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INCREASE OF VORTEX RESOLUTION IN COMPUTATIONAL FLUID MECHANICS BY A COMBINATION OF GRID- AND PARTICLE- BASED METHODS

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Abstract. The paper describes the further development of a hybrid grid free and grid based computational method. Application of grid free method allows one to reduce the artificial viscosity and to increase the resolution of fine vortices. Derivation of main equations and interaction between flow scales is discussed.

1 INTRODUCTION

Insufficient resolution of vortex structures is one of the key problems in Computational Fluid Dynamics (CFD). In our recent papers [1] and [2] we propose the hybrid grid- and particle based method, based on a combination of the finite volume and computational vortex element [3] methods. In fact, the idea to combine the grid based and vortex methods is not quite new. As noted in [3]: "The motivation for such techniques stems from the observation that the strengths and the weaknesses of grid-based and vortex schemes can be seen as complementary, depending on the physical problem". Among existing hybrid grid based and grid free method one can distinguish two approaches. The first one is based on the domain- decomposition technique which subdivides the computational domain into two overlapping regions [4]. The grid based method is applied in the region close to the body whereas the vortex method is utilized in the wake region. The matching of the solution between two domains is attained by Schwarz alternating method. The domain-decomposition technique is implemented in pure velocity- vorticity and velocity-pressure

and velocity-vorticity [5] formulations. Second approach is the vortex- in- cell (VIC) method. In VIC the grid is utilized for fast calculation of the velocities and for remeshing. First, due to application of the Poisson equation and Fast Fourier Transformation (FFT) instead of direct summation using the Biot-Savart integral the computations are sufficiently accelerated, especially when combined with Fast Multipole Method (FMM) for determination of boundary conditions. Second, and it is probably more important, the instability of the numerical simulation is sufficiently damped by using the remeshing procedure resulting in the redistribution of irregularly located vortex elements onto regular grid. In both cases the grid caused numerical diffusion, which absence is considered as the main advantage of vortex methods, is involved to reduce the stochastization of numerical solution. Although the grid introduction the VIC method is classified as Lagrangian or semi-Lagrangian approach since the vortices are tracked in Lagrangian way. However, the loss of the most important advantages of pure Lagrangian methods, i.e. grid independency, raises the big question about the efficiency and competitiveness of VIC with respect to common grid based methods.
The present method differs principally from all vortex and hybrid methods mentioned above. It is based on the decomposition of the velocity and vorticity fields into the distributed large scale and concentrated small scale fields. The large scale field is represented on the grid, whereas the small scale one is calculated using the grid-free computational vortex method. The domain decomposition is not applied. The method is pure Lagrangian one for small structures and pure grid based one for large scale structures. The simulation with CVM is embedded into the grid simulation. There exist a permanent exchange between grid and particle represented vortices. Since we use the formalism different from the classical CVM method many of its weaknesses become irrelevant.

The equations describing small and large scales introduced in [1] and [2] suffers from stiffness because the large scale transport equation contains non averaged small scale terms. In this paper we revise the derivation of main equations presented in [1] and [2]. There is a two way coupling between scales which is also discussed below in details.

2 EQUATIONS OF COUPLED EVOLUTION OF GRID- AND PARTICLE REPRESENTED FLOW FIELDS

2.1 Splitting the Navier- Stokes equations

In this work we revise the derivation of a system of two coupled transport equations [1] and [2] which were obtained by splitting of Navier Stokes equation according to scales.
Substitution of the decomposition $\mathbf{u} = \mathbf{u}^g + \mathbf{u}^v$ and $\boldsymbol{\omega} = \boldsymbol{\omega}^g + \boldsymbol{\omega}^v$ into the vorticity transport equation gives:

$$\frac{\partial(\boldsymbol{\omega}^v + \boldsymbol{\omega}^g)}{\partial t} + ((\mathbf{u}^v + \mathbf{u}^g)\nabla)(\boldsymbol{\omega}^v + \boldsymbol{\omega}^g) = ((\boldsymbol{\omega}^v + \boldsymbol{\omega}^g)\nabla)(\mathbf{u}^v + \mathbf{u}^g) + \nu\Delta(\boldsymbol{\omega}^g + \boldsymbol{\omega}^v) \quad (1)$$

