

# MULTISCALE ANALYSIS OF DAMAGE USING DUAL AND PRIMAL DOMAIN DECOMPOSITION TECHNIQUES

ORIOL LLOBERAS-VALLS<sup>1</sup>, FRANK P. X. EVERDIJ<sup>2</sup>,  
DANIEL J. RIXEN<sup>3</sup>, ANGELO SIMONE<sup>2</sup> AND LAMBERTUS J. SLUYS<sup>2</sup>

<sup>1</sup> International Center for Numerical Methods in Engineering (CIMNE)  
Campus Nord UPC, Edifici C-1, C/Jordi Girona 1-3, 08034 Barcelona, Spain  
olloberas@cimne.upc.edu

<sup>2</sup> Faculty of Civil Engineering and Geosciences  
Delft University of Technology  
P.O. Box 5048, 2600 GA Delft, The Netherlands

<sup>3</sup> Faculty of Mechanical Engineering  
Technische Universität München  
Boltzmannstr. 15, D - 85748 Garching, Germany

**Key words:** Multiscale analysis, dual and primal domain decomposition methods, strain localization, strain softening

**Abstract.** In this contribution, dual and primal domain decomposition techniques are studied for the multiscale analysis of failure in quasi-brittle materials. The multiscale strategy essentially consists in decomposing the structure into a number of non-overlapping domains and considering a refined spatial resolution where needed. In multiscale analysis of damage, the spatial refinement is performed where damage nucleation and propagation take place. The domain decomposition approach turns to be a computationally cheaper alternative to the direct numerical solution in which a fine scale model is considered throughout the complete sample. Dual and primal domain decomposition techniques are appropriate for such concurrent multiscale analyses and provide identical results. Parallel scalability of the multiscale analysis is studied using a moderate number of processors and a parallel direct solver for the system obtained through the assembly of all domains.

## 1 INTRODUCTION

Multiscale analysis of materials has gained momentum in the last years thanks to the breakthroughs in computer technology. Many new numerical multiscale techniques have recently emerged and the topic is in continuous evolution. An attempt to classify different

multiscale methods for the simulation of constitutive behaviour of complex materials has been done in [3] where the scale separation and structure of the computation serve as leading criteria for the classification. Accordingly one can distinguish between hierarchical and concurrent techniques depending on the scale separation, and decoupled, weakly coupled and strongly coupled micro-macro computations.

The use of domain decomposition methods [8] for multiscale analysis is a good example of concurrent strongly coupled techniques and has captured the attention of many researchers due to its flexibility, generality and adequacy for the solution of large structural problems. Examples of such approaches can be found in the work of Amini et al. [9] where a FETI-DP framework is employed as a basis for the multiscale analysis, or in Ladevèze et al. [10] and Guidault et al. [11] where a more complex interface between domains is employed.

The current contribution revises the main features of the multiscale framework based on dual domain decomposition methods [1] introduced in [3, 4], and presents the equivalent primal formulation assessed in multiscale analysis of non-linear softening materials.

## 2 DUAL AND PRIMAL MULTISCALE DOMAIN DECOMPOSITION

The main idea behind a multiscale domain decomposition strategy is to lower the burden of a full fine scale approach by decomposing a structure into a number of domains and consider a higher resolution where needed. The kernel of such an approach resides in the strategy to link the different domains and the type of compatibility between coarse and fine scale substructures.

### 2.1 General formulation

Consider a homogeneous body  $\Omega$  decomposed into a number  $N_s$  of domains  $\Omega^{(s)}$  (Figure 1). In a dual assembly, the interface compatibility is enforced through the signed Boolean matrices  $\mathbf{B}^{(s)}$  as

