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PARALLEL SOLUTION OF ELASTOPLASTIC PROBLEMS WITH NUMERICAL EXPERIMENTS

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Abstract. In this paper we present efficient parallel implementation of solvers for elastoplastic problems based on the so-called Total Finite Element Tearing and Interconnecting (TFETI) domain decomposition method. We propose a set of numerical experiments to illustrate effectivity of the presented solvers. We consider an associated elastoplastic model with the von Misses plastic criterion and the isotropic or kinematic hardening law. The semi smooth Newton method is applied to solve this nonlinear system. Corresponding linearized problems arising in the Newton iterations are solved in parallel by the above mentioned TFETI domain decomposition method. The proposed TFETI based algorithm was implemented using PETSc and its performance is illustrated on a 3D elastoplastic benchmark. Numerical results for different scalability and mesh levels are presented and discussed.

1 INTRODUCTION

Behavior of solid continuum beyond reversible elastic deformations is described by elastoplastic processes. They are typically described by hysteresis models with a time memory [1]. The rigorous mathematical analysis of elastoplastic problems and numerical methods for their solution started to appear in the late 1970s and in the early 1980s by the work of C. Johnson [2], H. Matthies [3], V. Korneev and U. Langer [4], and others. Since then a lot of mathematical papers contributing to computational plasticity have been published. Mayor papers in this areas are: monographs by J. Simo and T. Hughes [5] , W. Han and B. Reddy [6], and the book of Blaheta [7].

In this paper we consider an associated elastoplasticity with the von Misses plastic criterion and the isotropic or the kinematic hardening law (see e.g. [6, 7, 8]). This approach together with the balance equation, the small strain assumption and a combination of the Dirichlet and Neumann boundary conditions leads to the solution of a nonlinear variational equation with respect to the primal unknown displacement. Such an equation can also be equivalently formulated as a minimization problem with a potential energy functional (see e.g. [9, 10]).

In this paper, we focus on the efficient parallel implementation of elastoplastic problems based on the Total-FETI (TFETI) [11] domain decomposition method. TFETI method is a modification of standard FETI method originally introduced by Farhat and Roux [12]. In FETI method, a body is partitioned into non-overlapping subdomains, an elliptic problem with Neumann boundary conditions is defined for each subdomain, and intersubdomain field continuity is enforced via Lagrange multipliers. Later, Farhat, Mandel, and Roux introduced a "natural coarse problem" which solution was implemented by auxiliary projectors leading to optimal algorithm [13]. In TFETI even the Dirichlet boundary conditions are enforced by Lagrange multipliers. By this all subdomain stiffness matrices are singular with a-priori known kernels which is a great advantage in the numerical solution.

By a finite element space discretization of the one time step problem, we obtain a system of nonlinear equations. The corresponding nonlinear operator is nondifferentiable but strongly semi smooth. Therefore, it is suitable to choose the semi smooth Newton method for solving the system since the strong semi smoothness together with other properties ensure local quadratic convergence. Semi smooth functions in finite dimensional spaces and the semi smooth Newton method were introduced by Qi and Sun [14]. In elastoplasticity, the semi smoothness was investigated for example in [9, 10, 15].

In each Newton iteration, it is necessary to solve the respective linearized problem. Different linear solvers including those based on multigrid have been successfully tested by Wieners [16]. Moreover, since the linear systems of equations corresponding to the elastic and elastoplastic problems are spectrally equivalent [17], all preconditioners for elastic problems can be applied to elastoplastic ones as well. A linear solver considered in this paper is based on a FETI type domain decomposition method enabling its efficient parallel implementation.

For numerical experiments we use our implementation of the TFETI algorithm in novel package FLLOP (FETI Light Layer On top of PETSc) [18] developed at IT4I. FLLOP is based on the PETSc framework [19]. PETSc (Portable, Extensible Toolkit for Scientific Computation) is a suite of data structures and routines for the parallel solution of scientific applications modeled by partial differential equations. PETSc contains built-in sequential direct solvers for in-place, symbolic, numeric LU and Cholesky factorizations of matrices. We use CG method with Lumped preconditioner (see [20]).

