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ASYMPTOTIC MODELING OF A THIN ANISOTROPIC PIEZOELECTRIC INTERPHASE

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Abstract. We study the electromechanical behavior of a thin interphase, constituted by a linearly piezoelectric anisotropic material with high rigidity, embedded between two generic three-dimensional piezoelectric bodies by means of the asymptotic expansion method. After defining a small real dimensionless parameter ε, which will tend to zero, we characterize the limit model and the associated limit problem. Moreover, we identify the non classical electromechanical transmission conditions at the interface between the two three-dimensional bodies.

1 INTRODUCTION

In the last decades the use of smart materials in aeronautical, mechanical and civil engineering has provided a new degree of design flexibility for advanced composite structural members. This kind of technology is based on the ability to allow the structure to sense and react in a desired fashion, improving its performances. The new concept of adaptive structure requires, for instance, the use of piezoelectric sensors and actuators for controlling the mechanical behavior of structural systems. Piezoelectric materials may be integrated into a host structure to change its shape and to enhance its mechanical properties with different configurations: for instance, a piezoelectric transducer can be embedded into the structure to be controlled or it can be glued on it, as in the case of piezo-patches. Moreover, the same piezoelectric actuators are often obtained by alternating different thin layers of material with highly contrasted electromechanical properties. This generates different types of complex multimaterial assemblies, in which each phase interacts with the others.

The successful application of the asymptotic methods to obtain a mathematical justification of linear and non linear plate models in elasticity [1] has stimulated the research toward a rational simplification of the modeling of complex structures obtained joining elements of different dimensions and/or materials of highly contrasted properties. The asymptotic analysis has been also used to formally derive simplified models for piezoelectric plates, taking into account both sensor and actuator functions, see, for instance, [2, 3, 4], as well as pyroelectric and pyroelastic effects, see [5]. The direct solution of a complex multimaterial problem by a standard finite element method is too expensive from a computational point of view and the presence of strong contrasts in the geometry and mechanical properties causes numerical instabilities. That is why specific asymptotic expansions are used and allow to replace the original problem by a set of problems in which the thin layer, for instance, is substituted by a two-dimensional surface. The thin inclusion of a third material between two other ones when the rigidity properties of the inclusion are highly contrasted with respect to those of the surrounding materials has been deeply investigated in different functional frameworks in the case of linear elasticity, see [6, 7, 8], and, also, in the case of thin conductor plates embedded into a piezoelectric matrix [9].

In this work we consider a particular piezoelectric assembly, constituted by two generic three-dimensional piezoelectric bodies separated by a thin piezoelectric interphase with high rigidity. By defining a small real parameter ε, associated with the thickness and the electromechanical properties of the middle layer, we perform an asymptotic analysis by letting ε tend to zero, following the approach by P.G. Ciarlet [1]. Then we characterize the limit model and its associated limit problem. Within the reduced model the intermediate interphase "disappears" and it is replaced by a specific electromechanical surface energy defined over the middle plane of the plate. This surface energy is then traduced in ad hoc transmission conditions at the interface between the two piezoelectric bodies in terms of the jump of stresses, electric displacements and electric potentials.

The paper is organized as follows. In Section 2 we define the notation and the position of the problem. The limit model is then deduced through an asymptotic analysis, as shown in Section 3. In Section 4 we determine the electromechanical interface problem.

2 THE PHYSICAL PROBLEM

In the sequel, Greek indices range in the set $\{1, 2\}$, Latin indices range in the set $\{1, 2, 3\}$, and the Einstein's summation convention with respect to the repeated indices is adopted.

Let us consider a three-dimensional Euclidian space identified by $\mathbb{R}^3$ and such that the three vectors $\mathbf{e}_i$ form an orthonormal basis. Let Ω^+ and Ω^- be two disjoint open domains with smooth boundaries $\partial\Omega^+$ and $\partial\Omega^-$. Let $\omega := \{\partial\Omega^+ \cap \partial\Omega^-\}^\circ$ be the interior of the common part of the boundaries which is assumed to be a non empty domain in $\mathbb{R}^2$ having a positive two-dimensional measure. We consider the assembly constituted by two solids bonded together by an intermediate thin plate-like body $\Omega^{m,\varepsilon}$ of thickness $2h^\varepsilon$, where $0 < \varepsilon < 1$ is a dimensionless small real parameter which will tend to zero. We suppose that the thickness h^ε of the middle layer depends linearly on ε, so that $h^\varepsilon = \varepsilon h$.

More precisely, we denote respectively with $\Omega^{\pm,\varepsilon} := \{x^\varepsilon := x \pm \varepsilon h \mathbf{e}_3;\ x \in \Omega^\pm\}$, the

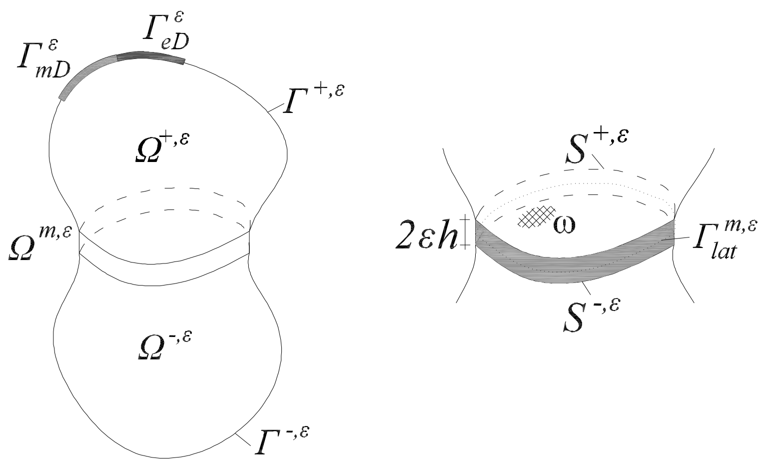

Figure 1: The reference configuration of the multimaterial and the geometry of the interphase.

translation of Ω^+ (resp. Ω^-) along the direction $\mathbf{e}_3$ (resp. $-\mathbf{e}_3$) of the quantity εh, with $\Omega^{m,\varepsilon} := \omega \times (-\varepsilon h, \varepsilon h)$, the central plate-like domain, and with $\Omega^\varepsilon := \Omega^{+,\varepsilon} \cup \Omega^{m,\varepsilon} \cup \Omega^{-,\varepsilon}$, the reference configuration of the assembly.

