ system for each Newton-Raphson iteration during the time integration. In this example, a $k = 20$ system has to be solved instead of a $n = 66$ system. so the speed up is counteracted by the fact that the nonlinear forces has to be formed with the full displacements $\mathbf{q}_f$. For larger models, large time savings are expected.



(a) Flexible robot arm, problem data. (b) Dynamic simulation snapshots.

Figure 5: The moment driven flexible robot arm.



(a) Horizontal displacement at the free end (b) Vertical displacement at the free end

Figure 6: Displacement history at the free end of the robot arm.

4.2 Moment driven flexible robot arm

We now consider a flexible robot arm , as shown in Fig.(5), subjected to a linear moment at the fixed end. All the required material properties could be found in Fig. (5).

Similarly, the results obtained with these four formulations could be observed in Fig. (6). We can see that except for the linear floating frame, all of the other three formulations could get a good agreement. It can be clearly seen that the nonlinear strain expression in the FFR must be adopted for this case. In this example, 10 elements are used in NLFFR formulation while only the first $m = 3$ VMs and the corresponding $p = 6$ MDs are used in the MOR formulation. The degrees of freedoms can reduce from 36 $(3 + 3 \times 11)$ to $k = 3 + m + p = 12$ $(3 + 3 + 6)$ in this example.

From Fig. (6) we can also see that if only the first 9 VMs are applied, the obtaiend solution is not as accurate as the one botaiend with a combination of VMs and MDs, even though the 4^{th} and 4^{th} VMs are axial modes. We can conclude that the MDs based MOR technique imore naturally capture the effects of geometrically nonlinearities as compared to a reduction based on *ad-hoc* selected vibration modes.

5 CONCLUSIONS

In this paper, a MOR approach is applied at the elastic displacement of a FFR, where the elastic deflection are derived from a moderate nonlinear kinematic expression for planar beam. In addition, a linear reduction basis of VMs is augmented with the relevant MDs and implemented in the FFR to extend to problems featuring arbitrary large rigid body rotations and finite elastic displacements. By comparing with the CFR formulation in two numerical examples, we can see that the proposed MOR method can not only yield good results, but also reduce the degrees of freedoms for the iteration calculation.

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