11th World Congress on Computational Mechanics (WCCM XI)
5th European Conference on Computational Mechanics (ECCM V)
6th European Conference on Computational Fluid Dynamics (ECFD VI)
E. Oñate, J. Oliver and A. Huerta (Eds)

THE CAUGHEY ABSORBING LAYER METHOD – IMPLEMENTATION AND VALIDATION IN ANSYS SOFTWARE

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Key Words: *Elastic Wave Propagation, Finite Element Method, Absorbing Boundary Layer, Spurious Reflections, Unbounded Domains.*

Abstract. The numerical analysis of the wave propagation problem, from elastic to electromagnetic waves, is often faced with the problem of dealing with unbounded media. Since the domain of finite-difference and finite element methods must be itself finite, various truncation techniques have been proposed over the last decades, such as absorbing boundary conditions (Lysmer and Kuhlemeyer [1]), infinite elements (Bettess [2]) and absorbing boundary layers (such as the Perfectly Matched Layer, or PML, introduced by Bérenger [3]).

In this paper, the Caughey Absorbing Layer Method (CALM), proposed by Semblat *et al.* [4], is implemented in the commercial finite element software Ansys, using an implicit dynamics formulation. It is tested for one- and two-dimensional problems and its efficiency is compared with that of the Lysmer-Kuhlemeyer absorbing boundaries. The dependency on material parameters, loss factor and load frequency is also tested.

To mitigate the problem of wave reflection at the interface between the medium of interest and the absorbing layer, different variations of damping along the layer's length are tested and their efficiency compared.

By analysing the maximum displacement and the L^2-norm of the displacement field, the implementation of the CALM in Ansys is shown to be effective at mitigating the problem of spurious wave reflection at the boundaries. Their performance is clearly superior to the Lysmer absorbing boundary conditions, but at a greater computational cost due to the additional degrees of freedom.

The quadratic variation of the Rayleigh damping has proved to be the most effective, and an estimate of the optimum loss factor as a function of the length of the layer in relation to the wavelength to absorb was proposed. Although the optimal damping is frequency dependent, it was shown to work well even if the frequency is overestimated or greatly underestimated.

1 INTRODUCTION

One of the most significant drawbacks in the numerical study of elastic wave propagation in solids, particularly when using the finite element (FE) method (Hughes [5]), is the difficulty to simulate a semi-infinite or unbounded domain.

This is the case in the analysis of soil vibrations: although the area of interest may be relatively small, it is neither confined to a closed space nor isolated from the surrounding soil. This means that simply modelling the area of interest without special considerations will result in spurious reflections of the elastic waves at the boundaries of the model. These reflections will become superimposed with the actual solution, making it inaccurate.

In an analytical analysis, it is common to admit the soil as a semi-infinite medium. This approach was employed by Boussinesq, who studied the stresses on the soil due to a static load (see Karol [6]). Since this is not possible in standard FE formulation, other approaches must be employed to simulate such an unbounded domain.

1.1 Infinite media truncation techniques

The main approaches to the problem of infinite and semi-infinite media truncation are: (i) local absorbing boundary conditions (Lysmer and Kuhlemeyer [1]); (ii) the boundary element method (Banerjee and Butterfield [7]); (iii) the infinite element method (Bettess [2]); (iv) absorbing layers, including Perfectly Matched Layers (PML, Bérenger [3]) and the Caughey Absorbing Layer Method (CALM, Semblat *et al.* [4]).

The local absorbing boundary conditions are among the simpler methods, but may lead to instabilities when there are discontinuities in the boundary (such as layers with different mechanical properties) and to rigid body motion. The rate of absorption depends on the angle of incidence of the wave, and is usually tuned to perfectly absorb only at a normal angle.

The boundary element method changes the nature of the numerical problem, from a volume discretisation to a boundary discretisation. Although it is very robust, the computational cost is much higher than traditional FE – for many problems where the surface to volume ratio is high, the boundary method may be less efficient than volume-discretisation methods (see Katsikadelis [8]).