which we split into two following equations:

$$\begin{cases} \dfrac{\partial \boldsymbol{\omega}^g}{\partial t} + (\mathbf{u}^g \nabla)\boldsymbol{\omega}^g = (\boldsymbol{\omega}^g \nabla)\mathbf{u}^g + \nu \Delta \boldsymbol{\omega}^g \\ \dfrac{\partial \boldsymbol{\omega}^v}{\partial t} + ((\mathbf{u}^v + \mathbf{u}^g)\nabla)\boldsymbol{\omega}^v = (\boldsymbol{\omega}^v \nabla)(\mathbf{u}^v + \mathbf{u}^g) + \nu \Delta \boldsymbol{\omega}^v - (\mathbf{u}^v \nabla)\boldsymbol{\omega}^g + (\boldsymbol{\omega}^g \nabla)\mathbf{u}^v \end{cases} \tag{2}$$

The first equation describes the evolution of the grid based vorticity whereas the second one is responsible for concentrated vortices. The equations (2) were derived from the following considerations:
• sum of two equations retrieves the original Navier Stokes equation (1),
• both equations contain the terms possessing the same temporal and spatial scales,
• both equations can be solved using existing methods and software.
Violation of the second rule results in a stiff equation which can be solved using any averaging, like it is done within RANS for combustion problems or averaging theorem [**?**] if the probability density function of the small scale fast function is known. To escape the averaging which usually results in the closure problem we write the equations separately for large scale slow variables $\mathbf{u}^g$ and fine scale fast variables $\mathbf{u}^v$. The first equation can be rewritten in $u-p$ variables using identities

$$\boldsymbol{\omega}^g = \nabla \times \mathbf{u}^g,\ (\boldsymbol{\omega}\nabla)\mathbf{u} - (\mathbf{u}\nabla)\boldsymbol{\omega} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}),$$

$$(\boldsymbol{\omega}\nabla)\mathbf{u} - (\upsilon\nabla)\boldsymbol{\omega} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}),\ (\mathbf{u} \times \boldsymbol{\omega}) = \frac{1}{2}\nabla u^2 - (\mathbf{u}\nabla)\mathbf{u}$$

Omitting intermediate calculations, we have the system:

$$\frac{\partial \mathbf{u}^g}{\partial t} + (\mathbf{u}^g \nabla)\mathbf{u}^g = \nabla P^g + \nu \Delta \mathbf{u}^g \tag{3}$$

$$\frac{d\boldsymbol{\omega}^v}{dt} = (\boldsymbol{\omega}^v \nabla)(\mathbf{u}^v + \mathbf{u}^g) + \nu \Delta \boldsymbol{\omega}^v - (\mathbf{u}^v \nabla)\boldsymbol{\omega}^g + (\boldsymbol{\omega}^g \nabla)\mathbf{u}^v \tag{4}$$

The scalar function P^g can be treated as the grid based pressure. The first equation is the common Navier Stokes equation written for grid based variables. The equations (3) and (4) are solved sequentially. The first equation is solved on the grid whereas the second one using the grid free computational vortex method. The sum of these two equations written in the same variables retrieves the original Navier- Stokes equation.

2.2 Coupling between large and small scales

The equations (3) and (4) seem to describe the one way coupling, i.e. the large scale motions influence the small scale ones and not vice versa. This would be incorrect. The feedback of small scale structures on large scale ones is physically relevant and should be represented mathematically. In this section we consider consequently all interaction mechanisms between scales.