$$\sum_{s=1}^{N_s} \mathbf{B}^{(s)} \mathbf{u}^{(s)} = \mathbf{0}. \quad (1)$$

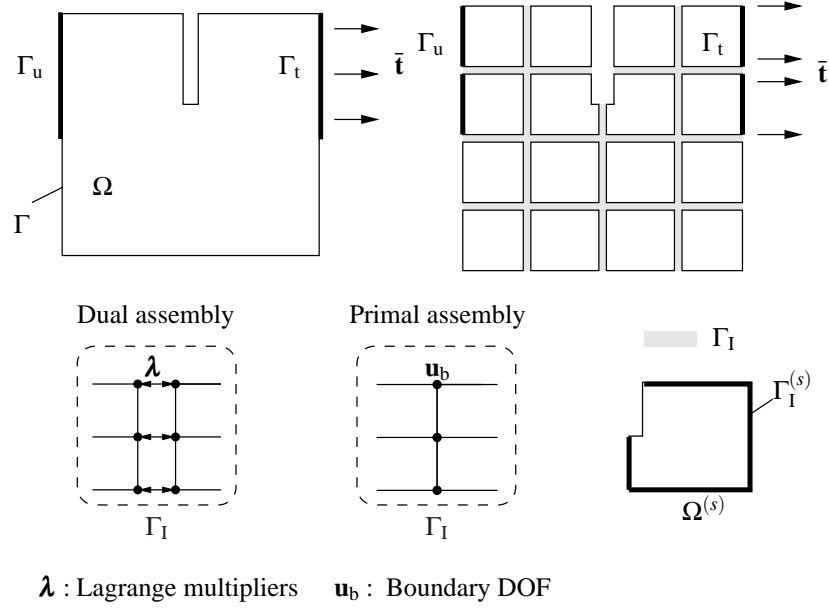
Consequently, the global system

$$\mathbf{K} \mathbf{u} = \mathbf{f} \quad (2)$$

arising from the discretization and linearization of the equilibrium equations is re-casted into the augmented system

$$\begin{bmatrix} \mathbf{K}^{(1)} & \mathbf{0} & \mathbf{0} & \mathbf{B}^{(1)\text{T}} \\ \mathbf{0} & \ddots & \mathbf{0} & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{K}^{(N_s)} & \mathbf{B}^{(N_s)\text{T}} \\ \mathbf{B}^{(1)} & \dots & \mathbf{B}^{(N_s)} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}^{(1)} \\ \vdots \\ \mathbf{u}^{(N_s)} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^{(1)} \\ \vdots \\ \mathbf{f}^{(N_s)} \\ \mathbf{0} \end{bmatrix}, \quad (3)$$

where  $\mathbf{K}$ ,  $\mathbf{u}$ ,  $\mathbf{f}$  and  $\boldsymbol{\lambda}$  denote the global stiffness matrix, displacement vector, force vector and Lagrange multipliers used to tie the different domains (bottom part of Figure 1).



**Figure 1:** Body  $\Omega$  decomposed into a number of non-overlapping domains.

In a primal assembly, the matching boundary nodes at the interface  $\mathbf{u}_b$  are assembled at the same position in the global system (bottom part of Figure 1) and therefore the interface degrees of freedom (DOF)  $\mathbf{u}_b$  are uniquely defined. The local DOF are identified from the global set  $\mathbf{u}$  as

$$\mathbf{u}^{(s)} = \mathbf{L}^{(s)} \mathbf{u}, \quad (4)$$

where  $\mathbf{L}^{(s)}$  contains identity coefficients which relate the internal DOF to the corresponding global DOF and Boolean coefficients expressing the assembly of the subdomains on the interface. The global system (2) can be therefore re-cast into

$$\left( \sum_{s=1}^{N_s} \mathbf{L}^{(s)\text{T}} \mathbf{K}^{(s)} \mathbf{L}^{(s)} \right) \mathbf{u} = \sum_{s=1}^{N_s} \mathbf{L}^{(s)\text{T}} \mathbf{f}^{(s)}. \quad (5)$$

It is important to realize that

$$\mathbf{B}\mathbf{L} = \mathbf{0}, \quad (6)$$

being

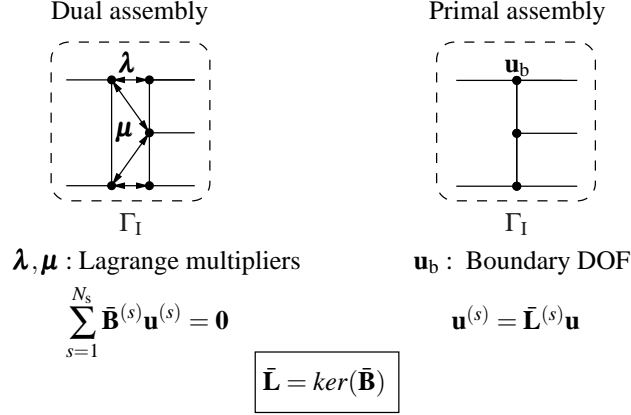
$$\mathbf{B} = [ \mathbf{B}^{(1)} \quad \dots \quad \mathbf{B}^{(N_s)} ] \quad (7)$$

$$\mathbf{L} = [ \mathbf{L}^{(1)} \quad \dots \quad \mathbf{L}^{(N_s)} ]^{\text{T}}. \quad (8)$$

For this reason one can obtain the localization operator  $\mathbf{L}$  by solving

$$\mathbf{L} = \ker(\mathbf{B}), \quad (9)$$

where  $\ker$  denotes the kernel or null space of the matrix  $\mathbf{B}$ .