Moreover, we define with $S^{\pm,\varepsilon} := \omega \times \{\pm\varepsilon h\} = \Omega^{\pm,\varepsilon} \cap \Omega^{m,\varepsilon}$, the upper and lower faces of the intermediate plate-like domain, $\Gamma^{\pm,\varepsilon} := \partial\Omega^{\pm,\varepsilon}/S^{\pm,\varepsilon}$, and $\Gamma^{m,\varepsilon}_{lat} := \partial\omega \times (-\varepsilon h, \varepsilon h)$, its lateral surface, see Figure 1.

Let $(\Gamma^\varepsilon_{mD}, \Gamma^\varepsilon_{mN})$ and $(\Gamma^\varepsilon_{eD}, \Gamma^\varepsilon_{eN})$ be two suitable partitions of $\partial\Omega^\varepsilon := \Gamma^{\pm,\varepsilon} \cup \Gamma^{m,\varepsilon}_{lat}$, with both Γ^ε_{mD} and Γ^ε_{eD} of strictly positive Lebesgue measure. The multimaterial is, on one hand, clamped along Γ^ε_{mD} and at an electrical potential $\varphi^\varepsilon_0 = 0$ on Γ^ε_{eD} and, on the other hand, subject to surface forces g^ε_i on Γ^ε_{mN} and electrical displacement d^ε on Γ^ε_{eN}. The assembly is also subject to body forces f^ε_i and electrical loadings F^ε acting in $\Omega^{\pm,\varepsilon}$. We suppose, without loss of generality, that $\Omega^{m,\varepsilon}$ and $\Gamma^{m,\varepsilon}_{lat}$ are both free of mechanical and electrical charges. The work of the external electromechanical loadings takes then the following form:

$$L^\varepsilon(r^\varepsilon) := \int_{\Omega^{\pm,\varepsilon}} (f^\varepsilon_i v^\varepsilon_i + F^\varepsilon \psi^\varepsilon) dx^\varepsilon + \int_{\Gamma^\varepsilon_{mN}} g^\varepsilon_i v^\varepsilon_i d\Gamma^\varepsilon + \int_{\Gamma^\varepsilon_{eN}} d^\varepsilon \psi^\varepsilon d\Gamma^\varepsilon$$

We suppose that $f^\varepsilon_i \in L^2(\Omega^{\pm,\varepsilon})$, $F^\varepsilon \in L^2(\Omega^{\pm,\varepsilon})$, $g^\varepsilon_i \in L^2(\Gamma^\varepsilon_{mN})$ and $d^\varepsilon \in L^2(\Gamma^\varepsilon_{eN})$. We finally assume that $\Omega^{\pm,\varepsilon}$ and $\Omega^{m,\varepsilon}$ are constituted by two homogeneous linearly piezoelectric materials, whose constitutive laws are defined as follows:

$$\begin{cases} \sigma^\varepsilon_{ij}(\mathbf{u}^\varepsilon, \varphi^\varepsilon) = C^\varepsilon_{ijk\ell} e^\varepsilon_{k\ell}(\mathbf{u}^\varepsilon) - P^\varepsilon_{kij} E^\varepsilon_k(\varphi^\varepsilon), \\ D^\varepsilon_i(\mathbf{u}^\varepsilon, \varphi^\varepsilon) = P^\varepsilon_{ijk} e^\varepsilon_{jk}(\mathbf{u}^\varepsilon) + H^\varepsilon_{ij} E^\varepsilon_j(\varphi^\varepsilon), \end{cases}$$

where $(\sigma^\varepsilon_{ij})$ is the classical Cauchy stress tensor, $(e^\varepsilon_{ij}(\mathbf{u}^\varepsilon)) := \left(\frac{1}{2}(\partial^\varepsilon_i u^\varepsilon_j + \partial^\varepsilon_j u^\varepsilon_i)\right)$ is the linearized strain tensor, (D^ε_i) is the electrical displacement field, φ^ε is the electrical potential and $E_i^\varepsilon(\varphi^\varepsilon) := -\partial_i^\varepsilon \varphi^\varepsilon$ its associated electrical field. $(C_{ijk\ell}^\varepsilon)$, (P_{ijk}^ε) and (H_{ij}^ε) represent, respectively, the classical fourth order elasticity tensor, the third order piezoelectric coupling tensor and the second order dielectric tensor related to $\Omega^{\pm,\varepsilon}$ and $\Omega^{m,\varepsilon}$.

Tensors $(C_{ijk\ell}^\varepsilon)$, (H_{ij}^ε) and (P_{ijk}^ε) satisfy the following coercivity properties: for any symmetric matrix field (b_{ij}), there exists a constant $c > 0$ such that $C_{ijk\ell}^\varepsilon b_{k\ell} b_{ij} \geq c \sum_{i,j} |b_{ij}|^2$; for any vector field (a_i), there exists a constant $c > 0$ such that $H_{ij}^\varepsilon a_j a_i \geq c \sum_i |a_i|^2$. Moreover, we have the symmetries $C_{ijk\ell}^\varepsilon = C_{k\ell ij}^\varepsilon = C_{jik\ell}^\varepsilon$, $H_{ij}^\varepsilon = H_{ji}^\varepsilon$ and $P_{kji}^\varepsilon = P_{kij}^\varepsilon$.