2.2.1 Interaction through boundary conditions

The boundary conditions are explicitly formulated only for the grid solution. According to Gresho and Sani [6] the boundary conditions for the velocity are sufficient to allow the determination of both velocity and pressure from the NS equation. The no slip condition at the wall reads

$$\mathbf{u}^v + \mathbf{u}^g = 0 \rightarrow \mathbf{u}^g = -\mathbf{u}^v \tag{5}$$

The necessary and sufficient boundary condition for the pressure, which is used in Poisson equation, is the Neumann BC obtained by the projection of the Navier Stokes equation onto the normal direction:

$$\frac{\partial P^g}{\partial n} = \nu \Delta u_n^g - \big(\frac{\partial u_n^g}{\partial t} + (\mathbf{u}^g \nabla) u_n^g\big) \tag{6}$$

Commonly, for high Reynolds numbers the first term on r.h.s. of (6) is neglected. The Neumann BC for the grid based part reads

$$\frac{\partial P^g}{\partial n} = -\big(\frac{\partial u_n^g}{\partial t} + (\mathbf{u}^g \nabla) u_n^g\big) = \big(\frac{\partial u_n^v}{\partial t} + (\mathbf{u}^g \nabla) u_n^v\big) \tag{7}$$

There are no explicit boundary conditions for the vortex part. The interaction of fine vortices with boundaries is considered in boundary conditions for the grid based solution. The interaction between scales is represented by terms on the right hand sides of boundary conditions (5) and (7).

2.2.2 Interaction through generation of small vortices due to evolution of large grid based vorticity

Due to instability the vorticity represented on grid can tend to creation of concentrated vortex structures. As soon as the size of vortex is getting comparable with the cell size, small vortices are generated from large grid based ones according to the algorithm described in [1]. This interaction mechanism is illustrated schematically in Fig. 1.

2.2.3 Influence of large grid based vortices on small vortices

This influence is taken by terms $(\boldsymbol{\omega}^v \nabla)\mathbf{u}^g$ and $(\mathbf{u}^g \nabla)\boldsymbol{\omega}^v$ in Eq. (4). The large vortices represented on grid contributes to the small vortex convection, rotation and amplification.

2.2.4 Influence of small vortices on large ones by generation of additional small vortices in grid based field

This process is described by the terms $-(\mathbf{u}^v \nabla)\boldsymbol{\omega}^g + (\boldsymbol{\omega}^g \nabla)\mathbf{u}^v$ in Eq. (4). The first term describes the transport of the grid based vorticity $\boldsymbol{\omega}^g$ by the velocity induced by concentrated vortices $\mathbf{u}^v$ whereas the second term is responsible for the rotation and

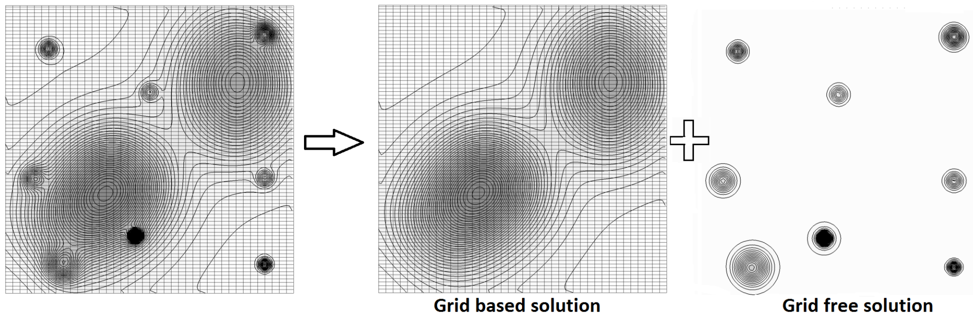

Figure 1: Interaction between scales by generation of small vortices due to evolution of large grid based vorticity

amplification of the grid based vorticity in field of $\mathbf{u}^v$. Analysis of terms $(\mathbf{u}^v\nabla)\boldsymbol{\omega}^g$ and $(\boldsymbol{\omega}^g\nabla)\mathbf{u}^v$ shows that the second term is much larger:

$$-(\mathbf{u}^v\nabla)\boldsymbol{\omega}^g \sim u^v_{max}\frac{\partial \omega^g}{\partial x},\ (\boldsymbol{\omega}^g\nabla)\mathbf{u}^v \sim \frac{u^v_{max}}{\sigma}\omega^g$$

where σ is the fine vortex size. If the linear approximation is used for velocities within the grid based method, the first term is zero $(\mathbf{u}^v\nabla)\boldsymbol{\omega}^g = 0$. Anyway it is much less than the second term, since the latter is proportional to σ^{-1}. Therefore, the local interaction between the vortices and grid based flow can be captured using the only second term $(\boldsymbol{\omega}^g\nabla)\mathbf{u}^v$.