**Figure 2:** Dual and primal assembly at the non-conforming interface  $\Gamma_I$ .

## 2.2 Algebraic multiscale extensions

The main ingredient in the multiscale extension consists in enforcing a certain compatibility condition at a non-conforming interface between different resolution domains (top part of Figure 2). Different types of compatibility have been recently reported in [4]. A full collocation condition of the solution field  $\mathbf{u}$  at the interface  $\Gamma_I$  between fine  $f$  and coarse  $c$  adjacent domains reads

$$\mathbf{u}^f = \mathbf{u}^c \quad \text{at } \Gamma_I, \quad (10)$$

and can be enforced using the shape functions of the coarse domain restricted at the interface. These conditions can be written using the compatibility matrices  $\mathbf{C}^{(s)}$  to obtain

$$\sum_{s=1}^{N_s} \mathbf{C}^{(s)} \mathbf{u}^{(s)} = \mathbf{0}. \quad (11)$$

The previous relations can be simply enforced in a dual approach by adding the compatibility equations to the Boolean matrices as

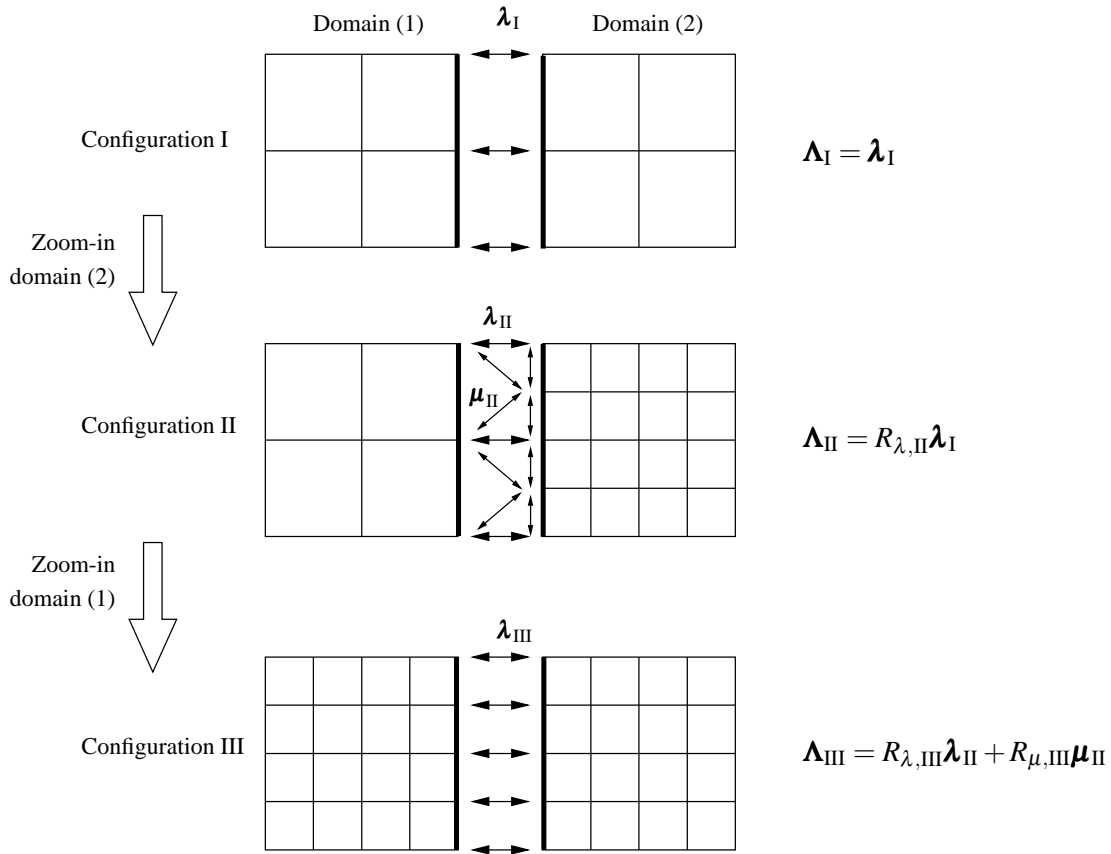
$$\left[ \bar{\mathbf{B}}^{(1)} \quad \dots \quad \bar{\mathbf{B}}^{(N_s)} \right] = \left[ \begin{array}{ccc} \mathbf{B}^{(1)} & \dots & \mathbf{B}^{(N_s)} \\ \mathbf{C}^{(1)} & \dots & \mathbf{C}^{(N_s)} \end{array} \right]. \quad (12)$$