The infinite element method is closer to the traditional FE approach. Essentially, it consists in modelling the interior domain with conventional finite elements, and using elements with a special shape function at the infinite boundary. These special shape functions grow without bound as the coordinate approaches infinity, therefore simulating an infinite element. Unfortunately, it is still not the norm for commercial FE software to include this formulation.

Absorbing layer methods have been widely used since the introduction of the PML by Bérenger [3], but the author reports previous work on other absorbing layers (see Holland and Williams [9]). In essence, the absorbing layer method applies a layer of material with some damping capability at the boundaries of the medium of interest. Waves behave normally inside the medium, but decay as they travel inside the absorbing layer, attenuating or preventing reflections at the boundaries of the model. However, some reflection is expected to occur at the interface between the normal medium and the absorbing layers.

The PML in particular can be implemented with a complex coordinate stretching (see Chew and Weedon [10]). The analytical formulation does not introduce reflections at the interface between the two materials (hence *perfectly matched*), but this property is partially lost after discretisation. The main drawback of the PML is that its implementation is not straightforward, particularly in the time domain – it requires a split-field formulation or convolution operations. This makes it difficult to use in FE commercial software.

The CALM, on the other hand, is not perfectly matched, but much simpler to implement. The absorbing layer has the mechanical properties of the medium of interest, but exhibits Caughey (or Rayleight) damping tuned to ensure that the rate of absorption for the desired frequency is above an arbitrary value. It has the advantage of being intrinsically multi-directional, unlike local absorbing boundaries and the PML. Since it only requires manipulation of the FE damping matrix, it is easily implemented in FE commercial software.

1.2 Case study

In the present paper, the authors implement the Rayleigh formulation of the CALM in the commercial FE software Ansys, using implicit time integration, for one- and two-dimensional problems. Its performance is compared with that of the Lysmer-Kuhlemeyer local absorbing boundary condition by means of the maximum relative displacement and the L^2-norm of the displacement field. The CALM is shown to be clearly superior, but at a greater computational cost due to the additional degrees of freedom.

To mitigate the reflections at the interface between the medium of interest and the absorbing layer, different variations of damping along the layer's length are tested and their efficiency compared. Unlike the results obtained by Semblat *et al.* [4], the higher order polynomial variation lead to better absorption in the case study. This is accordance with what was observed for the PML and other absorbing layers (see Festa and Nielsen [11] and Oskooi and Johnson [12]). As a result, an estimate of the optimum loss factor for the quadratic variation as a function of the length of the layer in relation to the wavelength is proposed.

The dependency on material parameters, loss factor and load frequency is also tested. Although the optimal damping is frequency dependent, the method is shown to work well even if the frequency is overestimated or greatly underestimated.

2 THE CAUGHEY ABSORBING LAYER METHOD

The method proposed by Jean-François Semblat *et al.* [4] was to employ absorbing layers with multi-direction attenuating properties. Simplicity of the formulation was an important factor for its development.

The chosen approach is to define a finite elastic medium with an absorbing layer at its boundaries. This absorbing layer is modelled with the same element technology and material properties as the interior of the medium of interest, but it includes damping properties that greatly reduced spurious wave reflections occur at the boundaries.

To ensure that the absorbing layer exhibits the desired properties, some considerations about the damping parameters to employ must be made.

2.1 The Rayleigh damping formulation

One of the simplest ways to define damping in FE analysis is to consider the Rayleigh formulation [13]: the damping matrix (**C**) is assumed to be a linear combination of the stiffness (**K**) and mass (**M**) matrixes:

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K} \quad (1)$$

α and β are known as the Rayleigh coefficients.

This approach is convenient because it can be easily computed, since FE methods already require the assembly of the mass and stiffness matrices.