Let us consider this term within framework of the vortex method [3]. If K is the smoothing function of the vortex elementwith the strength $\boldsymbol{\gamma}$), the following formula are valid:

$$\mathbf{u}^v = (\boldsymbol{\gamma} \times \mathbf{x})K(x)$$

$$(\boldsymbol{\omega}^g\nabla)\mathbf{u}^v = (\boldsymbol{\omega}^g\mathbf{x})(\boldsymbol{\gamma} \times \mathbf{x})\frac{\partial K}{\partial r} + (\boldsymbol{\omega}^g \times \boldsymbol{\gamma})K(x) \tag{8}$$

where $r = \sqrt{x_i x_i}$. The vector $(\boldsymbol{\omega}^g\nabla)\mathbf{u}^v$ describes the local impact of fine vortices on grid based vorticity. It is the local change of the grid based vorticity $\boldsymbol{\omega}^g$ at the place of the fine vortex with the strength $\boldsymbol{\gamma}$.

$$\frac{\partial \boldsymbol{\omega}^g}{\partial t} = (\boldsymbol{\omega}^g\mathbf{x})(\boldsymbol{\gamma} \times \mathbf{x})\frac{\partial K}{\partial r} + (\boldsymbol{\omega}^g \times \boldsymbol{\gamma})K(x) \tag{9}$$

For the vorton introduced in Sec 2.1 of [1] the last expression takes the form:

$$\frac{d\boldsymbol{\omega}^g}{dt} = (-\Omega^2(\boldsymbol{\omega}^g\mathbf{x})(\boldsymbol{\gamma} \times \mathbf{x}) + (\boldsymbol{\omega}^g \times \boldsymbol{\gamma}))\exp(-x^2\Omega^2/2) \tag{10}$$

Within the computational vortex method the governing equations are satisfied at element centers, i.e. at $\mathbf{x} = 0$. Since $K(0) = 1$ we get

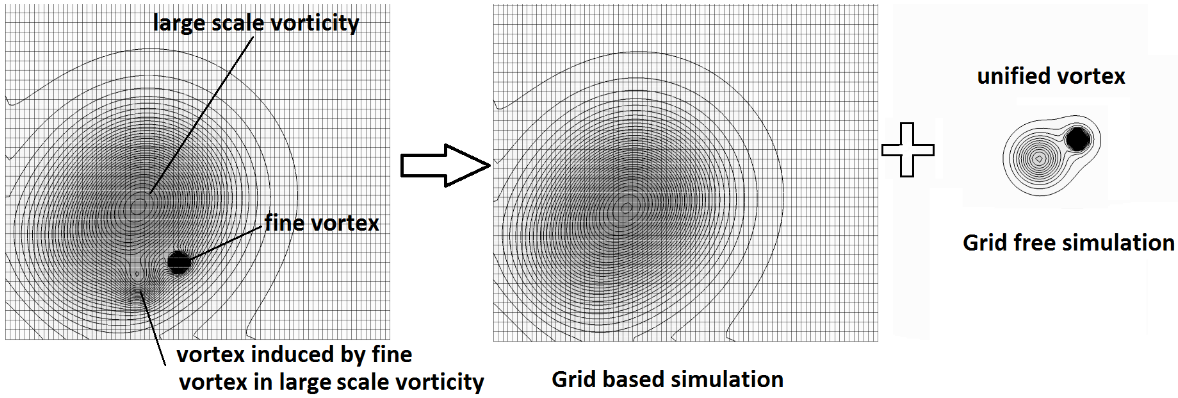

Figure 2: Interaction between scales by generation of additional small vortices in grid based field.

$$\frac{\partial \boldsymbol{\omega}^g}{\partial t} = \boldsymbol{\omega}^g \times \boldsymbol{\gamma} \tag{11}$$