The electromechanical state at the equilibrium is determined by the pair $s^\varepsilon := (\mathbf{u}^\varepsilon, \varphi^\varepsilon)$. We define the functional spaces

$$\begin{aligned}
&\mathbf{V}^\varepsilon(\Omega^\varepsilon, \Gamma^\varepsilon) := \{\mathbf{v}^\varepsilon \in H^1(\Omega^\varepsilon; \mathbb{R}^3);\ \mathbf{v}^\varepsilon = \mathbf{0} \ \text{ on } \Gamma^\varepsilon\},\\
&V^\varepsilon(\Omega^\varepsilon, \Gamma^\varepsilon) := \{v^\varepsilon \in H^1(\Omega^\varepsilon);\ v^\varepsilon = 0 \ \text{ on } \Gamma^\varepsilon\}.
\end{aligned}$$

The physical variational problem $\mathcal{P}^\varepsilon$ defined over the variable domain Ω^ε reads as follows:

$$\left\{\begin{array}{l}
\text{Find } s^\varepsilon \in \mathbf{V}^\varepsilon(\Omega^\varepsilon, \Gamma_{mD}^\varepsilon) \times V^\varepsilon(\Omega^\varepsilon, \Gamma_{eD}^\varepsilon) \ \text{ such that}\\
A^{-,\varepsilon}(s^\varepsilon, r^\varepsilon) + A^{+,\varepsilon}(s^\varepsilon, r^\varepsilon) + A^{m,\varepsilon}(s^\varepsilon, r^\varepsilon) = L^\varepsilon(r^\varepsilon),
\end{array}\right. \tag{1}$$

for all $r^\varepsilon \in \mathbf{V}^\varepsilon(\Omega^\varepsilon, \Gamma_{mD}^\varepsilon) \times V^\varepsilon(\Omega^\varepsilon, \Gamma_{eD}^\varepsilon)$, where the bilinear forms $A^{\pm,\varepsilon}(\cdot,\cdot)$ and $A^{m,\varepsilon}(\cdot,\cdot)$ are defined by

$$\begin{aligned}
A^{\pm,\varepsilon}(s^\varepsilon, r^\varepsilon) := \int_{\Omega^{\pm,\varepsilon}} &\big\{C_{ijk\ell}^{\pm,\varepsilon} e_{k\ell}^\varepsilon(\mathbf{u}^\varepsilon) e_{ij}^\varepsilon(\mathbf{v}^\varepsilon) + H_{ij}^{\pm,\varepsilon} E_j^\varepsilon(\varphi^\varepsilon) E_i^\varepsilon(\psi^\varepsilon) +\\
&+ P_{ihk}^{\pm,\varepsilon}(E_i^\varepsilon(\psi^\varepsilon) e_{hk}^\varepsilon(\mathbf{u}^\varepsilon) - E_i^\varepsilon(\varphi^\varepsilon) e_{hk}^\varepsilon(\mathbf{v}^\varepsilon))\big\}\, dx^\varepsilon,\\
A^{m,\varepsilon}(s^\varepsilon, r^\varepsilon) := \int_{\Omega^{m,\varepsilon}} &\big\{C_{ijk\ell}^{m,\varepsilon} e_{k\ell}^\varepsilon(\mathbf{u}^\varepsilon) e_{ij}^\varepsilon(\mathbf{v}^\varepsilon) + H_{ij}^{m,\varepsilon} E_j^\varepsilon(\varphi^\varepsilon) E_i^\varepsilon(\psi^\varepsilon) +\\
&+ P_{ihk}^{m,\varepsilon}(E_i^\varepsilon(\psi^\varepsilon) e_{hk}^\varepsilon(\mathbf{u}^\varepsilon) - E_i^\varepsilon(\varphi^\varepsilon) e_{hk}^\varepsilon(\mathbf{v}^\varepsilon))\big\}\, dx^\varepsilon.
\end{aligned}$$

By virtue of the $\mathbf{V}^\varepsilon(\Omega^\varepsilon, \Gamma_{mD}^\varepsilon) \times V^\varepsilon(\Omega^\varepsilon, \Gamma_{eD}^\varepsilon)$-coercivity of the bilinear forms and thanks to the Lax-Milgram lemma, problem (1) admits one and only one solution.

3 ASYMPTOTIC ANALYSIS

In order to study the asymptotic behavior of the solution of problem (1) when ε tends to zero, we rewrite the problem on a fixed domain Ω independent of ε. By using the approach of [1], we consider the bijection $\pi^\varepsilon : x \in \overline{\Omega} \mapsto x^\varepsilon \in \overline{\Omega}^\varepsilon$ given by

$$\left\{\begin{array}{ll}
\pi^\varepsilon(x_1, x_2, x_3) = (x_1, x_2, x_3 - (1 - \varepsilon h)), & \text{for all } x \in \overline{\Omega}_{tr}^{+},\\
\pi^\varepsilon(x_1, x_2, x_3) = (x_1, x_2, \varepsilon x_3), & \text{for all } x \in \overline{\Omega}^{m},\\
\pi^\varepsilon(x_1, x_2, x_3) = (x_1, x_2, x_3 + (1 - \varepsilon h)), & \text{for all } x \in \overline{\Omega}_{tr}^{-},
\end{array}\right.$$

where $\Omega_{tr}^{\pm} := \{x \pm \mathbf{e}_3,\ x \in \Omega^{\pm}\}$, $\Omega^m := \omega \times (-h, h)$ and $S^{\pm} := \omega \times \{\pm h\}$. In order to simplify the notation, we identify $\Omega_{tr}^{\pm}$ with $\Omega^{\pm}$, and $\overline{\Omega}$ with $\overline{\Omega}^{\pm} \cup \overline{\Omega}^{m}$. Likewise, we note $\Gamma_\pm := \partial\Omega^\pm / S^\pm$, $\Gamma^m_{lat} := \partial\omega \times (-h,h)$, $(\Gamma_{mD}, \Gamma_{mN})$ and $(\Gamma_{eD}, \Gamma_{eN})$, the partitions of $\partial\Omega := \Gamma^\pm \cup \Gamma^m_{lat}$.

Consequently, $\partial^\varepsilon_\alpha = \partial_\alpha$ and $\partial^\varepsilon_3 = \frac{1}{\varepsilon}\partial_3$ in Ω^m.