It is well known [14] that the loss factor (η, the ratio of energy dissipated to the energy stored in the system for every oscillation) is approximately double the damping ratio (ζ, the ratio of the damping coefficient to the critical damping coefficient), which, for Rayleigh damping, relates to the frequency of excitation (ω) and to the Rayleigh coefficients:

$$\eta \approx 2\zeta = \alpha/\omega + \beta\omega \quad (2)$$

Therefore, the loss factor will be minimum when the frequency of excitation is

$$\frac{d\eta}{d\omega} = 0 \Leftrightarrow \omega = \sqrt{\frac{\alpha}{\beta}} \quad (3)$$

Since this is the minimum absorption possible, in theory if the Rayleigh coefficients is defined to get a desired loss factor η_{min} for a certain frequency ω_r, all excitations will be damped at least as much as the value defined. As the frequency of excitation moves away from ω_r, the more the damping will be felt.

The desired Rayleigh coefficients can be obtained from equations (2) and (3):

$$\begin{cases} \eta_{min} = \alpha/\omega + \beta\omega \\ \omega_r = \sqrt{\alpha/\beta} \end{cases} \Rightarrow \begin{cases} \alpha = \eta_{min}\omega_r/2 \\ \beta = \eta_{min}/2\omega_r \end{cases} \quad (4)$$

It is straightforward to apply these conclusions to the absorbing layers in study: to define Rayleigh damping for those layers, the coefficients to adopt can be calculated using equation (4), by defining the minimum loss factor as a desired value of absorption and the target frequency as the expected frequency of excitation for the problem at hand.

2.2 Limitations

In theory this methodology should work with any kind of dynamic analysis, both for time and frequency domain – the use of Rayleigh damping, a particular case of Caughey damping, ensures that the system has classical normal modes, as proved by Caughey and O'Kelly [15].

Preliminary tests have shown that the explicit central difference time integration method used by the LS-Dyna module of the Ansys software [16] requires an unreasonable computational cost to solve this problem – the time-step is five to six orders of magnitude lower than what is recommended for the same problem without the desired damping.

This limitation is confirmed by the authors in their own FE implementation, using both the LS-Dyna the Verlet [17] integration methods – either the time-step has to be extremely small, or the loss factor has to be reduced to a value that makes it unsuitable to the purpose at hand.

All models are therefore implemented in Ansys' implicit dynamics module. This choice comes with its own limitations: the pre-packaged Rayleigh damping implementation applies the same α coefficient to the whole model. To define the desired damping, six discrete damping elements have to be added for each quadrilateral element of the absorbing layer.

3 ONE-DIMENSIONAL ABSORBING LAYER

To test the effectiveness of the CALM, a one-dimensional problem is considered. The model is analogous to a rod with axial deformation only, but it is modelled as a mesh of quadrilateral plane stress elements, so the implementation can later be applied to a two-dimensional model. The degrees of freedom perpendicular to the rod's axis are constrained, and the objective is to simulate a semi-infinite rod (see Figure 1).

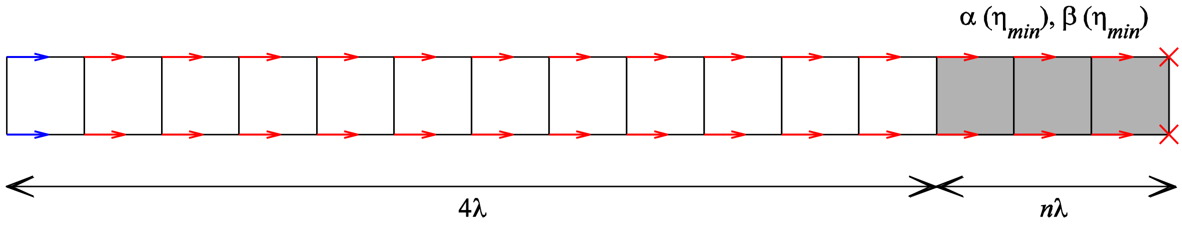


Figure 1: One dimensional model with CALM (based on Semblat *et al.* [4])

The length of the model is equal to four times the wavelength to absorb, plus the length of the absorbing layer, which is also a multiple of the wavelength.