Consequently, the extended field of Lagrange multipliers reads

$$\mathbf{\Lambda} = \left[ \begin{array}{c} \lambda \\ \mu \end{array} \right] \quad (13)$$

In a primal approach, the localization operators  $\mathbf{L}^{(s)}$  need to be modified in order to incorporate the new constraints arising from the coarse-to-fine displacement compatibility. The primal system (5) can be recovered by considering the modified localization operator

$$\mathbf{u}^{(s)} = \bar{\mathbf{L}}^{(s)} \mathbf{u}. \quad (14)$$



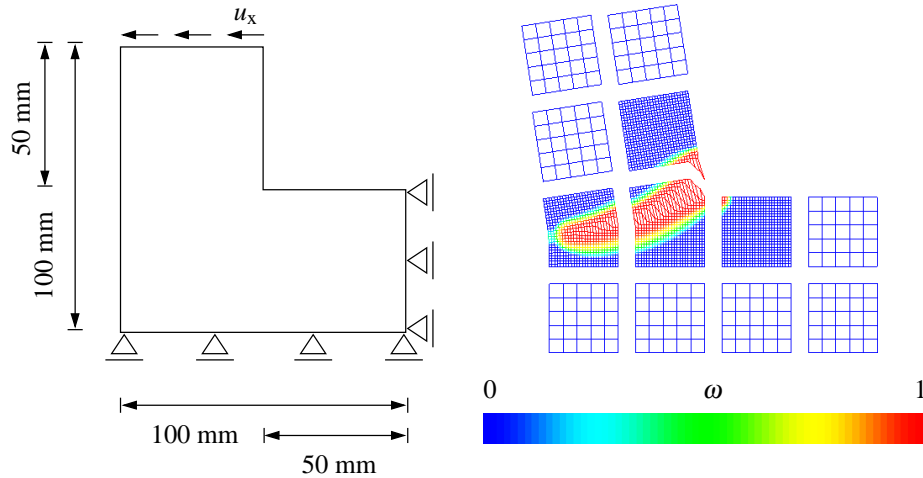
**Figure 3:** Force resolution migration in a multiscale dual approach.

Other ingredients for an adaptive multiscale analysis consist in a criterion to trigger the zoom-in events, the solution of local boundary value problems (BVP) and a set of global relaxation steps as described in detail in [3, 4]. It is important to note that in a dual multiscale approach one needs to keep track of the interface tying forces to set up the appropriate force field at the boundary of a newly refined domain. This force migration scheme is accomplished through the resize matrices  $\mathbf{R}$  (Figure 3) and is explained in more detail in [3]. Such a procedure is obviously not needed in a primal multiscale approach due to the absence of tying forces in the formulation.

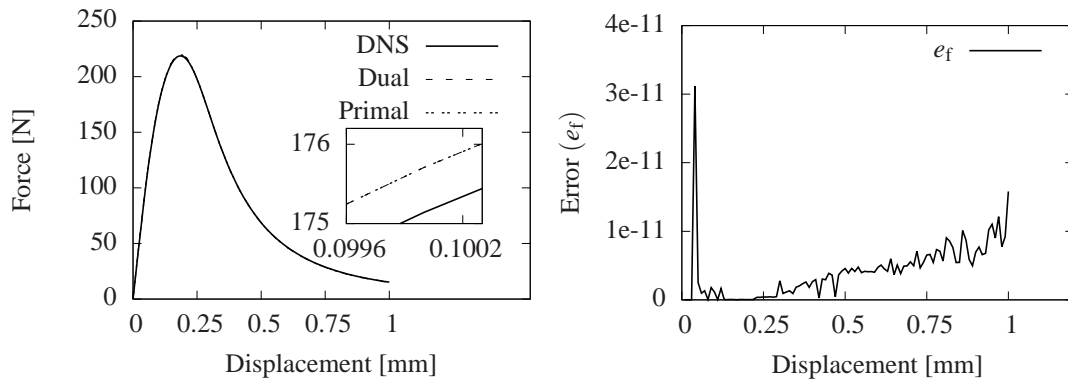
### 3 EXAMPLES

#### 3.1 Dual and primal multiscale analysis

Dual and primal multiscale analyses are performed on the L-shaped homogeneous specimen depicted in Figure 4. The specimen is decomposed in 12 regular domains and zoom-in is performed at those domains affected by damage growth. A gradient enhanced damage model [7] is employed in the computations and the adopted material parameters are