2 INITIAL VALUES OF ELASTOPLASTIC CONSTITUTIVE MODEL

Model of elastoplasticity is time dependent model where the history of loading is taken into account. In our implementation we consider associated elastoplasticity with von Mises plastic criterion and isotropic or kinematic hardening law (see e.g. [6, 7, 8]).

Let us consider a deformable body occupying a domain $\Omega \subset \mathbb{R}^3$ with a Lipschitz continuous boundary $\Gamma = \partial\Omega$. State of the body during the loading process could be described by the Cauchy stress tensor $\sigma \in S$, the displacement $u \in \mathbb{R}^3$ and the small strain tensor $\varepsilon \in S$. Here $S = \mathbb{R}^{3\times 3}_{sym}$ is the space of all symmetric second order tensors. More details can be found in [8].

The above mentioned variables depend on the spatial variable $x \in \Omega$ and the time variable $t \in [t_0, t^*]$. The equilibrium equation in the quasistatic case could be described as

$$-\mathrm{div}(\sigma(x,t)) = g(x,t) \quad \forall (x,t) \in \Omega \times [t_0, t^*]. \tag{1}$$

The reader can find the notation and description of the elastoplastic initial-value constitutive model for von Mises yield criterion with isotropic or kinematic hardening, e.g., in [22, 21, 20].

Let the boundary Γ be fixed on the part Γ_D that has a nonzero Lebesgue measure with respect to Γ, i.e. we prescribe the homogeneous Dirichlet boundary condition on Γ_D

$$u(x,t) = 0 \quad \forall (x,t) \in \Gamma_D \times [t_0, t^*]. \tag{2}$$

On the rest of the boundary $\Gamma_N = \Gamma \setminus \Gamma_D$ we prescribe the Neumann boundary conditions

$$\sigma(x,t)n(x) = F(x,t) \quad \forall (x,t) \in \Gamma_N \times [t_0, t^*], \tag{3}$$

where $n(x)$ denotes the exterior unit normal and $F(x,t)$ denotes the prescribed surface forces at the point $x \in \Gamma_N$ and the time $t \in [t_0, t^*]$. Similarly, we can consider other boundary conditions, e.g. symmetry and periodic conditions.

For a weak formulation of the investigated problems, it is sufficient to introduce the space of kinematically admissible displacements

$$V = \left\{ v \in [H^1(\Omega)]^3 : \ v = 0 \ \text{ on } \ \Gamma_D \right\}. \tag{4}$$

Then the conditions (1), (2), and (3) can be written in a weak form

$$\int_\Omega \langle \sigma, \varepsilon(v) \rangle_F dx = \int_\Omega g^T v dx + \int_{\Gamma_N} F^T v ds \quad \forall v \in V, \ \forall t \in [t_0, t^*], \tag{5}$$

where $\varepsilon(v)$ is a small strain tensor, $\langle .,. \rangle_F$ and $\|.\|_F$ denote the Frobenius scalar product and the corresponding norm on the space $S = \mathbb{R}^{3\times 3}_{sym}$, respectively. We assume that the functions σ, F, g are sufficiently smooth so that the integrals in (5) are correctly defined in the Lebesgue sense. The weak formulation of the corresponding elasto-plastic problem can be found in [6, 8]. The reader can find details about a time discretized model in [20, 21, 22].

3 FINITE ELEMENT DISCRETIZATION AND SEMISMOOTH NEWTON METHOD IN ELASTOPLASTICITY

In this section, approximation of time discretized elastoplastic problem by the finite element method and corresponding algebraic notation will be decribed. Later on the semismooth Newton method for the problem will be introduced as well.

3.1 Finite element discretization and algebraic formulation

Details of finite element implementation of elastoplastic problems can be found in [7, 9].

We consider a 3D polynomial domain Ω which will be discretized using linear simplex elements. The corresponding shape regular triangulation is denoted by $\mathcal{T}_h$. Thus the space V is approximated by its subspace V_h of piecewise linear and continuous functions. Therefore the spaces of the strains, the stress and the kinematic hardening are approximated by piecewise constant functions.