The material properties chosen are those of a hard soil, but with the Poisson ratio equal to zero, to model a true uniaxial problem, where there are no transversal deformations. The Young modulus (E) is 200 MPa and the mass density (ρ) is 2000 kg/m^3. This values lead to a pressure wave's speed (c_P) of 316.228 m/s (see Telford *et al.* [18]).

The free-end of the rod is subjected to an impulse displacement, with its time history equal to a second-order Ricker Wavelet (see Hosken [19]):

$$R_2(t) = -U_0\left(2\pi^2(t-t_s)^2/t_p^2 - 1\right)e^{-\pi^2(t-t_s)^2/t_p^2} \tag{5}$$

where U_0 is the maximum amplitude of the wave, t is the time coordinate, t_p is the fundamental period of the wavelet and t_s is the time shift and t is the time coordinate.

Assuming a fundamental frequency ω = 500 rad/s, the period of the wavelet is

$$t_p = 2\pi/\omega \approx 0.012566\ s \tag{6}$$

The time shift was assumed to be equal to the fundamental period, and the maximum amplitude equal to one millimetre.

Knowing the fundamental frequency of the excitation, the wavelength of the pressure waves can be estimated as

$$\lambda = \lambda_P = 2\pi c_P/\omega \approx 3.974\text{ m} \tag{7}$$

The size of the elements was chosen to be λ/24, to reduce numerical wave dispersion.

3.1 Absorbing layer properties

The next step is to define the properties of the absorbing layer. Preliminary tests show that using a constant loss factor leads to significant reflection at the interface. This reflection can be controlled by reducing the loss factor, but this increases the reflections at the boundary of the model – there is a trade-off between having adequate absorption inside de layer and minimizing reflection at the interface. Even by tuning the loss factor to the best possible value, the maximum amplitude of the reflected waves is about 10% of the original impulse.

To circumvent this problem, the loss factor can be assumed to grow from 0 at the interface to a prescribed value η_{min} at the boundary. Various implementations are possible: (i) a single element with a continuous variation of the loss factor already factored in; (ii) a discretisation of the layer in multiple elements, each with continuous variation of the loss factor; (iii) a discretisation of both the geometry of the layer and of the variation of the loss factor (i.e., each element has constant loss factor across its volume, but different values for each element).

The first approach leads to reflections due to the sudden change in element size (see Bazant [20]). The two other options have virtually indistinguishable results, at least for the level of refinement used (24 elements per wavelength). Since the discretised variation of the loss factor is simpler to implement, it is used exclusively in the rest of the paper.

3.3 Parametric optimization of the loss factor

A parametric test is then performed to find the optimum loss factor as a function of the absorbing layer length (h_{abs}), which varies approximately from one wavelength to five wavelengths at increments. The loss factor at the end of the absorbing layer, η_{min}, varies in increments of 0.25. Of the various different variations of the loss factor along the absorbing layer tested, the best two are presented: linear and quadratic.

To assess the effectiveness of each implementation, objective measures of the reflection must be defined. Two different quality parameters are considered: the maximum displacement inside the medium of interest ($u(x,t)$) after the Ricker wavelet has left it ($t > t_w$), expressed as a percentage of the maximum applied displacement

$$u_{\max} = \max_{x,t>t_w}\left(u\left(x,t\right)\right) \Big/ U_0 \tag{8}$$

and the time integration of the L^2-norm of the displacement field inside the medium of interest, for $t > t_w$, as a percentage of the maximum value of the L^2-norm of the displacement field inside the medium of interest for a simulation on a long model ($u_\infty(x,t)$). To preserve dimensional consistency, the numerator is normalized by dividing it by the time interval:

$$\mathrm{L}_t^2 = \left(\int_{t_w}^{t_f} \mathrm{L}^2\left(u\left(x,t\right)\right) dt \Big/ \left(t_f - t_w\right) \right) \square \Big/ \max\left(\mathrm{L}^2\left(u_\infty\left(x,t\right)\right)\right) \tag{9}$$