**Figure 4:** L-shaped specimen (left) and final damage contours of the adaptive multiscale analysis (right).



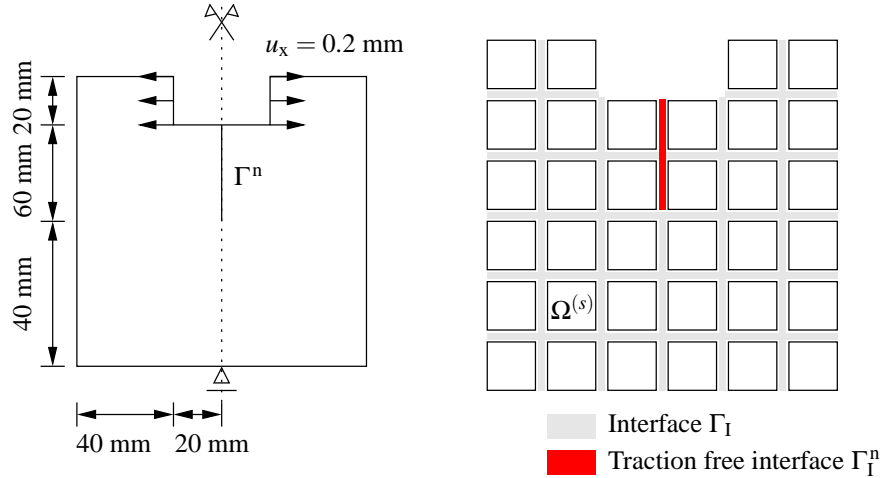
**Figure 5:** Force-displacement plot for the dual and primal analyses (left) and force field relative error (right).

identical to the ones reported in [3].

The displacement versus reaction at the upper edge is plotted in Figure 5 (left) for the analyses employing dual and primal approaches. The relative difference between reaction forces obtained with the dual and primal approaches is negligible and it is therefore concluded that both multiscale analyses provide identical results.

### 3.2 Parallel performance

In this example, the dual assembly is employed to perform a multiscale analysis simulating a wedge split test of a concrete specimen. The test setup is sketched in Figure 6 and the material parameters are adopted from [5, 6]. Three different decompositions with 34, 68 and 136 domains are considered (Figure 7) and the global assembled system is solved using a parallel solver designed for the solution of large sparse linear systems based



**Figure 6:** Boundary conditions (left) and domain decomposition (right) for the wedge split test.

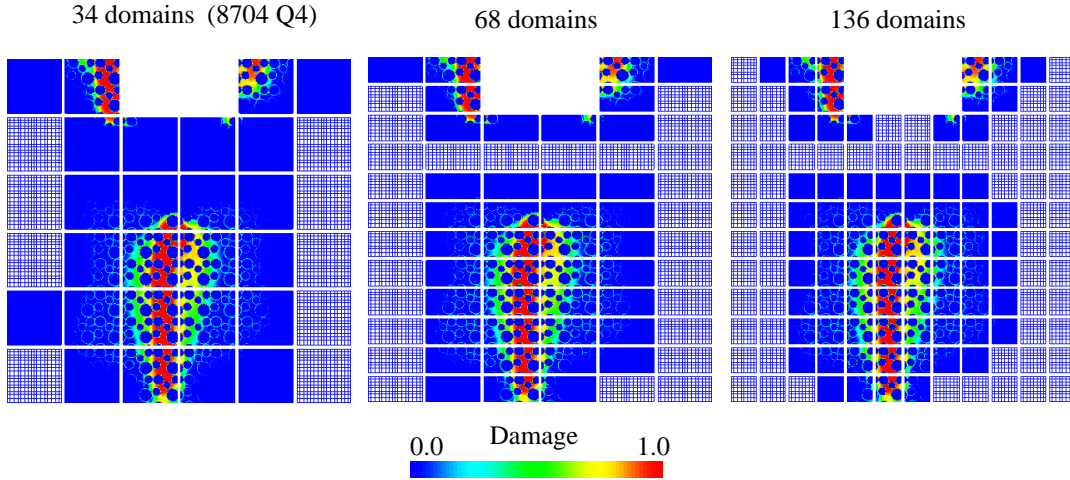
on direct methods. Automatic load-balancing and multi-threading are accounted for and, consequently, the performance of the solver is optimized according to the hardware used. All analyses have been performed on a Dell R710 PowerEdge machine (two X5570 quad-core Xeons, Core I7 Nehalem, clocked at 2.93 GHz with 24 GB of memory). Parallel tasks related to BVP solves of different zoomed-in domains and FE assemblies are distributed over one to eight cores using openMP. It is observed in Figure 8 that parallel scalability is found for a low number of processors. Solution strategies for massively parallel computers, typically employed in domain decomposition approaches, need robust iterative solvers and are considered out of the scope of this contribution.