With the unknown electromechanical state $s^\varepsilon = (\mathbf{u}^\varepsilon, \varphi^\varepsilon)$, we associate the scaled unknown electromechanical state $s(\varepsilon) := (\mathbf{u}(\varepsilon), \varphi(\varepsilon))$ defined by:

$$\begin{array}{ll} u^\varepsilon_\alpha(x^\varepsilon) = u_\alpha(\varepsilon)(x) \text{ and } u^\varepsilon_3(x^\varepsilon) = \varepsilon^{-1} u_3(\varepsilon)(x) & \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^{m,\varepsilon}, \\ \varphi^\varepsilon(x^\varepsilon) = \varepsilon\varphi(\varepsilon)(x) & \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^{m,\varepsilon}. \end{array}$$

We likewise associate with any test functions $r^\varepsilon = (\mathbf{v}^\varepsilon, \psi^\varepsilon)$, the scaled test functions $r = (\mathbf{v}, \psi)$, defined by the scalings:

$$\begin{array}{ll} v^\varepsilon_\alpha(x^\varepsilon) = v_\alpha(x) \text{ and } v^\varepsilon_3(x^\varepsilon) = \varepsilon^{-1} v_3(x) & \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^{m,\varepsilon}, \\ \psi^\varepsilon(x^\varepsilon) = \varepsilon\psi(x) & \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^{m,\varepsilon}. \end{array}$$

For ε sufficiently small, we associate with the constant functions $C^{\pm,\varepsilon}_{ijk\ell}$, $H^{\pm,\varepsilon}_{ij}$, $P^{\pm,\varepsilon}_{ijk}$: $\overline{\Omega}^{\pm,\varepsilon} \to \mathbb{R}$ the constant functions $C^\pm_{ijk\ell}$, $H^\pm_{ij}$, $P^\pm_{ijk} : \overline{\Omega}^\pm \to \mathbb{R}$ defined by

$$C^{\pm,\varepsilon}_{ijk\ell} := C^\pm_{ijk\ell}, \quad H^{\pm,\varepsilon}_{ij} := H^\pm_{ij}, \quad P^{\pm,\varepsilon}_{ijk} := P^\pm_{ijk} \quad \text{for all } x^\varepsilon = \pi^\varepsilon(x) \in \overline{\Omega}^{\pm,\varepsilon},$$

and we associate with the constant functions $C^{m,\varepsilon}_{ijk\ell}$, $H^{m,\varepsilon}_{ij}$, $P^{m,\varepsilon}_{ijk} : \overline{\Omega}^{m,\varepsilon} \to \mathbb{R}$ the constant functions $C^m_{ijk\ell}$, H^m_{ij}, $P^m_{ijk} : \overline{\Omega}^m \to \mathbb{R}$ defined by

$$C^{m,\varepsilon}_{ijk\ell} := \tfrac{1}{\varepsilon} C^m_{ijk\ell}, \quad H^{m,\varepsilon}_{ij} := \tfrac{1}{\varepsilon} H^m_{ij}, \quad P^{m,\varepsilon}_{ijk} := \tfrac{1}{\varepsilon} P^m_{ijk} \quad \text{for all } x^\varepsilon = \pi^\varepsilon(x) \in \overline{\Omega}^{m,\varepsilon}.$$

We also make the following assumptions on the applied mechanical and electrical forces:

$$\begin{array}{ll} f^\varepsilon_i(x^\varepsilon) = f_i(x) \ \text{ and } \ g^\varepsilon_i(x^\varepsilon) = g_i(x) & \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^{\pm,\varepsilon}, \\ F^\varepsilon(x^\varepsilon) = F(x) \ \text{ and } \ d^\varepsilon(x^\varepsilon) = d(x) & \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^{\pm,\varepsilon}, \end{array}$$

where functions $f_i \in L^2(\Omega^\pm)$, $F \in L^2(\Omega^\pm)$, $g_i \in L^2(\Gamma_{mN})$ and $d \in L^2(\Gamma_{eN})$ are independent of ε. Thus $L^\varepsilon(r^\varepsilon) = L(r)$.

We define the spaces

$$\begin{array}{l} \mathbf{V}(\Omega, \Gamma) := \{\mathbf{v} \in H^1(\Omega; \mathbb{R}^3); \ \mathbf{v} = \mathbf{0} \ \text{ on } \Gamma\}, \\ V(\Omega, \Gamma) := \{v \in H^1(\Omega); \ v = 0 \ \text{ on } \Gamma\}. \end{array}$$

According to the previous assumptions, problem (1) can be reformulated on a fixed domain Ω independent of ε. Thus we obtain the following scaled problem $\mathcal{P}(\varepsilon)$:

$$\left\{ \begin{array}{l} \text{Find } s(\varepsilon) \in \mathbf{V}(\Omega, \Gamma_{mD}) \times V(\Omega, \Gamma_{eD}) \ \text{ such that} \\ A^-(s(\varepsilon), r) + A^+(s(\varepsilon), r) + A^m(\varepsilon)(s(\varepsilon), r) = L(r), \end{array} \right. \tag{2}$$

for all $r \in \mathbf{V}(\Omega, \Gamma_{mD}) \times V(\Omega, \Gamma_{eD})$, where the bilinear forms $A^{\pm}(\cdot,\cdot)$ and $A^m(\varepsilon)(\cdot,\cdot)$ are given by

$$\begin{aligned}
A^{\pm}(s(\varepsilon), r) &:= \int_{\Omega^{\pm}} \big\{ C^{\pm}_{ijk\ell} e_{k\ell}(\mathbf{u}(\varepsilon)) e_{ij}(\mathbf{v}) + H^{\pm}_{ij} \partial_j \varphi(\varepsilon) \partial_i \psi + \\
&\qquad + P^{\pm}_{ihk} (E_i(\psi) e_{hk}(\mathbf{u}(\varepsilon)) - E_i(\varphi(\varepsilon)) e_{hk}(\mathbf{v})) \big\}\, dx, \\
A^m(\varepsilon)(s(\varepsilon), r) &:= \tfrac{1}{\varepsilon^4} a^m_{-4}(s(\varepsilon), r) + \tfrac{1}{\varepsilon^3} a^m_{-3}(s(\varepsilon), r) + \tfrac{1}{\varepsilon^2} a^m_{-2}(s(\varepsilon), r) + \tfrac{1}{\varepsilon} a^m_{-1}(s(\varepsilon), r) + \\
&\qquad + a^m_0(s(\varepsilon), r) + \varepsilon a^m_1(s(\varepsilon), r) + \varepsilon^2 a^m_2(s(\varepsilon), r),
\end{aligned}$$