The two quality parameters are computed for each combination of loss factor, variation and absorbing layer length. The best value of each as a function of the layer length is presented in Figure 2. It can be seen that the amplitude of the reflected waves decreases asymptotically with the layer's length: the longer the path of absorption, the smoother the variation of the damping, and therefore less reflections occur at the interface and between elements.

It is also clear that the quadratic variation of the loss factor leads to better results than the linear one. For this reason, the quadratic variation is used from here on. However, it should be noted that the difference is not very significant, and the linear variation is a valid approach.

Previous works (Festa and Nielsen [11], Oskooi and Johnson [12]) have reported that high order polynomial variation of the damping (both for PML and other absorbing layers) leads to better wave absorption. Future work will test if that is also the case for the CALM.

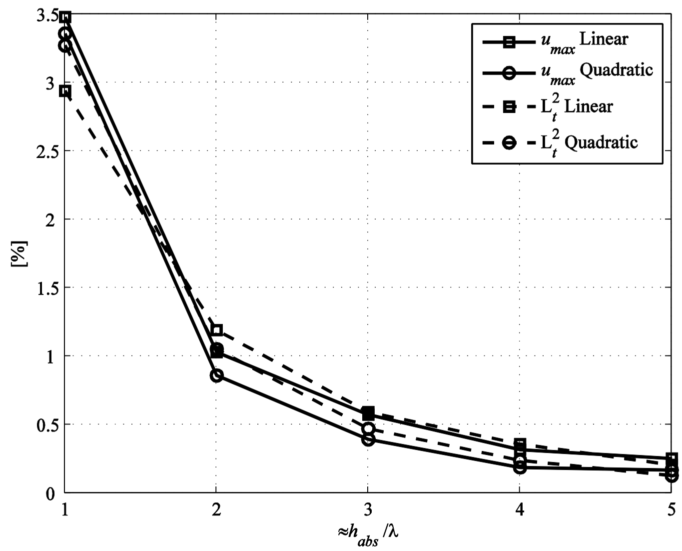


Figure 2: Best value of the quality parameters as a function of the layer's length

3.4 Optimum loss factor as a function of the layer's length

Having the optimum values of the loss factor as a function of the layer's length in proportion to the wavelength allows generalization of the obtained results to other material proprieties. Since the properties of the material are already taken into account in the calculation of the wave speed (and therefore the wavelength), it seems reasonable to expect the optimum loss factor to be independent of the material parameters.

Since the L_t^2 quality parameter gives a more accurate impression of the total reflection of the incident waves, the loss factor considered to be optimum was the one that minimized this parameter. These values are presented in Table 1.

Table 1: Optimum loss factor as a function of the layer's length and estimated value

h_{abs} [m]	4	8	12	16	20
h_{abs}/λ	1.007	2.013	3.020	4.026	5.033
η_{min}	2.50	1.50	1.25	1.00	0.75
η_{min}^*	2.528	1.550	1.164	0.950	0.812

To express the optimum loss factor as a function of the layer's length, different regression techniques are tested. The best fit is the power law, which yields the following expression:

$$\eta_{min}^* = 2.540\left(h_{abs}/\lambda\right)^{-0.706} \tag{10}$$

Table 1 includes the estimated loss factor using equation (10).

The power law is a good approach from a theoretical point of view: as the length of the layer tends to zero, the loss factor needed to absorb the incident elastic waves grow to infinity; as the length grows to infinity, the loss factor diminishes until no damping is needed at all.