### 3.3 3D opportunities

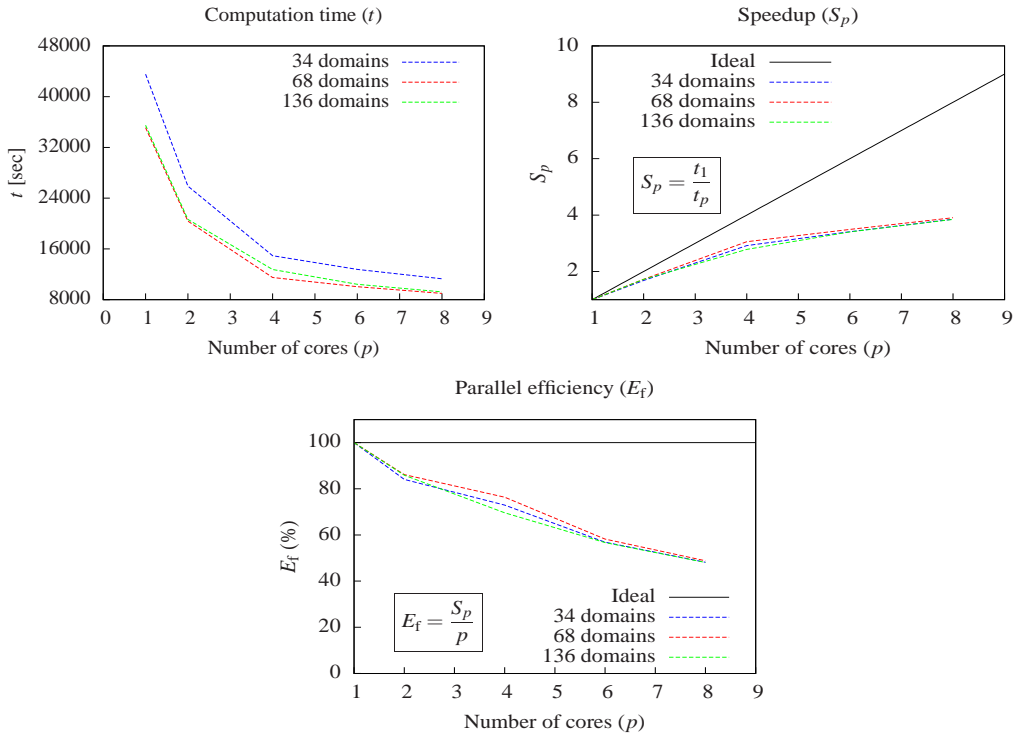
To illustrate the performance of the adaptive multiscale framework, a simple tensile test is performed on a homogeneous bar with variable height (Figure 9). The sample is decomposed into seven domains according to the planes  $x = 20$ ,  $x = 40$ ,  $x = 60$ ,  $x = 80$ ,  $x = 100$  and  $x = 120$  mm, respectively. The coarse domains are discretized with one 8-noded hexahedron while the fine domains contain 375 8-noded hexahedra. The constitutive model is linear elastic with Young’s modulus  $E = 35$  GPa and Poisson’s ratio  $\nu = 0.2$ . Zoom-in is performed when the strain component  $\varepsilon_{xx} > 8.0 \times 10^{-4}$  at any integration point within the domain. Horizontal displacement contours on the deformed configuration at different loading stages are depicted in Figure 10.