with

$$\begin{aligned}
a^m_{-4}(s, r) &:= \int_{\Omega^m} C^m_{3333} e_{33}(\mathbf{u}) e_{33}(\mathbf{v}) dx, \\
a^m_{-3}(s, r) &:= \int_{\Omega^m} 2C^m_{\alpha 333} (e_{33}(\mathbf{u}) e_{\alpha 3}(\mathbf{v}) + e_{\alpha 3}(\mathbf{u}) e_{33}(\mathbf{v})) dx, \\
a^m_{-2}(s, r) &:= \int_{\Omega^m} \big\{ C^m_{\alpha\beta 33} (e_{33}(\mathbf{u}) e_{\alpha\beta}(\mathbf{v}) + e_{\alpha\beta}(\mathbf{u}) e_{33}(\mathbf{v})) + 4C^m_{\alpha 3\beta 3} e_{\alpha 3}(\mathbf{u}) e_{\beta 3}(\mathbf{v}) + \\
&\qquad + P^m_{333} (\partial_3 \varphi e_{33}(\mathbf{v}) - e_{33}(\mathbf{u}) \partial_3 \psi) \} dx, \\
a^m_{-1}(s, r) &:= \int_{\Omega^m} \{ 2C^m_{\alpha\beta\sigma 3} (e_{\sigma 3}(\mathbf{u}) e_{\alpha\beta}(\mathbf{v}) + e_{\alpha\beta}(\mathbf{u}) e_{\sigma 3}(\mathbf{v})) + P^m_{\alpha 33} (\partial_\alpha \varphi e_{33}(\mathbf{v}) - e_{33}(\mathbf{u}) \partial_\alpha \psi) + \\
&\qquad + 2P^m_{3\alpha 3} (\partial_3 \varphi e_{\alpha 3}(\mathbf{v}) - e_{\alpha 3}(\mathbf{u}) \partial_3 \psi) \} dx, \\
a^m_0(s, r) &:= \int_{\Omega^m} \{ C^m_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\mathbf{u}) e_{\alpha\beta}(\mathbf{v}) + 2P^m_{\alpha\beta 3} (\partial_\alpha \varphi e_{\beta 3}(\mathbf{v}) - e_{\beta 3}(\mathbf{u}) \partial_\alpha \psi) + H^m_{33} \partial_3 \varphi \partial_3 \psi + \\
&\qquad + P^m_{3\alpha\beta} (\partial_3 \varphi e_{\alpha\beta}(\mathbf{v}) - e_{\alpha\beta}(\mathbf{u}) \partial_3 \psi) \} dx, \\
a^m_1(s, r) &:= \int_{\Omega^m} \big\{ P^m_{\alpha\beta\sigma} (\partial_\alpha \varphi e_{\beta\sigma}(\mathbf{v}) - e_{\beta\sigma}(\mathbf{u}) \partial_\alpha \psi) + H^m_{\alpha 3} (\partial_\alpha \varphi \partial_3 \psi + \partial_\alpha \psi \partial_3 \varphi) \big\}\, dx, \\
a^m_2(s, r) &:= \int_{\Omega^m} H^m_{\alpha\beta} \partial_\beta \varphi \partial_\alpha \psi dx.
\end{aligned}$$

The rescaled variational problem (2) has a unique solution in $\mathbf{V}(\Omega, \Gamma_{mD}) \times V(\Omega, \Gamma_{eD})$ by virtue of the Lax-Milgram lemma. In the sequel, only if necessary, we will note, respectively, with $(\mathbf{v}^{\pm}, \psi^{\pm})$ and $(\mathbf{v}^m, \psi^m)$, the restrictions of functions $(\mathbf{v}, \psi)$ to $\Omega^{\pm}$ and Ω^m.

We can now perform an asymptotic analysis of the rescaled problem (2). Since the rescaled problem (2) has a polynomial structure with respect to the small parameter ε, we can look for the solution $s(\varepsilon) = (\mathbf{u}(\varepsilon), \varphi(\varepsilon))$ of the problem as a series of powers of ε:

$$s(\varepsilon) = s^0 + \varepsilon s^1 + \varepsilon^2 s^2 + \ldots \;\Rightarrow\; \left\{ \begin{array}{l} \mathbf{u}(\varepsilon) = \mathbf{u}^0 + \varepsilon \mathbf{u}^1 + \varepsilon^2 \mathbf{u}^2 + \ldots \\ \varphi(\varepsilon) = \varphi^0 + \varepsilon \varphi^1 + \varepsilon^2 \varphi^2 + \ldots \end{array} \right. , \tag{3}$$

with $s^q = (\mathbf{u}^q, \varphi^q) \in \mathbf{V}(\Omega, \Gamma_{mD}) \times V(\Omega, \Gamma_{eD})$, $q \geq 0$. By substituting (3) into the rescaled problem (2), and by identifying the terms with identical power of ε, we obtain, as customary, a set of variational problems to be solved in order to characterize the limit electromechanical state $s^0 = (\mathbf{u}^0, \varphi^0)$ and its associated limit problem.

3.1 The limit model

When dealing with the asymptotic models for piezoelectric plates, the scaling that we use for the electric potential is commonly associated with a piezoelectric actuator plate (see [3]). By identifying the terms with identical power, we define the following set of variational problems:

$$
\begin{aligned}
\mathcal{P}_{-4}: \quad & a_{-4}^m(s^0, r) = 0,\\
\mathcal{P}_{-3}: \quad & a_{-4}^m(s^1, r) + a_{-3}^m(s^0, r) = 0,\\
\mathcal{P}_{-2}: \quad & a_{-4}^m(s^2, r) + a_{-3}^m(s^1, r) + a_{-2}^m(s^0, r) = 0,\\
\mathcal{P}_{-1}: \quad & a_{-4}^m(s^3, r) + a_{-3}^m(s^2, r) + a_{-2}^m(s^1, r) + a_{-1}^m(s^0, r) = 0,\\
\mathcal{P}_{0}: \quad & a_{-4}^m(s^4, r) + a_{-3}^m(s^3, r) + a_{-2}^m(s^2, r) + a_{-1}^m(s^1, r) + a_{0}^m(s^0, r) +\\
& + A^+(s^0, r) + A^-(s^0, r) = L(r).
\end{aligned}
$$

To proceed with the asymptotic analysis we need to solve each variational subproblem above and characterize the limit electromechanical state $s^0 = (\mathbf{u}^0, \varphi^0)$ and its associated limit problem.