We define the approximated bilinear form $a_{k,h}(u_h)$ for $u_h \in V_h$ by

$$a_{k,h}(u_h)(w_h, v_h) = \int_\Omega \langle T^o_{k,h}(\varepsilon(u_h))\varepsilon(w_h), \varepsilon(v_h)\rangle dx, \quad v_h, w_h \in V_h. \tag{6}$$

Each function $v_h = (v_{h,1}, v_{h,2}, v_{h,3}) \in V_h$ can be represented by a vector

$$\boldsymbol{v} \in \mathbb{R}^n, \ \boldsymbol{v} := (v_{h,j}(x_i))_{i\in\{1,\dots,\mathcal{N}\}, j\in\{1,2,3\}},$$

where $\mathcal{N}$ denotes the number of vertices of the triangulation $\mathcal{T}_h$ and $n = 3\mathcal{N}$. The homogeneous Dirichlet boundary condition is represented by a restriction matrix $\boldsymbol{B}_U \in \mathbb{R}^{m\times n}$, i.e.

$$\boldsymbol{B}_U \boldsymbol{u} = \boldsymbol{o}. \tag{7}$$

Let $\boldsymbol{R}_T \in \mathbb{R}^{12\times n}$ be a restriction operator for a displacement vector $\boldsymbol{u} \in \mathbb{R}^n$ on a local element $T \in \mathcal{T}_h$, i.e.

$$\boldsymbol{u}_T = \boldsymbol{R}_T \boldsymbol{u}. \tag{8}$$

We denote by $\boldsymbol{u}_k$ and $\triangle \boldsymbol{u}_{k+1}$ the displacement vector and the searching displacement increment at the time step k, respectively. We denote the load vector representing the volume and surface forces F_k, g_k by $\boldsymbol{f}_k$ and its increment by $\triangle \boldsymbol{f}_k$.

Further, we use a vector representation in $\mathbb{R}^6$ of the stress and strain tensors that is typical for an implementation of elastic problem, i.e.

$$\boldsymbol{\sigma} = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{13})^T, \quad \boldsymbol{\varepsilon} = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{23}, 2\varepsilon_{13})^T. \tag{9}$$

Notice that the stress and strain vectors have different structures in comparison to the above tensor notation.

We define the non-linear operator $\boldsymbol{F}_k : \mathbb{R}^n \to \mathbb{R}^n$,

$$\boldsymbol{F}_k(\boldsymbol{v}) = \sum_{T\in\mathcal{T}_h} (\boldsymbol{T}_{k,T}(\boldsymbol{G}\boldsymbol{R}_T\boldsymbol{v}))^T \boldsymbol{G}\boldsymbol{R}_T, \quad \boldsymbol{v} \in \mathbb{R}^n, \tag{10}$$

which represents the left hand side, and the stiffness matrix $\boldsymbol{K}_k(\boldsymbol{v}) \in \mathbb{R}^{n\times n}$, $\boldsymbol{v} \in \mathbb{R}^n$,

$$\boldsymbol{K}_k(\boldsymbol{v}) = \sum_{T\in\mathcal{T}_h} \left(\boldsymbol{T}^o_{k,T}(\boldsymbol{G}\boldsymbol{R}_T\boldsymbol{v})\boldsymbol{G}\boldsymbol{R}_T\right)^T \boldsymbol{G}\boldsymbol{R}_T, \tag{11}$$

which represents the bilinear form $a_{k,h}(v_h)$. The reader can find the exact definition of variables $\boldsymbol{T}_{k,T}$, $\boldsymbol{T}^o_{k,T}$, $\boldsymbol{G}_T$, $\boldsymbol{R}_T$ in [20, 21, 22]. In particular, we denote the matrix $\boldsymbol{K}_k(\triangle \boldsymbol{u}_{k+1,i})$ briefly by $\boldsymbol{K}_{k,i}$, where the meaning of $\triangle \boldsymbol{u}_{k+1,i} \in \mathbb{R}^n$ will be explained in the next section.