## 4 CONCLUSIONS

The performance of both dual and primal multiscale techniques is assessed for damage analysis in quasi-brittle materials. Results turn to be identical and it is concluded that both formulations are equivalent when utilized in a multiscale context. Due to the lack of

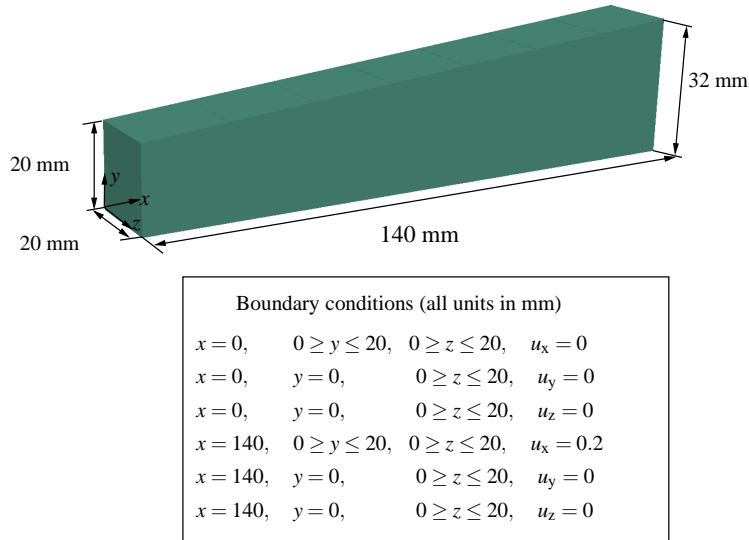


**Figure 7:** Final damage distributions for a decomposition using 34, 68 and 136 domains.



**Figure 8:** Computation times  $t$ , speed-up  $S_p$ , i.e. ratio between the time measured with one processor  $t_1$  and  $p$  processors  $t_p$ , and parallel efficiency  $E_f$  for up to eight cores.





**Figure 9:** Geometry and boundary conditions for the three-dimensional test

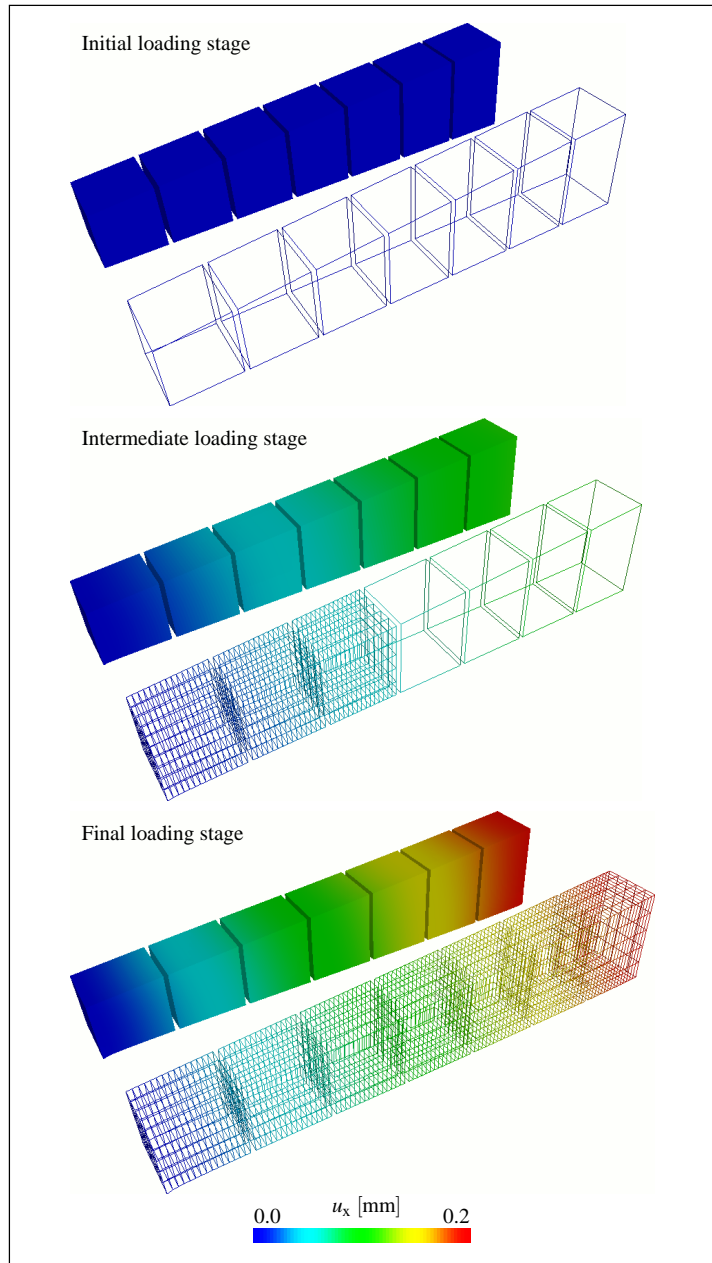
interconnecting forces in the primal assembly there is no need for a force migration scheme as introduced in the dual framework. It is observed that parallel scalability is found for a low number of processors when parallel direct solvers are applied at the fully assembled system of equations. The proposed multiscale techniques show promising features for the three dimensional analysis of large structures. However, further research is required in order to devise robust iterative solvers needed for the computations with massively parallel computers typically employed in domain decomposition approaches.

