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DYNAMIC RESPONSE OF RAILWAY TRACKS IN TUNNEL

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Abstract. Periodically supported beams subjected to a moving load are often used for modeling the railway dynamics and analytical solutions have been developed for such modeling [3, 4]. More complex models can be constructed by including supports with damping or non-linear stiffness elements. This study deals with the dynamical modeling of non-ballasted railways, especially railways in tunnels. The model is developed as a dynamical system of multi-degree of freedom. Under the periodic assumption on the reaction force of the supports, the equation of motion for a periodically supported beam subjected to a moving load has been written. Then the Fourier transform has been used to solve this equation in case of damped supports. Analytical solutions have been established for the motion of the wheel and rail and also for the reaction force of the supports. The analytical solutions have been compared with in situ experimental measurements. The comparison shows that the theoretical results agree well with the measured results if damped supports are included in the model.

1 INTRODUCTION

In dynamic analysis of railway tracks, the analytical models of periodically supported beams have been often used to approximate the response of the system. In 2002, V.H. Nguyen [2] developed an analytical model by considering the railway as a continuously supported beam and by neglecting the mass of supports. In order to take account the vertical displacement of non-continued supports, X.Sheng and all. [3] used Fourier series while G. Bonnet and R. Lassoused [4] applied the superposition of Bloch waves. These methods are difficult to use if the mechanical behaviour of railway supports is more

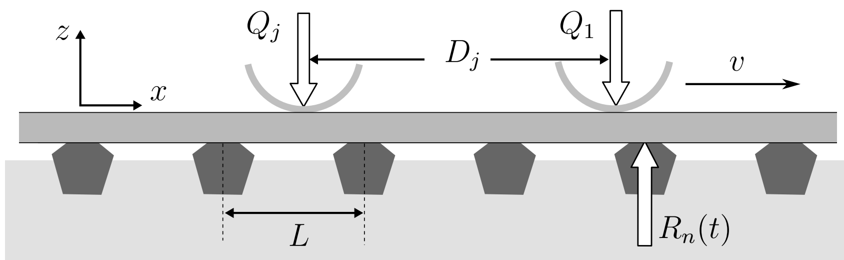


Figure 1: Forces applied on a periodically supported beam

complex (ex. hyperelastic or non-linear). In this paper, we will establish a general relation between displacements and forces of railway supports. This relation holds for periodically supported beams with any kind of supports and platform behaviour (ballast or non-ballast). Then we will apply this relation to the case of linear supports to find out a response of the system and a validation with experiment results.

2 AN ANALYTICAL MODEL FOR PERIODICALLY SUPPORTED BEAM

In this study, we consider a general case of periodically supported beam. It means that the supports have the same mechanical constitutive law and they are distributed periodically, separated by a same length L as shown in figure 1.

The beam is subjected to moving forces Q_j characterized by the distance to the first moving force D_j ($j = 1..K$ with K is the number of moving forces). The reaction force of support at the coordinate $x = nL$ (with $n = -\infty..\infty$) is denoted by $R_n(t)$. The total force is calculated as following:

$$F(x,t) = \sum_{n=-\infty}^{\infty} R_n(t)\delta(x - nL) - \sum_{j=1}^{K} Q_j \delta(x + D_j - vt) \tag{1}$$

For the Euler-Bernoulli beam, the vertical displacement $w_r(x,t)$ of the beam under the force $F(x,t)$ is solution of the following dynamic equation:

$$EI\frac{\partial^4 w_r(x,t)}{\partial x^4} + \rho S \frac{\partial^2 w_r(x,t)}{\partial t^2} - F(x,t) = 0 \tag{2}$$

where ρ and E are density and Young's modulus of the beam.