To test the applicability of the proposed formula, four different combinations of material properties were tested (including softer materials and different Poisson ratios), as well as a plane strain formulation. All results were very close to the ones obtained before.

3.5 Detuned absorbing layer

Since it is not always possible to clearly define the frequency content of the loads, or it may happen that a wide range of frequencies are relevant, it is important to test if misjudging the prevailing frequency does not lead to a drastic drop in the efficiency of the CALM.

To that effect, the value assumed as the load frequency for the absorbing layer (ω_r) was changed to take different values from the Ricker wavelet (ω). Figure 3 shows the maximum displacement as a function of the ratio of the load frequency to the layer frequency.

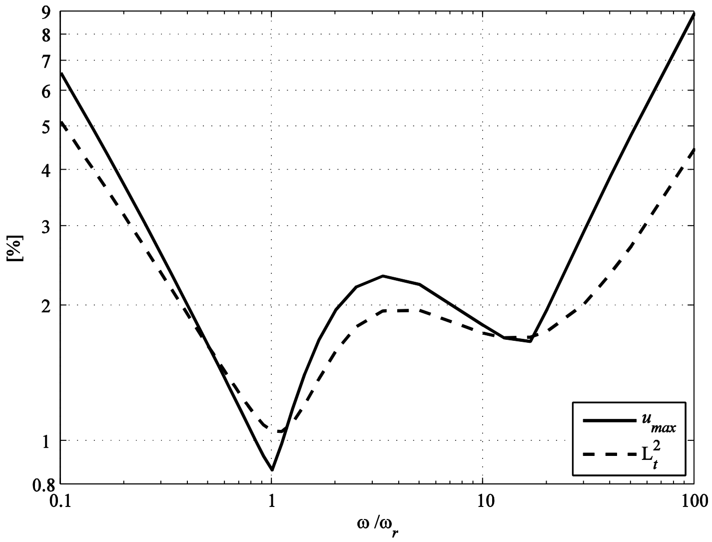


Figure 3: Amplitude of the reflected waves as a function of the ratio of the load to layer frequency

From the analysis of the results, one can confirm that the damping parameters defined in equation (4) lead to maximum absorption of the desired frequency.

Furthermore, it becomes evident that the efficiency of the absorbing layer suffers the most when the frequency of the load is lower than what the absorbing layer is prepared for. As will be shown in the next section, the absorbing layer is not as efficient in absorbing low frequency elastic waves, which explains these results. It is therefore preferable to underestimate the dominating frequency of the loads than overestimating it.

4 TWO-DIMENSIONAL ABSORBING LAYER

A two-dimensional plane stress problem is considered next. The model represents and elastic half-space with a horizontal free-surface where a vertical point load is applied. The intensity of the load as a function of time follows the Ricker wavelet (equation (5)). The maximum intensity is 1 kN, and the frequency is once again equal to 500 rad/s.

The material properties are the same as for the one-dimensional problem, except for the Poisson ratio, that is now $\nu = 0.3$. This means that a different pressure wave speed should be considered. It should also be noted that shear waves and Rayleigh waves are also to be expected, due to the two-dimensional nature of the problem and the existence of a interface (see Telford *et al.* [18]). Table 2 presents the wave speed and wavelength for the three types.

Table 2: Wave speed and wavelength for the three types of waves, considering ν = 0.3

Wave type	Pressure	Shear	Rayleigh
***c* [m/s]**	366.900	196.116	181.935
λ [m]	4.611	2.464	2.286

The larger wavelength is that of the pressure waves, and therefore the absorbing layer thickness will be measured as a multiple of this value.

Due to the symmetrical nature of the problem, one of the boundaries of the FE grid is only constrained on the horizontal direction. The top boundary is free, and the two remaining boundaries, where the absorbing layer ends, are fixed, as can be seen in Figure 4.