Let

$$\boldsymbol{V} := \{\boldsymbol{v} \in \mathbb{R}^n | \boldsymbol{B}_U \boldsymbol{v} = \boldsymbol{o}\}.$$

Then by using (10), we can write: find $\triangle \boldsymbol{u}_{k+1} \in \boldsymbol{V}$ such that

$$\boldsymbol{v}^T \left(\boldsymbol{F}_k(\triangle \boldsymbol{u}_{k+1}) - \triangle \boldsymbol{f}_{k+1}\right) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{12}$$

where $\triangle \boldsymbol{f}_{k+1}$ is the increment of the load vector. Let $\tilde{\boldsymbol{u}}_k \in \mathbb{R}^{n-m}$, $\tilde{\boldsymbol{f}}_k \in \mathbb{R}^{n-m}$, $\tilde{\boldsymbol{K}}_{k,i} \in \mathbb{R}^{(n-m)\times(n-m)}$, and $\tilde{\boldsymbol{F}}_k : \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ denote the restrictions of $\boldsymbol{u}_k$, $\boldsymbol{f}_k$, $\boldsymbol{K}_{k,i}$, and $\boldsymbol{F}_k$ given by omitting the entries (degrees of freedom) corresponding to the prescribed Dirichlet boundary conditions. Then we can rewrite the equation (12) to the following system of non-linear equations:

$$\text{find } \triangle \boldsymbol{u}_{k+1} \in \boldsymbol{V} : \quad \tilde{\boldsymbol{F}}_k(\triangle \tilde{\boldsymbol{u}}_{k+1}) = \triangle \tilde{\boldsymbol{f}}_{k+1}. \tag{13}$$

3.2 Semismooth Newton method for one time step problem

The non-linear system of equations (13) is solved by the semismooth Newton method (see e.g. [14]). The corresponding algorithm is following:

Algorithm 1 (Semismooth Newton method)

1: *initialization:* $\triangle \boldsymbol{u}_{k,0} = \boldsymbol{o}$
2: ***for*** $i = 0, 1, 2, \ldots$ ***do***
3: *find* $\boldsymbol{\delta u}_i \in \boldsymbol{V}$*:* $\tilde{\boldsymbol{K}}_{k,i} \boldsymbol{\delta} \tilde{\boldsymbol{u}}_i = \triangle \tilde{\boldsymbol{f}}_{k+1} - \tilde{\boldsymbol{F}}_k(\triangle \tilde{\boldsymbol{u}}_{k,i})$
4: *compute* $\triangle \boldsymbol{u}_{k,i+1} = \triangle \boldsymbol{u}_{k,i} + \boldsymbol{\delta u}_i$
5: **if** $\|\triangle \boldsymbol{u}_{k,i+1} - \triangle \boldsymbol{u}_{k,i}\| / (\|\triangle \boldsymbol{u}_{k,i+1}\| + \|\triangle \boldsymbol{u}_{k,i}\|) \leq \epsilon_{Newton}$ **then stop**
6: ***end for***
7: *set* $\triangle \boldsymbol{u}_{k+1} = \triangle \boldsymbol{u}_{k,i+1}$

Here $\epsilon_{Newton} > 0$ is the relative stopping tolerance and $\boldsymbol{\delta} \tilde{\boldsymbol{u}}_{\boldsymbol{i}} \in \mathbb{R}^{n-m}$ is the restriction of $\boldsymbol{\delta u_i}$ given by omitting the entries (degrees of freedom) corresponding to the prescribed Dirichlet boundary conditions. The systems of linear equations, which are considered in each Newton iteration, will be solved by the TFETI method introduced in the next section.

4 TFETI METHOD

In the TFETI domain decomposition method [11], we tear the body from the part of the boundary with the Dirichlet boundary condition, decompose it into subdomains, assign each subdomain with a unique number, and introduce new "gluing" conditions on the artificial intersubdomain boundaries and on the boundaries with imposed Dirichlet condition. In particular, the domain $\Omega_h \equiv \Omega$ is decomposed into a system of s disjoint polyhedral subdomains $\Omega^p \subset \Omega$, $p = 1, 2, \ldots, s$.