The solution of equations (1) and (2) with initial conditions describe the dynamics of the beam with multi-degree of freedom. In general, these equations can not be solved analytically. However, we can find a general solution when the response of the system is stationary by using the following hypothesis:

Hypothesis: *When the dynamic response of a periodically supported beam is stationary, the reaction forces of all supports are described by a same function but with a delay equal to the time for a moving load from a support to an other.*

In other words, the reaction force repeats when a moving force passes from one to an other support: $R_n(t) = R(t - \frac{nL}{v})$ where $R(t)$ is the reaction force of the support at the origin. Thus:

$$F(x,t) = \sum_{n=-\infty}^{\infty} R(t - \frac{x}{v})\delta(x - nL) - \sum_{j=1}^{K} Q_j \delta(x + D_j - vt) \tag{3}$$

By injecting this expression of the total force in equation (2) and by taking the Fourier transform, we obtain:

$$EI\frac{\partial^4 \hat{w}_r(x,\omega)}{\partial x^4} - \rho S\omega^2 \hat{w}_r(x,\omega) + \sum_{j=1}^{K} \frac{Q_j}{v} e^{-i\frac{\omega}{v}(x+D_j)} - \hat{R}(\omega) \sum_{n=-\infty}^{\infty} e^{-i\frac{\omega}{v}x}\delta(x - nL) = 0 \tag{4}$$

where $\hat{w}_r(x,\omega)$ and $\hat{R}(\omega)$ are Fourier transforms of $w_r(x,t)$ and $R(t)$ respectively.

By taking the Fourier transform of (4) with regard to the variable x, we obtain:

$$(EI\lambda^4 - \rho S\omega^2)\Pi(\lambda,\omega) + 2\pi\delta(\lambda + \frac{\omega}{v}) \sum_{j=1}^{K} \frac{Q_j}{v} e^{-i\frac{\omega}{v}D_j} - \hat{R}(\omega) \sum_{n=-\infty}^{\infty} e^{-i(\lambda+\frac{\omega}{v})nL} = 0 \tag{5}$$

where $\Pi(\lambda,\omega)$ is the Fourier transform of $\hat{w}_r(x,\omega)$ with regard to the variable x. The last term of (5) is a Dirac comb which has a following propriety:

$$\sum_{n=-\infty}^{\infty} e^{-i\left(\lambda+\frac{\omega}{v}\right)nL} = \frac{2\pi}{L} \sum_{n=-\infty}^{\infty} \delta\left(\lambda + \frac{\omega}{v} + \frac{2\pi}{L}n\right) \tag{6}$$

Then from equation (5), the expression of $\Pi(\lambda,\omega)$ can be obtained:

$$\Pi(\lambda,\omega) = \frac{2\pi}{EI(\lambda^4 - k_b^4)} \left[\frac{\hat{R}(\omega)}{L} \sum_{n=-\infty}^{\infty} \delta\left(\lambda + \frac{\omega}{v} + \frac{2\pi}{L}n\right) - \delta\left(\lambda + \frac{\omega}{v}\right) \sum_{j=1}^{K} \frac{Q_j}{v} e^{-i\frac{\omega}{v}D_j} \right] \tag{7}$$

where $k_b = \sqrt[4]{\frac{\rho S\omega^2}{EI}}$.

Finally, the expression of $\hat{w}_r(x,\omega)$ is deduced by making the inverse Fourier transform of $\Pi(\lambda,\omega)$:

$$\hat{w}_r(x,\omega) = \frac{\hat{R}(\omega)}{LEI} \sum_{n=-\infty}^{\infty} \frac{e^{-i\left(\frac{\omega}{v}+\frac{2\pi n}{L}\right)x}}{\left(\frac{\omega}{v} + \frac{2\pi n}{L}\right)^4 - k_b^4} - \sum_{j=1}^{K} \frac{Q_j e^{-i\frac{\omega}{v}(x+D_j)}}{vEI\left[\left(\frac{\omega}{v}\right)^4 - k_b^4\right]} \tag{8}$$

For instance, the vertical displacement on the top of a support ($x = 0$) is:

$$\hat{w}_r(0,\omega) = \hat{R}(\omega)\eta_E(\omega) - \sum_{j=1}^{K} \frac{Q_j e^{-i\omega\frac{D_j}{v}}}{vEI\left[\left(\frac{\omega}{v}\right)^4 - k_b^4\right]} \tag{9}$$