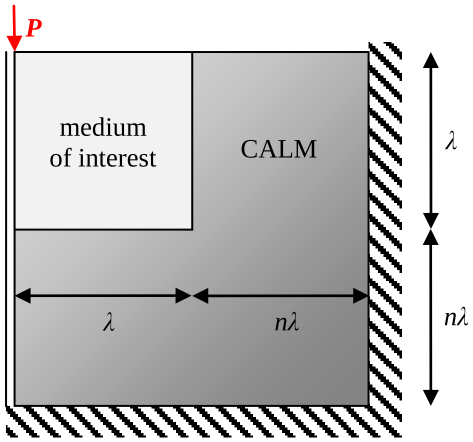


Figure 4: Two dimensional model with CALM (based on Semblat *et al.* [4])

The size of the medium of interest is restrained to only one multiple of the wavelength, due to the considerable increase in the computational cost of the problem compared to the one-dimensional model. The thickness of the absorbing layer varies from one to four wavelengths.

For comparison, a model with the same dimensions but no damping properties is tested – for a sufficiently large model, no reflections should be observed inside the medium of interest.

To verify how the CALM compares to other classical methods for grid truncation, the model was also implemented with Lysmer-Kuhlemeyer absorbing boundary conditions [1]. This method consists of adding a damper to each degree of freedom at the boundary with damping coefficient defined by the properties of the material and the size of the finite elements. To prevent rigid body motion, springs are also added at the bottom, a technique already employed by the authors with good results [21,22].

4.1 Results

The three types of model described above are tested. For the CALM, the loss factor at the end is the value estimated using equation (10) with the wavelength of the pressure waves.

The displacements are obtained at the surface of the medium, where the Rayleigh waves have a bigger influence (pressure and shear waves are important throughout the model).

The quality parameters are similar to the ones used before, with some adaptations. First, there is no longer a single direction of displacement, but two. Due to the contribution of the

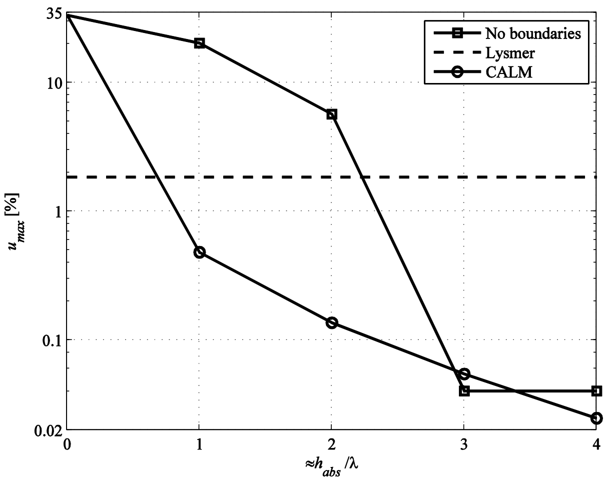
Rayleigh waves to the behaviour in the area of interest, it is important to also compare the total displacement of the nodes at the surface:

$$u_{total} = \sqrt{u_x^2 + u_y^2} \tag{11}$$

The other main difference from the one-dimensional case is that the maximum displacement after the waves have left the elastic medium is now expressed as a percentage of the maximum displacement in the largest model without absorbing layer:

$$u_{\max} = \max_{x,t>t_p}\left(u\left(x,t\right)\right) \Big/ \max_{x}\left(u_{\infty}\left(x,t\right)\right) \tag{12}$$

Results are presented in Figures 5 for total displacement (vertical and horizontal displacement show very similar trends). Although the Lysmer boundaries provide a good solution, the CALM approach leads to better results. As before, the thickness of the boundary has a very clear influence in the results. A layer with length equal to the wavelength leads to better results than the Lysmer boundaries, but by increasing the length, there is a considerable improvement in the absorption of the elastic waves. Naturally, the CALM approach has the disadvantage of requiring a bigger model, increasing considerably the solution time.