Let N_p denote the primal dimension (number of degrees of freedom of the decomposed problem) and N_c the number of cores being at disposal for our computation. The displacement vector $\mathbf{v} \in \mathbb{R}^{N_p}$ has the following structure

$$\mathbf{v} = \left(\mathbf{v}_1^T, \mathbf{v}_2^T, \ldots, \mathbf{v}_s^T\right)^T,$$

where $\mathbf{v}_p$ denotes the displacement vector on the subdomain Ω^p. Let us define the space

$$\mathcal{V} = \left\{\mathbf{v} \in \mathbb{R}^{N_p} \mid \mathbf{B}\mathbf{v} = \mathbf{o}\right\}. \tag{14}$$

Here the equality constraint matrix $\mathbf{B} \in \mathbb{R}^{N_d \times N_p}$ represents the gluing conditions between neighbouring subdomains and the Dirichlet boundary conditions. It has orthonormal rows. By $N_d = N_{d,G} + N_{d,D}$ we denote the dimension of the dual problem where $N_{d,G}$ is the total number of gluing conditions and $N_{d,D}$ is the total number of Dirichlet boundary conditions.

Note that the structure of the global matrix $\mathbf{K}_{k,i} \in \mathbb{R}^{N_p \times N_p}$, $k = 1, 2, \ldots, N$, $i = 0, 1, 2, \ldots$, is block diagonal. Its diagonal is composed of the subdomain stiffness matrices $\boldsymbol{K}_{k,i}^p$ defined for each Ω^p, $p = 1, \ldots, s$. Therefore, we can solve the inner problem (13) in the same way as a problem of linear elasticity, see e.g. [23].

In this paper we use the TFETI domain decomposition method to solve (13). For more details see e.g. [20] and [22]. The method is based on enforcing the "gluing" and Dirichlet conditions by Lagrange multipliers $\boldsymbol{\lambda} \in \mathbb{R}^{N_d}$. Then the Lagrangian associated with problem (13) reads

$$L_{k,i}(\mathbf{v}, \boldsymbol{\lambda}) = \frac{1}{2}\mathbf{v}^T\mathbf{K}_{k,i}\mathbf{v} - \triangle\mathbf{f}_k^T\mathbf{v} + \boldsymbol{\lambda}^T\mathbf{B}\mathbf{v}, \quad \mathbf{v} \in \mathbb{R}^{N_p},\ \boldsymbol{\lambda} \in \mathbb{R}^{N_d}. \tag{15}$$

Using the convexity of the cost function and constraints, we can use the classical duality theory to reformulate the problem (13) to get

$$J_k(\delta\mathbf{u}^i) = \min_{\mathbf{v}\in\mathcal{V}} J_{k,i}(\mathbf{v}) = \min_{\mathbf{v}\in\mathbb{R}^{N_d}} \sup_{\boldsymbol{\lambda}\in\mathbb{R}^{N_d}} L_{k,i}(\mathbf{v}, \boldsymbol{\lambda}) = \tag{16}$$

$$= \max_{\boldsymbol{\lambda}\in\mathbb{R}^{N_d}} \inf_{\mathbf{v}\in\mathbb{R}^{N_p}} L_{k,i}(\mathbf{v}, \boldsymbol{\lambda}) = \max_{\boldsymbol{\lambda}\in\mathbb{R}^{N_d}} \{-\boldsymbol{\Theta}_{k,i}(\boldsymbol{\lambda})\}, \tag{17}$$

with

$$\boldsymbol{\Theta}_{k,i}(\boldsymbol{\lambda}) = \begin{cases} \frac{1}{2}\boldsymbol{\lambda}^T\mathbf{B}\mathbf{K}_{k,i}^{\dagger}\mathbf{B}^T\boldsymbol{\lambda} - \boldsymbol{\lambda}^T\mathbf{B}\mathbf{K}_{k,i}^{\dagger}\triangle\mathbf{f}_k, & \mathbf{R}_k^T(\triangle\mathbf{f}_k - \mathbf{B}^T\boldsymbol{\lambda}) = \mathbf{o}, \\ +\infty, & \text{otherwise}, \end{cases}$$