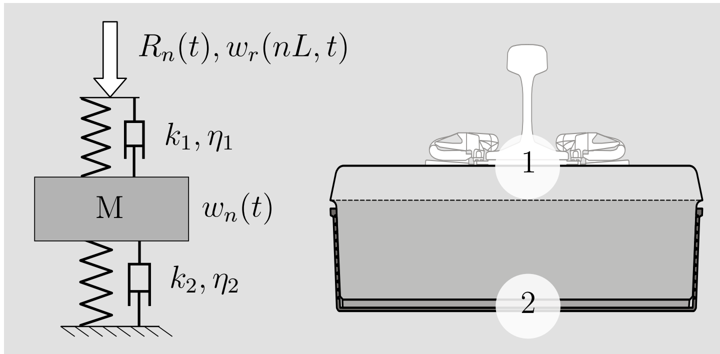


Figure 2: Model of non-ballasted railway support where (1) is rail pad and (2) is support pad

with:

$$\eta_E(\omega) = \frac{1}{LEI} \sum_{n=-\infty}^{\infty} \frac{1}{\left(\frac{\omega}{v} + \frac{2\pi n}{L}\right)^4 - k_b^4} \tag{10}$$

It is also possible to write $\eta_E(\omega)$ in the following way:

$$\eta_E(\omega) = \frac{1}{4k_b^3 EI} \left[\frac{\sin(Lk_b)}{\cos(Lk_b) - \cos(L\frac{\omega}{v})} - \frac{\sinh(Lk_b)}{\cosh(Lk_b) - \cos(L\frac{\omega}{v})} \right] \tag{11}$$

Equation (9) is a general relationship between the vertical displacement and the reaction force on the top of a support for a periodically supported beam under moving forces. The final solution of the displacement and the reaction force can be found by combining this relationship with the constitutive law of the support itself. In the next section, we will solve this equation in case of support with linear behaviour and apply it to a non-ballasted railway track.

3 APPLICATION IN CASE OF LINEAR BEHAVIOUR

In this section, supports with linear behavior will be considered. Damping will be taken into account and Kelvin-Voigt viscoelastic model will used as an example (see figure 2).

Let $w_n(t)$ denote the vertical displacement of block of the support n. The reaction force of support on the rail is given by:

$$R_n(t) = -\eta_1 \frac{d(w_r(nL,t) - w_n(t))}{dt} - k_1(w_r(nL,t) - w_n(t)) \tag{12}$$

where η_1, k_1 are the damping and spring coefficients of rail pad.

The displacement of a block is governed by the following equation:

$$M\frac{d^2 w_n(t)}{dt^2} + (\eta_1 + \eta_2)\frac{dw_n(t)}{dt} + (k_1 + k_2)w_n = \eta_1 \frac{dw_r(nL,t)}{dt} + k_1 w_r(nL,t) \tag{13}$$

where η_2, k_2 are damping and spring coefficients of the equivalent elastic pad under the block and M is mass of the block.

Making the Fourier transform of the last equation leads to the following results:

$$\begin{aligned} \hat{w}_n(\omega) &= \frac{A(\omega)}{A(\omega)+B(\omega)} \hat{w}_r(nL,\omega) \\ \hat{R}_n(\omega) &= -\theta(\omega)\hat{w}_r(nL,\omega) \end{aligned} \tag{14}$$

where:

$$\begin{aligned} A(\omega) &= i\omega\eta_1 + k_1 \\ B(\omega) &= -M\omega^2 + i\omega\eta_2 + k_2 \\ \theta(\omega) &= \frac{A(\omega)B(\omega)}{A(\omega)+B(\omega)} \end{aligned} \tag{15}$$

We see that the hypothesis $R_n(t) = R(t - \frac{nL}{v})$ is equivalent to the following expression:

$$\hat{R}_n(\omega) = e^{-i\omega\frac{nL}{v}} \hat{R}(\omega) \quad \forall n = -\infty..\infty \tag{16}$$

From equation (14), we show the last equation is satisfied if and only if:

$$\hat{w}_r(nL,\omega) = e^{-i\omega\frac{nL}{v}} \hat{w}_r(0,\omega) \quad \forall n = -\infty..\infty \tag{17}$$

In the next section, we will prove that the response of the system under the hypothesis satisfies equation (17).