We start by solving problem $\mathcal{P}_{-4}$. Let us choose test functions $r = s^0 \in \mathbf{V}(\Omega, \Gamma_{mD}) \times V(\Omega, \Gamma_{eD})$:

$$\int_{\Omega^m} C_{3333}^m e_{33}(\mathbf{u}^0) e_{33}(\mathbf{u}^0) dx = 0.$$

Since $C_{3333}^m > 0$, we have $e_{33}(\mathbf{u}^0) = 0$ and, thus, $u_3^{m,0} = w^0(\tilde{x})$, with $\tilde{x} = (x_\alpha) \in \omega$.

Let us consider problem $\mathcal{P}_{-3}$. Since $e_{33}(\mathbf{u}^0) = 0$, we get

$$\int_{\Omega^m} \left\{ C_{3333}^m e_{33}(\mathbf{u}^1) + 2C_{\alpha 333}^m e_{\alpha 3}(\mathbf{u}^0) \right\} e_{33}(\mathbf{v}) dx = 0,$$

which is satisfied when $e_{33}(\mathbf{u}^1) = -\frac{2C_{\alpha 333}^m}{C_{3333}^m} e_{\alpha 3}(\mathbf{u}^0)$.

Let us consider problem $\mathcal{P}_{-2}$ with test functions $v_3 = 0$:

$$\int_{\Omega^m} \left(C_{\alpha 333}^m - \frac{C_{3333}^m C_{\alpha 3\beta 3}^m}{2C_{\beta 333}^m} \right) e_{33}(\mathbf{u}^1) \partial_3 v_\alpha dx = 0.$$

This implies that $e_{33}(\mathbf{u}^1) = 0$ and, thus, $e_{\alpha 3}(\mathbf{u}^0) = 0$, i.e., $u_\alpha^{m,0}(\tilde{x}, x_3) = \bar{u}_\alpha^0(\tilde{x}) - x_3 \partial_\alpha w^0(\tilde{x})$. The displacement field $\mathbf{u}^{m,0}$ corresponds to a Kirchhoff-Love-type kinematics and it belongs to $\mathbf{V}_{KL}(\Omega^m, \Gamma_{mD}^0) := \{\mathbf{v} \in H^1(\Omega^m; \mathbb{R}^3);\ e_{i3}(\mathbf{v}) = 0,\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_{mD}^0\} = \{\mathbf{v} \in H^1(\omega; \mathbb{R}^2) \times H^2(\omega);\ v_i = \partial_\nu v_3 = 0 \text{ on } \gamma_0\}$, where $\Gamma_{mD}^0 := \gamma_0 \times (-h, h) \subset \Gamma_{lat}^m$ denote the fixed part of the boundary Γ_{lat}^m of the plate-like body.

By choosing test functions $v_\alpha = 0$, problem $\mathcal{P}_{-2}$ is verified when

$$C_{3333}^m e_{33}(\mathbf{u}^2) + 2C_{\alpha 333}^m e_{\alpha 3}(\mathbf{u}^1) + C_{\alpha\beta 33}^m e_{\alpha\beta}(\mathbf{u}^0) + P_{333}^m \partial_3 \varphi^0 = 0. \tag{4}$$

By taking into account test functions $v_3 = 0$ in problem $\mathcal{P}_{-1}$, we obtain that:

$$C^m_{\tau 333} e_{33}(\mathbf{u}^2) + 2C^m_{\tau 3\sigma 3} e_{\sigma 3}(\mathbf{u}^1) + C^m_{\alpha\beta\tau 3} e_{\alpha\beta}(\mathbf{u}^0) + P^m_{3\tau 3}\partial_3\varphi^0 = 0. \tag{5}$$

The linear system constituted by equations (4)-(5)

$$\begin{cases} C^m_{3333} e_{33}(\mathbf{u}^2) + 2C^m_{\alpha 333} e_{\alpha 3}(\mathbf{u}^1) = -C^m_{\alpha\beta 33} e_{\alpha\beta}(\mathbf{u}^0) - P^m_{333}\partial_3\varphi^0, \\ C^m_{\tau 333} e_{33}(\mathbf{u}^2) + 2C^m_{\tau 3\sigma 3} e_{\sigma 3}(\mathbf{u}^1) = -C^m_{\alpha\beta\tau 3} e_{\alpha\beta}(\mathbf{u}^0) - P^m_{3\tau 3}\partial_3\varphi^0, \end{cases}$$

admits one and only one solution, such that

$$\begin{cases} e_{\sigma 3}(\mathbf{u}^1) = \Delta_{\sigma k}(C^m_{\alpha\beta k3} e_{\alpha\beta}(\mathbf{u}^0) + P^m_{3k3}\partial_3\varphi^0), \\ e_{33}(\mathbf{u}^2) = 2\Delta_{3k}(C^m_{\alpha\beta k3} e_{\alpha\beta}(\mathbf{u}^0) + P^m_{3k3}\partial_3\varphi^0). \end{cases} \tag{6}$$

Coefficients $\Delta_{ik} = \Delta_{ki}$ are shown below

$$\begin{array}{ll} \Delta := \epsilon_{ijk} C^m_{i313} C^m_{j323} C^m_{k333}, & \Delta_{1k} := -\frac{1}{\Delta}\epsilon_{kij} C^m_{i323} C^m_{j333}, \\ \Delta_{2k} := -\frac{1}{\Delta}\epsilon_{kij} C^m_{j313} C^m_{i333}, & \Delta_{3k} := -\frac{1}{\Delta}\epsilon_{kij} C^m_{i313} C^m_{j323}, \end{array}$$

and ϵ_{kij} denotes the Ricci's alternator symbol.