The fine vortex influence is reduced to the rotation of the grid based vorticity $\boldsymbol{\omega}^g$ by the vector $\boldsymbol{\gamma}$. This rotation occurs in the local volume occupied by the vortex element. Indeed, the vorticity of the vortex element

$$\boldsymbol{\omega}^v = 2\boldsymbol{\gamma}K - ((\boldsymbol{\gamma} \times \mathbf{x}) \times \mathbf{x})\frac{\partial K}{\partial r}$$

has the same asymptotic behavior at large $r \sim r^2\frac{\partial K}{\partial r}$ as the vector $\frac{d\boldsymbol{\omega}^g}{dt}$. Therefore, the vortex element and the vorticity (9) occupy approximately the same volume.
Concluding, the influence of a small vortex results in the appearance of the local disturbance of the background field $\boldsymbol{\omega}^g$. The most simple way to account for the local rotation of the grid based vorticity is to keep $\boldsymbol{\omega}^g$ unchanged and to introduce a new element with the vorticity $\boldsymbol{\omega}^g$ and size of σ turned around the vector $\boldsymbol{\gamma}$ with the angular velocity $\gamma\Delta t$. To keep the number of vortices constant it is worth to update the original vortex element by change its strengths according to

$$\gamma_{update} = \boldsymbol{\gamma} + |\boldsymbol{\omega}^g|\frac{\boldsymbol{\omega}^g + (\boldsymbol{\omega}^g \times \boldsymbol{\gamma})\Delta t}{|\boldsymbol{\omega}^g + (\boldsymbol{\omega}^g \times \boldsymbol{\gamma})\Delta t|} - \boldsymbol{\omega}^g \tag{12}$$

This interaction mechanism is illustrated schematically in Fig. 2.

2.2.5 Influence of small vortices on large ones by mapping of small vortices to grid

The mapping is performed in two cases:
• Diffusion of small vortices according to the algorithm described in Sec 2.1 of [1] and mapping them back to the grid.

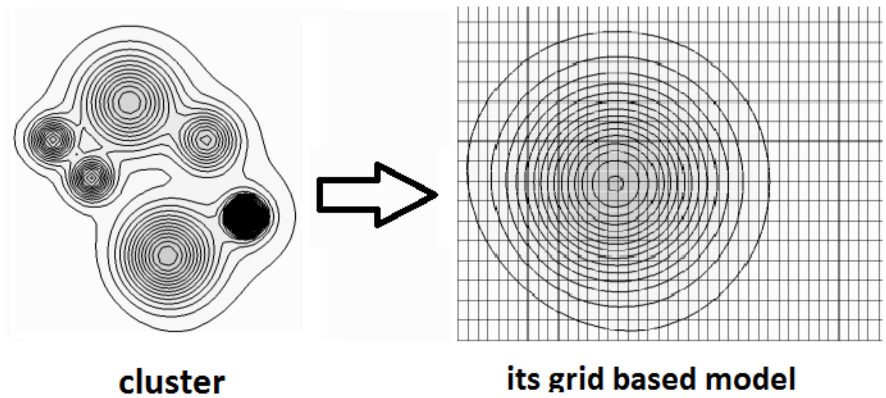

Figure 3: Interaction between scales by mapping of the fine vortex clusters to grid.

• Fine vortices can create the clusters. For this case a special algorithm should be developed to recognize the clusters, their approximation and mapping them back to the grid (Fig. 3).

3 CONCLUSIONS

The paper presents main equations of the hybrid method combining grid free and grid based numerical simulations. The flow motion is decomposed into small scales motions calculated using the grid free computational vortex method and large scale motions treated with grid based methods. Application of grid free method allows one to reduce the artificial viscosity and to increase the resolution of vortices. Equation for small scales contains explicitly the terms describing the influence of large scales whereas the governing equation for large scale motion is formally independent on small scales. This avoids the stiffness of large scale motion equations. The interaction between scales is taken into account in boundary conditions and by algorithms of mapping of small vortices to grid and vice versa.

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