3.1 Response of railway track and verification of hypothesis

The following solution is deduced from equations (9) and (14) (with $x = 0$):

$$\hat{w}_r(0,\omega) = \frac{-\sum_{j=1}^{K} Q_j e^{-i\frac{\omega}{v}D_j}}{vEI\left[1+\theta(\omega)\eta_E(\omega)\right]\left[\left(\frac{\omega}{v}\right)^4 - k_b^4\right]} \tag{18}$$

By injecting equations (9) and (14) into equation (8), we obtain:

$$\hat{w}_r(x,\omega) = \hat{w}_r(0,\omega)e^{-i\frac{\omega}{v}x}\left(1+\theta(\omega)\eta_E(\omega) - \frac{\theta(\omega)}{LEI}\sum_{n=-\infty}^{\infty}\frac{e^{-i\frac{2\pi n}{L}x}}{\left(\frac{\omega}{v}+\frac{2\pi n}{L}\right)^4 - k_b^4}\right) \tag{19}$$

By using equation (10), the following result can be obtained from the last equation:

$$\hat{w}_r(x,\omega) = \hat{w}_r(0,\omega)e^{-i\frac{\omega}{v}x}\left(1+\frac{\theta(\omega)}{LEI}\sum_{n=-\infty}^{\infty}\frac{1-e^{-i\frac{2\pi n}{L}x}}{\left(\frac{\omega}{v}+\frac{2\pi n}{L}\right)^4 - k_b^4}\right) \tag{20}$$

We see that equation (20) satisfies equation (17) when $x = nL$. Thus, the hypothesis is validated.

The motion of the rail can be calculated by taking the inverse Fourier transform of the last equation:

$$w_r(x,t) = w_r\left(0, \frac{x}{v} - t\right) + \sum_{n=-\infty}^{\infty} \frac{1 - e^{-i\frac{2\pi n}{L}x}}{2\pi LEI} \int_{-\infty}^{\infty} \frac{\theta(\omega)\hat{w}_r(0,\omega)e^{-i\omega\frac{x-vt}{v}}}{\left(\frac{\omega}{v} + \frac{2\pi n}{L}\right)^4 - k_b^4} d\omega \tag{21}$$

Then the reaction force and the displacement of the block can be calculated via equation (14). In a similar way, we could find solutions for railways with other linear supports.

3.2 Vertical vibration of wheels

From equation (21), the displacement of wheel - rail contact point at $x = vt$ can be deduced:

$$w_w(t) = w_r(0,0) + \sum_{n=-\infty}^{\infty} \frac{1 - e^{-i\frac{2\pi n}{L}vt}}{2\pi vEI} \int_{-\infty}^{\infty} \frac{\theta/LEI}{1+\theta\eta_E} \frac{\sum_{j=1}^{K} Q_j e^{-i\omega\frac{D_j}{v}} d\omega}{\left[\left(\frac{\omega}{v}\right)^4 - k_b^4\right]\left[\left(\frac{\omega}{v} + \frac{2\pi n}{L}\right)^4 - k_b^4\right]} \tag{22}$$

This displacement is a periodical motion of frequency $f = \frac{v}{L}$ described by the second term of equation (22). The amplitude A_0 of this motion can be obtained at the middle point between two supports ($t = \frac{L}{2v}$).

$$A_0 = \frac{1}{2\pi vEI} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1 - e^{-i\pi n}}{\left[\left(\frac{\omega}{v} + \frac{2\pi n}{L}\right)^4 - k_b^4\right]} \frac{\sum_{j=1}^{K} Q_j e^{-i\omega\frac{D_j}{v}}}{(1+\theta\eta_E)\left[\left(\frac{\omega}{v}\right)^4 - k_b^4\right]} d\omega \tag{23}$$

Let $\tilde{\eta}_E(\omega)$ define by:

$$\tilde{\eta}_E(\omega) = \sum_{n=-\infty}^{\infty} \frac{1 - e^{-i\pi n}}{\left[\left(\frac{\omega}{v} + \frac{2\pi n}{L}\right)^4 - k_b^4\right]}$$

The amplitude A_0 of the motion of wheel can be reduced from equation (23) as following:

$$A_0 = \frac{1}{2\pi vEI} \int_{-\infty}^{\infty} \frac{\theta\tilde{\eta}_E}{1+\theta\eta_E} \frac{\sum_{j=1}^{K} Q_j e^{-i\omega\frac{D_j}{v}}}{\left[\left(\frac{\omega}{v}\right)^4 - k_b^4\right]} d\omega \tag{24}$$

where $\tilde{\eta}_E(\omega)$ can be also written in the following form:

$$\tilde{\eta}_E(\omega) = \frac{1}{4k_b^3 EI}\left[\frac{\sin(\frac{Lk_b}{2})}{\cos(\frac{Lk_b}{2}) + \cos(\frac{L\omega}{2v})} - \frac{\sinh(\frac{Lk_b}{2})}{\cosh(\frac{Lk_b}{2}) + \cos(\frac{L\omega}{2v})}\right] \tag{25}$$