where $\mathbf{K}_{k,i}^{\dagger}$ denotes a generalized inverse matrix satisfying $\mathbf{K}_{k,i}\mathbf{K}_{k,i}^{\dagger}\mathbf{K}_{k,i} = \mathbf{K}_{k,i}$ and $\mathbf{R}_{k,i} \in \mathbb{R}^{N_p \times N_n}$ represents the null space of $\mathbf{K}_{k,i}$. More details about implementation of $\mathbf{B}\mathbf{K}_{k,i}^{\dagger}\mathbf{B}^T$ can be found in [13]. Thus the corresponding dual problem has the form

$$\text{find } \boldsymbol{\lambda}^i \in \mathbb{R}^{N_d}: \quad \boldsymbol{\Theta}_{k,i}(\boldsymbol{\lambda}^i) \leq \boldsymbol{\Theta}_{k,i}(\boldsymbol{\lambda}) \quad \forall \boldsymbol{\lambda} \in \mathbb{R}^{N_d}. \tag{18}$$

In the following part we focus on the efficient solving of linearized problem in one Newton iteration.

Let us establish the following notation

$$\mathbf{F} = \mathbf{B}\mathbf{K}_{k,i}^{\dagger}\mathbf{B}^T,\ \mathbf{H} = \mathbf{R}_{k,i}^T\mathbf{B}^T,\ \mathbf{d} = \mathbf{B}\mathbf{K}_{k,i}^{\dagger}\triangle\mathbf{f}_k,\ \mathbf{e} = \mathbf{R}_{k,i}^T\triangle\mathbf{f}_k,$$

where $\mathbf{H}$ is the natural coarse space matrix and $\mathbf{R}_{k,i}$ is block diagonal matrix containing basis vectors of the kernel of $\mathbf{K}_{k,i}$. We obtain a new minimization problem

$$\min \frac{1}{2}\boldsymbol{\lambda}^T\mathbf{F}\boldsymbol{\lambda} - \boldsymbol{\lambda}^T\mathbf{d} \quad \text{s.t.} \quad \mathbf{H}\boldsymbol{\lambda} = \mathbf{e}. \tag{19}$$

Further, the equality constraints $\mathbf{H}\boldsymbol{\lambda} = \mathbf{e}$ can be homogenized to $\mathbf{H}\boldsymbol{\lambda} = \mathbf{o}$ by splitting $\boldsymbol{\lambda}$ into $\boldsymbol{\mu} + \tilde{\boldsymbol{\lambda}}$ where $\tilde{\boldsymbol{\lambda}}$ satisfies $\mathbf{H}\tilde{\boldsymbol{\lambda}} = \mathbf{e}$ (e.g. $\tilde{\boldsymbol{\lambda}} = \mathbf{H}^T(\mathbf{G}\mathbf{G}^T)^{-1}\mathbf{e}$) which implies $\boldsymbol{\mu} \in \operatorname{Ker}\mathbf{H}$. We then substite $\boldsymbol{\lambda} = \boldsymbol{\mu} + \tilde{\boldsymbol{\lambda}}$, omit terms without $\boldsymbol{\mu}$, minimize over $\boldsymbol{\mu}$ and add $\tilde{\boldsymbol{\lambda}}$ to $\boldsymbol{\mu}$.

Finally, the equality constraints $\mathbf{H}\boldsymbol{\lambda} = \mathbf{o}$ can be enforced by the projector $\mathbf{P} = \mathbf{I} - \mathbf{Q}$ onto the null space of $\mathbf{H}$, where $\mathbf{Q} = \mathbf{H}^T(\mathbf{G}\mathbf{G}^T)^{-1}\mathbf{H}$ is the projector onto the image space of $\mathbf{H}^T$ ($\operatorname{Im}\mathbf{Q} = \operatorname{Im}\mathbf{H}^T$ and $\operatorname{Im}\mathbf{P} = \operatorname{Ker}\mathbf{H}$). We call the action of $(\mathbf{G}\mathbf{G}^T)^{-1}$ the *coarse problem* (CP) of FETI. It holds that $\mathbf{P}\boldsymbol{\mu} = \boldsymbol{\mu}$ because $\boldsymbol{\mu} \in \operatorname{Ker}\mathbf{H}$, so the final problem reads

$$\mathbf{P}\mathbf{F}\boldsymbol{\mu} = \mathbf{P}(\mathbf{d} - \mathbf{F}\tilde{\boldsymbol{\lambda}}). \tag{20}$$

The problem (20) may be solved efficiently by the CG method thanks to the classical estimate by Farhat, Mandel and Roux.