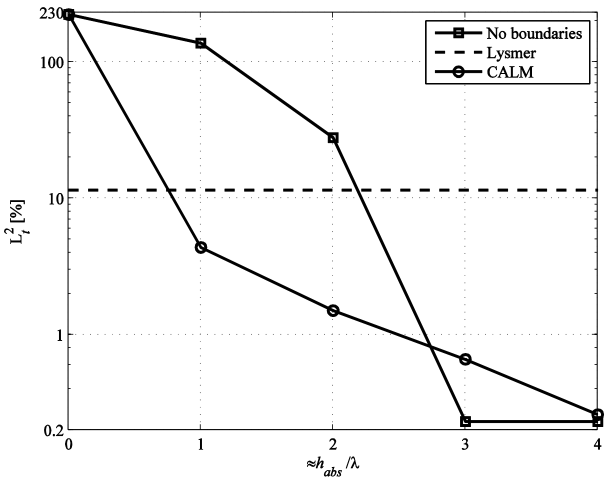


Figure 5: Maximum displacement of the reflected waves, two-dimensional model

Comparison of the vertical displacement at the point where the load is applied shows that the frequency of the reflected waves goes down with the increase of the absorbing layer's length, as does their amplitude.

4.2 Optimum loss factor

As before, a detailed optimization of the loss factor was performed, but only for the absorbing layer with thickness equal to the wavelength. Results are summarized in Table 9.

Table 3: Optimum loss factor values for the considered quality parameters, compared to the predicted value

	η_{min}*	u_{max} [%]	η_{min}	u_{max} [%]	η_{min}*	L_t^2 [%]	η_{min}	L_t^2 [%]
u_x	2.528	1.13%	1.9	0.87%	2.528	3.72%	1.8	2.22%

u_y	0.48%	2.3	0.48%	4.64%	2.6	4.64%
u_{total}	0.48%	2.35	0.48%	4.38%	2	4.26%

It is easy to verify that the optimum loss factor is not the same for the horizontal, vertical and total displacement, neither for all the parameters considered. The optimum value for the horizontal displacement is lower than the predicted value. The optimum value for the vertical and total displacement is closer to the prediction, even though the absorbing layer was optimized for the pressure waves which, at the surface of the elastic medium, are expected to travel in the horizontal direction. However, it is clear that the predicted values do not lead to results much worse than the optimum value, and from the figures it can be seen that exceeding the actual optimum value is preferable than to underestimate it.

5 CONCLUSIONS

The Caughey Absorbing Layer Method has been shown to work effectively to mitigate the problem of spurious wave reflections at the boundaries, for one and two-dimensional models. The absorbing layer not only greatly reduces the amplitude of the reflected waves, but also filters their high frequency content.

It was shown that the quadratic variation of the loss factor leads to better results than a constant value and linear variation.

An estimated value for the optimum loss factor as a function of the length of the layer in relation to the wavelength to absorb was proposed, and shown to lead to good results.

It is important to note that, although the CALM is more efficient than the Lysmer boundaries, it could not be implemented with standard explicit time integration, and it has the disadvantage of requiring a higher number of degrees of freedom that make the analysis slower when compared to absorbing boundary methods.

5.1 Future work

The next logical step is to test the two-dimensional model with plain strain elements. Since the results in the one-dimensional model with plain strain were very close to the ones obtained with plane stress, it is expected that the same will happen in two-dimensions.

Higher order polynomial variation of the damping of the absorbing layer should be tested, since previous works [11,12] have suggested that this lowers reflections at the interfaces.

The optimization process that led to equation (10), although successful, was relatively coarse, so a more refined analysis could lead to a more exact approximation. This optimization could also be done for the two-dimensional model, in which it was carried out only for a layer with length equal to the wavelength.

An important development to pursue would be to implement the finite elements with independent Rayleigh damping in the Ansys software, so the method can be applied to more complex problems, including three-dimensional models.

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