We define the following functional spaces:

$$\begin{array}{l} \widetilde{\mathbf{V}} := \{\mathbf{v}^\pm \in \mathbf{V}(\Omega^\pm, \Gamma_{mD}),\ \mathbf{v}^m \in \mathbf{V}_{KL}(\Omega^m, \Gamma^0_{mD});\ \mathbf{v}^\pm|_{S^\pm} = \mathbf{v}^m|_{S^\pm}\}, \\ \Psi := \{\psi \in L^2(\Omega^m);\ \partial_3\psi \in L^2(\Omega^m)\}, \\ \widetilde{\Psi} := \{\psi^\pm \in V(\Omega^\pm, \Gamma_{eD}),\ \psi^m \in \Psi;\ \psi^\pm|_{S^\pm} = \psi^m|_{S^\pm}\}. \end{array}$$

Let us consider problem $\mathcal{P}_0$. By choosing test functions $r \in \widetilde{\mathbf{V}} \times \widetilde{\Psi}$, and by means of the relations (6) among $\mathbf{u}^2$, $\mathbf{u}^0$ and φ^0, one obtains the following limit problem:

$$\begin{cases} \text{Find } s^0 \in \widetilde{\mathbf{V}} \times \widetilde{\Psi} \text{ such that} \\ A^+(s^0, r) + A^-(s^0, r) + A^m_{KL}(s^0, r) = L(r) \text{ for all } r \in \widetilde{\mathbf{V}} \times \widetilde{\Psi}, \end{cases} \tag{7}$$

where

$$A^m_{KL}(s^0, r) := \int_{\Omega^m} \Big\{ \widetilde{C}^m_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\mathbf{u}^0) + \widetilde{P}^m_{3\alpha\beta}\partial_3\varphi^0) e_{\alpha\beta}(\mathbf{v}) - (\widetilde{P}^m_{3\alpha\beta} e_{\alpha\beta}(\mathbf{u}^0) - \widetilde{H}^m_{33}\partial_3\varphi^0)\partial_3\psi \Big\}\, dx.$$

The reduced electromechanical coefficients $\widetilde{C}^m_{\alpha\beta\sigma\tau}$, $\widetilde{P}^m_{3\alpha\beta}$ and $\widetilde{H}^m_{33}$ are listed below

$$\begin{array}{l} \widetilde{C}^m_{\alpha\beta\sigma\tau} := C^m_{\alpha\beta\sigma\tau} + 2\Delta_{pq} C^m_{\alpha\beta p3} C^m_{\sigma\tau q3}, \quad \widetilde{P}^m_{3\alpha\beta} := P^m_{3\alpha\beta} + 2\Delta_{pq} C^m_{\alpha\beta p3} P^m_{3q3}, \\ \widetilde{H}^m_{33} := H^m_{33} - 2\Delta_{pq} P^m_{3p3} P^m_{3q3}. \end{array}$$

The limit electric potential $\varphi^{m,0}$ can be explicitly characterized as a second order polynomial function of x_3. Indeed let us consider the limit problem (7) and choose test functions $r = (\mathbf{0}, \psi) \in \widetilde{V} \times \widetilde{\Psi}$. By integrating by parts, we obtain the following equation with its associated continuity conditions of the electric potential at the interfaces $S^{\pm}$:

$$\begin{cases} \partial_{33}\varphi^{m,0} = -\dfrac{\widetilde{P}^m_{3\alpha\beta}}{\widetilde{H}^m_{33}}\partial_{\alpha\beta}w^0 & \text{in } \Omega^m, \\ \varphi^{m,0}(x_1,x_2,h) = \varphi^{+,0}, \quad \varphi^{m,0}(x_1,x_2,-h) = \varphi^{-,0} & \text{on } \; S^{\pm}, \end{cases}$$

where $\varphi^{\pm,0} := \varphi^{\pm,0}(x_1,x_2,\pm h)$. The electric potential can be written as follows

$$\varphi^{m,0}(x_1,x_2,x_3) = \sum_{k=0}^{2} \phi^k(x_1,x_2)x_3^k \tag{8}$$

with

$$\phi^0 = \frac{\varphi^{+,0}+\varphi^{-,0}}{2} + \frac{h^2\widetilde{P}^m_{3\alpha\beta}}{2\widetilde{H}^m_{33}}\partial_{\alpha\beta}w^0, \quad \phi^1 = \frac{\varphi^{+,0}-\varphi^{-,0}}{2h}, \quad \phi^2 = -\frac{\widetilde{P}^m_{3\alpha\beta}}{2\widetilde{H}^m_{33}}\partial_{\alpha\beta}w^0.$$

Remark. The previous characterization of the electric potential $\varphi^{m,0}$ is a rigorous justification of the a priori assumptions conjectured by Bernadou and Hanel [10] and it represents the complete anisotropic generalization with respect to the paper [11]. We can also notice that the regularity of the electric potential only depends on the regularities of $\varphi^{\pm,0}$ and of w^0. Hence, $\varphi^{\pm,0} \in L^2(\omega)$ and $w^0 \in H^2(\omega)$ imply that $\varphi^{m,0} \in \Psi$. The space Ψ with the norm $\|\psi\|^2_\Psi := |\psi|^2_{0,\Omega^m} + |\partial_3\psi|^2_{0,\Omega^m}$ can be identified with the space $H^1(-h,h;L^2(\omega))$ endowed with the usual norm. Therefore, the trace of the elements of Ψ on $S^{\pm}$ makes sense in $L^2(S^{\pm})$.

3.2 A different form of the limit problem

By taking advantage of the explicit form (8) of the electric potential $\varphi^{m,0}$ and of the Kirchhoff-Love displacement field $\mathbf{u}^{m,0} := (u^{m,0}_\alpha = \bar{u}^0_\alpha(\tilde{x}) - x_3\partial_\alpha w^0(\tilde{x}), u^{m,0}_3 = w^0(\tilde{x}))$, with $\mathbf{u}^0_H = (\bar{u}^0_\alpha)$, we rewrite problem (7) in a different way: the bilinear form, defined on Ω^m, will be associated with a two-dimensional appropriate bilinear form defined over the middle plane ω of the plate, which represents the interface between Ω^+ and Ω^-.