5 NUMERICAL EXPERIMENTS

All numerical experiments were performed using Anselm supercomputer located at IT4Innovations, Czech Republic. The Anselm cluster consists of 209 compute nodes, totaling 3344 compute cores with 15 TB of RAM and giving a theoretical peak performance of over 94 Tflop/s. Each node is a powerful x86-64 computer, equipped with 16 cores, at least 64GB RAM, and 500GB harddrive. Nodes are interconnected by fully non-blocking fat-tree Infiniband network and equipped with Intel Sandy Bridge processors.

The performance of the proposed algorithm is demonstrated on an elastoplastic homogeneous brick of sizes $8 \times 4 \times 4$ which is subject to the zero Dirichlet conditions on the left lateral face and Neuman boundary condition in the form $g(t) = 150t$, $t \in [t_0, t^*]$, $t_0 = 0$, $t^* = 1$ on the part of upper face, see Figure 1.

The elastoplastic body Ω is made of homogeneous isotropic material with the parameters $E = 207$ GPa, $\nu = 0.29$, $\sigma_y = 450$ MPa, and $c_0 = 6,667$ MPa, $H_m = 10,000$ MPa

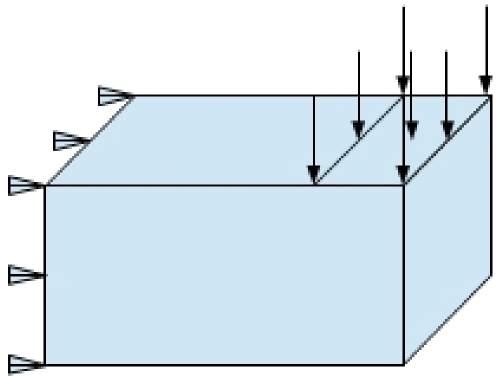

Figure 1: Geometry of the body

where E and ν are the Young modulus and the Poisson ratio, respectively. The value of variables c_0 and H_m are in relation $c_0 = \frac{2}{3}H_m$.

To test a numerical scalability of our TFETI based algorithm we consider two sets of subdomain discretizations. The domain Ω was partitioned into $2 \times 1 \times 1$, $4 \times 2 \times 2$, 8×4, and $16 \times 8 \times 8$ subdomains. In the case of the first discretization the number of primal variables per subdomain was $N_{p_1} = 3,993$, in the case of the second one $N_{p_2} = 27,783$. Therefore the total number of primal variables ranges from $7,986$ to $4,088,832$ and from $55,566$ to $28,449,792$, respectively.

The stopping tolerances of the Newton and the CG algorithms are

$$\epsilon_{Newton} = 10^{-4} \text{ and } \epsilon_{CG} = 10^{-7}. \tag{21}$$

In Tables 1 and 2, the number of subdomain, the number of primal and dual variables, the number of plastic elements, the number of Newton iterations, the total number of the CG iterations, and the total time are reported for each decomposition. Note, that the average number of CG iterations is kept approximately constant, according to the theory.

Distributions of the von Mises stress $\|\text{dev}(\sigma)\|_F$ and the total displacement $\|u\|$ at the final time t^* are depicted in Figures 2 and 3, respectively.

Comparing isotropic and kinematic hardening for various discretizations we see that the resulting number of plastic elements, solution time, and the average number of CG iterations are almost the same. The difference is less than 0.01% and is caused by the roundoff errors.