3.3 Approximation of the railway tracks response under the load of a train

The load by a train can be considered as a series of point load. The charges of each wagon on the train are often equal to the limit charge. It can be approximately by a series of identical charges ($Q_j = Q$) at a distance from the first wheel $D_j = jH$ (for front wheels) and $D_j = jH + D$ (for back wheels) where H and D are the length of each wagon and distance between the font and the back wheel of a boogie. Using the propriety of Dirac comb, one can write:

$$\sum_{j=-\infty}^{\infty} Q_j e^{-i\omega \frac{D_j}{v}} = \frac{2\pi v}{H} \sum_{n=-\infty}^{\infty} Q\delta\left(\omega + \frac{2\pi v}{H} j\right) \tag{26}$$

By injecting the last equation into equation (18) and making the inverse Fourier transform, the following analytical solution can be found for the response of the railway track:

$$w_r(0,t) = \frac{-2QLk_1k_2}{H(k_1+k_2)} - \sum_{j\neq 0} e^{i2\pi j \frac{v}{H} t} \left\{ \frac{\frac{Q}{HEI}\left(1 + e^{-i\omega \frac{D}{v}}\right)}{\left[\left(\frac{\omega}{v}\right)^4 - k_b^4\right]\left[1 + \theta\eta_E\right]} \right\}_{\omega = j\frac{2\pi v}{H}} \tag{27}$$

In the same way, we can get the displacement and the reaction force of a block by taking the inverse Fourier transform of $\hat{w}_0(\omega)$ and $\hat{R}(\omega)$ which are deduced from equations (14) and (18) as following:

$$w_0(t) = \frac{-2QL}{Hk_2} - \sum_{j\neq 0} e^{i2\pi j \frac{v}{H} t} \left\{ \frac{\frac{Q}{HEI}\left(1 + e^{-i\omega \frac{D}{v}}\right) A}{\left[\left(\frac{\omega}{v}\right)^4 - k_b^4\right]\left[A + B + AB\eta_E\right]} \right\}_{\omega = j\frac{2\pi v}{H}} \tag{28}$$

$$R(t) = \frac{2QL}{H} + \sum_{j\neq 0} e^{i2\pi j \frac{v}{H} t} \left\{ \frac{\frac{Q}{HEI}\left(1 + e^{-i\omega \frac{D}{v}}\right) \theta}{\left[\left(\frac{\omega}{v}\right)^4 - k_b^4\right]\left[1 + \theta\eta_E\right]} \right\}_{\omega = j\frac{2\pi v}{H}} \tag{29}$$

where $A(\omega), B(\omega)$ in equation (15).

These expressions show that the response of railway is a periodical motion with the frequency $f = \frac{v}{H}$.

3.4 Example

In this section we will compare the analytical solution with experimental results for a non-ballasted railway. The transverse force and rotating motion are ignored. The experiments were conducted in 2005 in the Channel tunnel. The parameters used in the calculation are given in table 1.

The displacement of rail track is computed by using equation (21). Figure 3 shows this displacement of the track in the length of wagon H when the first wheel is on the top of

Table 1: Parameters of a non-ballasted railway tracks

Charge of each wheels (Q)	kN	75
Train speed (v)	m/s	45
Length of boogie (D)	m	3
Length of wagon (H)	m	18
Rail mass (ρS)	kg/m	60
Rail stiffness (EI)	MNm^2	6.3
Block mass (M)	kg	100
Sleeper length (L)	m	0,6
Damping factor of rail pad (η_1)	MNs/m	1.0
Stiffness of rail pad (k_1)	MN/m	200
Damping coeff. under support (η_2)	MNs/m	0.35
Stiffness under support (k_2)	MN/m	18

a support. In other positions of the wheels, the displacement curves have similar form but they are below this curve (the amplitude of this change calculated by equation (24) is $A_0 = 0.0065$ mm). It is noticeable that the curve has a symmetrical form if damping is not included and the damping reduces the displacement at the first wheel position.