Thus, by choosing test functions $\psi^m(\tilde{x},x_3) = \psi^0(\tilde{x}) + x_3\psi^1(\tilde{x}) + x_3^2\psi^2(\tilde{x}) \in \Psi^m$, with $\psi^1 = \frac{[\![\psi]\!]}{2h} \in L^2(\omega)$, and $\mathbf{v}^m := (v^m_\alpha = \bar{v}^0_\alpha(\tilde{x}) - x_3\partial_\alpha v_3(\tilde{x}), v^m_3 = v_3(\tilde{x})) \in \mathbf{V}_{KL}$, the bilinear form $A^m_{KL}(s^0,r)$, defined over Ω^m, can be identified with an equivalent two-dimensional bilinear form $\widetilde{A}^m_{KL}(s^0,r)$, defined over ω, as follows:

$$\begin{aligned} \widetilde{A}^m_{KL}(s^0,r) := 2h\int_\omega \Bigg\{ &\left(\widetilde{C}^m_{\alpha\beta\sigma\tau}e_{\sigma\tau}(\mathbf{u}^0_H) + \widetilde{P}^m_{3\alpha\beta}\frac{[\![\varphi^0]\!]}{2h}\right)e_{\alpha\beta}(\mathbf{v}_H) + \\ &+ \left(-\widetilde{P}^m_{3\alpha\beta}e_{\alpha\beta}(\mathbf{u}^0_H) + \widetilde{H}^m_{33}\frac{[\![\varphi^0]\!]}{2h}\right)\frac{[\![\psi]\!]}{2h}\Bigg\}d\tilde{x} + \frac{2h^3}{3}\int_\omega A^m_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}w^0\partial_{\alpha\beta}v_3 d\tilde{x}, \end{aligned}$$

where $A^m_{\alpha\beta\sigma\tau} := \widetilde{C}^m_{\alpha\beta\sigma\tau} - \frac{\widetilde{P}^m_{3\alpha\beta}\widetilde{P}^m_{3\sigma\tau}}{\widetilde{H}^m_{33}}$ and $[\![f]\!] := f^+ - f^-$ denotes the jump function at the interface ω between Ω^+ and Ω^-.

Hence the limit problem (7) takes the following equivalent form

$$\begin{cases} \text{Find } s^0 \in \widetilde{\mathbf{V}} \times \widetilde{\Psi} \text{ such that} \\ A^+(s^0, r) + A^-(s^0, r) + \widetilde{A}^m_{KL}(s^0, r) = L(r) \text{ for all } r \in \widetilde{\mathbf{V}} \times \widetilde{\Psi}. \end{cases}$$

4 THE ELECTROMECHANICAL INTERFACE PROBLEM

The aim of this section is to derive a coupled electromechanical interface problem between the two piezoelectric bodies Ω^+ and Ω^- with some ad hoc transmission conditions at the interface ω. By virtue of the asymptotic methods we replace the three-dimensional electromechanical energy of the intermediate piezoelectric layer with a specific two-dimensional surface energy defined over the middle plane of the plate. This surface energy generates non classical transmission conditions between the two three-dimensional bodies. We distinguish, respectively, between the electric and the mechanical interface problems, each one with their associated appropriate transmission conditions. By rewriting problem (7) in its differential form after an integration by parts and by using the expression (8) of the limit electric potential, we obtain:

Electrostatic problems in $\Omega^\pm$ **Elasticity problems in $\Omega^\pm$**

$$\begin{cases} \partial_i D_i^\pm(\mathbf{u}^0, \varphi^0) = F & \text{in } \Omega^\pm, \\ D_i^\pm(\mathbf{u}^0, \varphi^0) n_i = d & \text{on } \Gamma_{eN}, \\ \varphi^0 = 0 & \text{on } \Gamma_{eD}, \end{cases} \qquad \begin{cases} -\partial_j \sigma_{ij}^\pm(\mathbf{u}^0, \varphi^0) = f_i & \text{in } \Omega^\pm, \\ \sigma_{ij}^\pm(\mathbf{u}^0, \varphi^0) n_j = g_i & \text{on } \Gamma_{mN}, \\ \mathbf{u}^0 = \mathbf{0} & \text{on } \Gamma_{mD}, \end{cases}$$

Transmission conditions on ω

$$\begin{cases} [\![D_3(\mathbf{u}^0, \varphi^0)]\!] = 0 & \text{on } \omega, \\ [\![\sigma_{\alpha 3}(\mathbf{u}^0, \varphi^0)]\!] - \widetilde{P}^m_{3\alpha\beta}[\![\partial_\beta \varphi^0]\!] = \partial_\beta n_{\alpha\beta}(\mathbf{u}^0_H) & \text{on } \omega, \\ [\![\sigma_{33}(\mathbf{u}^0, \varphi^0)]\!] = \partial_{\alpha\beta} m_{\alpha\beta}(w^0) & \text{on } \omega, \\ [\![\mathbf{u}^0]\!] = \mathbf{0} & \text{on } \omega, \end{cases}$$

where $\sigma_{ij}^\pm(\mathbf{u}^0, \varphi^0) := C^\pm_{ijk\ell} e_{k\ell}(\mathbf{u}^0) - P^\pm_{ijk} E_k(\varphi^0)$ is the Cauchy stress tensor, $D_i^\pm(\mathbf{u}^0, \varphi^0) = P^\pm_{ijk} e_{jk}(\mathbf{u}^0) + H^\pm_{ij} E_j(\varphi^0)$ is the electric displacement field, $n_{\alpha\beta}(\mathbf{u}^0_H) := 2h\widetilde{C}^m_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\mathbf{u}^0_H)$ is the membrane stress tensor, while $m_{\alpha\beta}(w^0) := -\frac{2h^3}{3} A^m_{\alpha\beta\sigma\tau} \partial_{\sigma\tau} w^0$ is the moment tensor.

Remark 2. The previous electromechanical interface problem can be considered as a generalization in the case of piezoelectric assemblies of the transmission problem obtained in [7, 8] for thin elastic inclusions with high rigidity. The particular jump conditions at the interface yield to a non standard transmission problem which can be solved by an adapted Neumann-Neumann domain decomposition algorithm [12].

5 CONCLUDING REMARKS

In the present work we derive an interface model corresponding to a generic piezoelectric assembly with a piezoelectric interphase through an asymptotic analysis. The middle layer is replaced by a particular surface energy which is associated with ad hoc transmission conditions at the interface of the two bodies. This model is extremely versatile because it can capture the electromechanical behavior of different assemblies, just by varying the nature of the constituent materials. Here, we propose the more general situation, in which the multimaterial is constituted by three different anisotropic piezoelectric materials. However, we can adapt the model by using other material combinations: for instance, we can choose two elastic and conductor bodies separated by an intermediate piezoelectric layer, which could describe the behavior of a piezoelectric actuator embedded within a certain structural member.

As future developments, we would like to prove the strong convergence of the solution of the physical problem towards the solution of the limit problem, in order to mathematically justify the limit model. Moreover, we would like to study more complex interface problems taking into account thermo-electromagnetoelastic couplings and time-dependent phenomena.

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