6 CONCLUSIONS

In this paper we presented a TFETI based parallel solver for problems of elastoplasticity with isotropic and kinematic hardening. We demonstrated a feasible numerical scalability and a reasonable weak parallel scalability up to 128 cores. The increase of computational time for $1,024$ cores could be explained by the increased dimension of the coarse problem

Problem setting				
No. of subdomains	2	16	128	1,024
No. of cores	2	16	128	1,024
No. of elements	12,000	96,000	768,000	6,144,000
Primal variables	7,986	63,888	511,104	4,088,832
Dual variables	726	10,968	107,664	939,552
Performace with isotropic hardening				
No. of plastic elems.	1,121	11,265	95,495	777,547
No. of Newton iters.	5	5	6	6
Total No. of CG iters.	168	265	489	611
Avg. No. of CG iters.	35	53	82	102
Avg. time/CG [s]	0.065	0.24	0.76	18.58
Avg. time/CG iter. [s]	0.002	0.004	0.009	0.182
Total time [s]	2.74	4.53	11.39	144.14
Performace with kinematic hardening				
No. of plastic elems.	1,121	11,265	95,495	777,546
No. of Newton iters.	5	5	6	6
Total No. of CG iters.	168	267	487	611
Avg. No. of CG iters.	34	53	81	102
Avg. time/CG [s]	0.063	0.24	0.73	18.73
Avg. time/CG iter. [s]	0.002	0.005	0.009	0.184
Total time [s]	2.78	4.58	11.28	145.42

Table 1: Results of the benchmark for one time-step with 3,993 primal variables per subdomain

and the use of default PETSc LU solver for its solution. This drawback can be overcome by using, e.g., distributed SuperLU or MUMPS solvers [24]. Usage of these two solvers is currently a work in progress and will be presented in a next paper.

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Problem setting				
No. of subdomains	2	16	128	1024
No. of cores	2	16	128	1024
No. of elements	96,000	768,000	6,144,000	49,152,000
Primal variables	55,566	444,528	3,556,224	28,449,792
Dual variables	2,646	41,088	406,944	3,565,632
Performace with isotropic hardening				
No. of plastic elems.	11,265	11,265	777,539	6,253,554
No. of Newton iters.	5	6	6	6
Total No. of CG iters.	219	425	659	846
Avg. No. of CG iters.	44	71	110	141
Avg. time/CG [s]	2.01	5.13	9.70	24.03
Avg. time/CG iter. [s]	0.045	0.072	0.088	0.170
Total time [s]	116.15	195.51	260.00	406.26
Performace with kinematic hardening				
No. of plastic elems.	11,265	11,265	777,536	6,253,562
No. of Newton iters.	5	6	6	6
Total No. of CG iters.	221	422	657	845
Avg. No. of CG iters.	44	70	110	141
Avg. time/CG [s]	2.03	5.15	9.81	23.78
Avg. time/CG iter. [s]	0.046	0.073	0.090	0.169
Total time [s]	116.55	194.03	263.71	410.96

Table 2: Results of the benchmark for one time-step with 27,783 primal variables per subdomain

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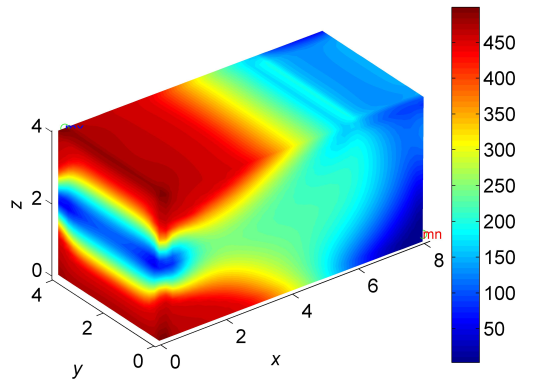

Figure 2: Distribution of von Mises stress $\|\mathrm{dev}(\sigma)\|_F$ at t_4

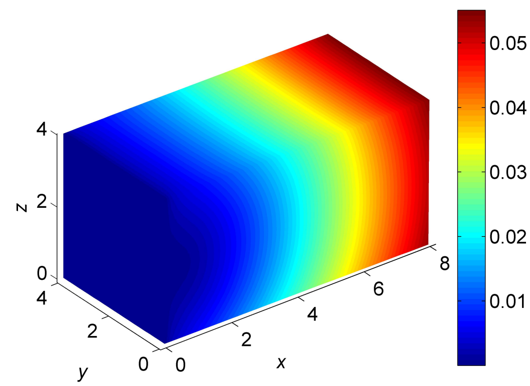

Figure 3: Total displacement $\|u\|$ at t_4

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