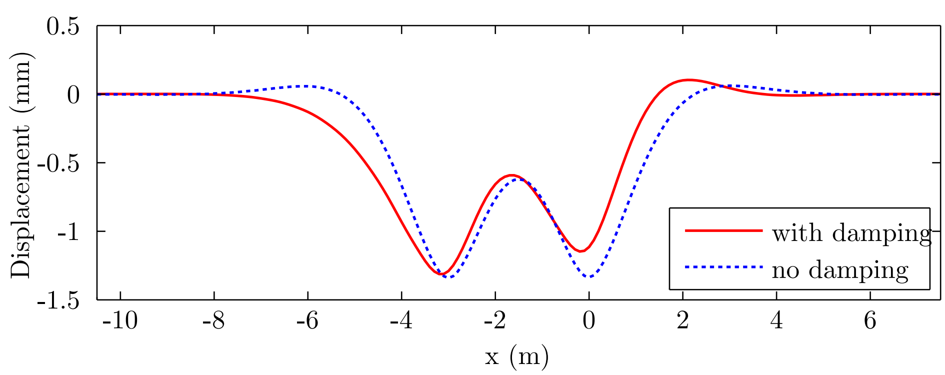
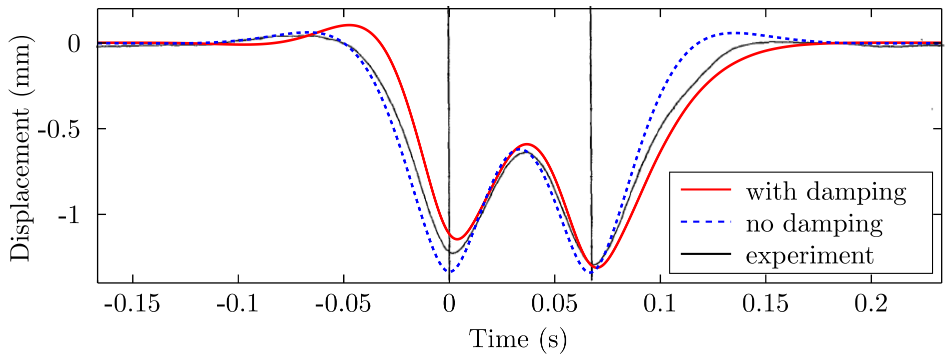


Figure 3: Vertical displacement of rail

The dynamic response of rail track was measured in 2005. Sensors were positioned under the rail at the rail-support contact point. The analytical solution and experimental results are compared in figure 4 and show good agreement for both displacement and reaction force if damping is included in the model.



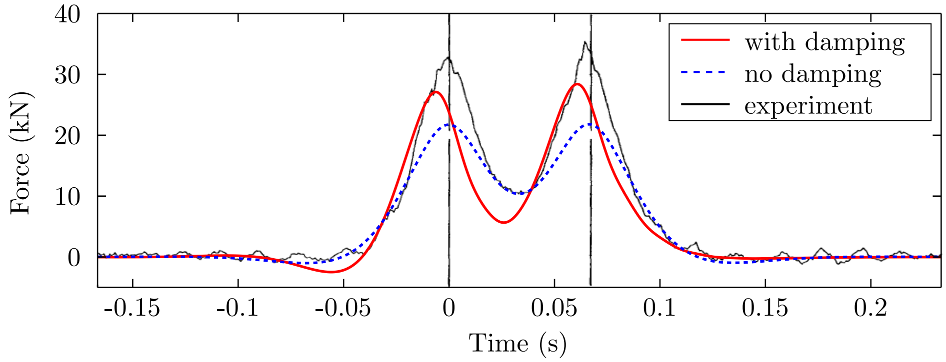


Figure 4: Displacement and reaction force at rail - support contact point

4 CONCLUSION

The railway response under moving trains has been studied by using a periodically supported beam model. First a general relationship between the rail track response, the support reaction forces and the moving loads has been established. Then the response of displacement and reaction forces have been calculated out for a railway with linear supports including damping. The analytical solution agrees well with the experimental results measured in the Channel tunnel. The model can be extended to Timoshenko beam to take into account shear forces. Extensions may also be made for including longitudinal forces or rotating motions of rail tracks and supports.

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