## 5 ACKNOWLEDGEMENTS

Oriol Lloberas-Valls gratefully acknowledges the funding received from the MICINN of the Spanish Government through a "Juan de la Cierva" Postdoctoral grant and partial funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n. 320815 (ERC Advanced Grant Project "Advanced tools for computational design of engineering materials" COMP-DES-MAT).

## REFERENCES

- [1] D. J. Rixen. Parallel processing. In S. G. Braun, editor, *Encyclopedia of Vibration*, pages 990-1001. Elsevier, Oxford, 2001.
- [2] O. Lloberas-Valls, D. J. Rixen, A. Simone and L. J. Sluys: Domain Decomposition techniques for the efficient modeling of brittle heterogeneous materials. *Computer Methods in Applied Mechanics and Engineering*, 200(13–16):1577–1590, 2011.



**Figure 10:** Horizontal displacement contours and deformed grids at different loading stages. 100× displacement magnification.

- [3] O. Lloberas-Valls, D. J. Rixen, A. Simone and L. J. Sluys: Multiscale domain decomposition analysis of quasi-brittle heterogeneous materials. *International Journal for Numerical Methods in Engineering*, 89(11):1337–1366, 2012.
- [4] O. Lloberas-Valls, D. J. Rixen, A. Simone and L. J. Sluys: On micro-to-macro connections in domain decomposition multiscale methods. *Computer Methods in Applied Mechanics and Engineering*, 225–228:177–196, 2012.
- [5] O. Lloberas-Valls, F. P. X. Everdij, D. J. Rixen, A. Simone and L. J. Sluys. Concurrent multiscale analysis of heterogeneous materials. In J. Eberhardsteiner and H. J. Böhm and F. G. Rammerstorfer, editors, Proceedings of the 6th European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS 2012), Vienna, Austria, 10 – 14 September, 2012.
- [6] O. Lloberas-Valls, F. P. X. Everdij, D. J. Rixen, A. Simone, and L. J. Sluys. Objective multiscale analysis of random heterogeneous materials. In E. Oñate, D. R. J. Owen, D. Peric, and B. Suárez, editors, Proceedings of the XII International Conference on Computational Plasticity – COMPLAS XII, Barcelona, Spain, 3–5 September 2013.
- [7] R.H.J. Peerlings, R. de Borst, W.A.M. Brekelmans and J.H.P. de Vree: Gradient-enhanced damage for quasi-brittle materials. *International Journal for Numerical Methods in Engineering*, 39(19):3391-3403, 1996.
- [8] C. Farhat and F. X. Roux: A method of finite element tearing and interconnecting and its parallel solution algorithm. *International Journal for Numerical Methods in Engineering*, 32(7):1205-1227, 1991.
- [9] A. Mobasher Amini and D. Dureisseix and P. Cartraud: Multi-scale domain decomposition method for large-scale structural analysis with a zooming technique. *International Journal for Numerical Methods in Engineering*, 79(4):417–443, 2009.
- [10] P. Ladevèze and O. Loiseau and D. Dureisseix: A micro-macro and parallel computational strategy for highly heterogeneous structures. *International Journal for Numerical Methods in Engineering*, 52(12):121–138, 2001.
- [11] P. A. Guidault and O. Allix and L. Champaney and J. P. Navarro: A two-scale approach with homogenization for the computation of cracked structures. *Computers & Structures*, 85(17-18):1360–1371, 2007.