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ANOTHER WAY OF SOLVING THE TAYLOR VORTEX AND THE DRIVEN CAVITY PROBLEM IN THE STREAM FUNCTION-VORTICITY FORMULATION

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Abstract. In this work, two problems will be presented: The Taylor Vortex problem and the driven cavity problem. Both problems are solved using the stream function-vorticity formulation of the Navier-Stokes equations in 2D. Results are obtained using two methods: A fixed point iterative method [1] and another one working with matrixes A and B resulting from the discretization of the laplacian and the advective term, respectively [2].

1 INTRODUCTION

In this work results for the Taylor Vortex problem and the driven cavity problem will be presented. The formulation used is the stream function-vorticity formulation of the Navier-Stokes equations in 2D. The equations are solved using finite differences and two methods: a fixed point iterative method and another one working with both matrixes A and B resulting from the discretization of the laplacian and the advective term, respectively. The iterative method has already been used for solving the Navier-Stokes and Boussinesq equations in different formulations, [3 - 6].

With the fixed point iterative method used in [1], the idea was to work with a symmetric positive definite matrix. This scheme worked very well as shown in [3 - 7], but the processing

time, was in general, very large, especially for high Reynolds numbers. With the second method we are working with a matrix which is not symmetric, but it can be proved that it is strictly diagonally dominant for Δt sufficiently small. The processing time was more or less 30% to 35% smaller when using this method.

2 MATHEMATICAL MODEL

Let $D = \Omega \times (0,T)$, $T > 0$, $\Omega \subset R^2$, be the region of the flow of an unsteady isothermal incompressible fluid and Γ its boundary. These flows are governed by the non-dimensional system of equations in D, defined by:

$$\mathbf{u}_t - \frac{1}{Re}\nabla^2\mathbf{u} + \nabla \mathrm{p} + \mathbf{u}\cdot\nabla\mathbf{u} = \mathrm{f}\,, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0. \tag{2}$$

These are the Navier-Stokes equations in primitive variables, where u is the velocity, p is the pressure and the dimensionless parameter Re is the Reynolds number. This system must be supplemented with appropriate initial and boundary conditions: $u(x,0) = u_0(x)$ in Ω and $u = g$ on Γ, respectively. In order to avoid the pressure variable and the incompressibility condition (2), the stream function-vorticity formulation is used here.

The stream function ψ is defined by:

$$u = \frac{\partial\psi}{\partial y}\,, \quad v = \frac{\partial\psi}{\partial x}, \tag{3}$$

where u and v are the velocities in x and y-axis, respectively. In this case, $(\mathbf{u}\cdot\nabla)\psi = 0$. The vorticity is defined as the curl of the velocity field, and in 2D it is defined as:

$$\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}. \tag{4}$$

So we get the following coupled system of equations:

$$\omega_t - \frac{1}{Re}\nabla^2\omega + \mathbf{u}\cdot\nabla\omega = 0\,, \tag{5}$$

$$\nabla^2\,\psi = -\,\omega. \tag{6}$$

These are the Navier-Stokes equations in the stream function-vorticity formulation.

3 THE NUMERICAL METHOD

We approximate the time derivative by the second-order scheme:

$$\omega_t(x,(n+1)\Delta t) \approx \frac{3\omega^{n+1}-4\omega^{n}+\omega^{n-1}}{2\Delta t}, \tag{7}$$

where $n \geq 1$, $x \in \Omega$ and $\Delta t > 0$ is the time step.

At each time level the following nonlinear system defined in Ω has to be solved:

$$\nabla^2 \psi = -\omega, \qquad \psi|_\Gamma = 0, \tag{8a}$$

$$\alpha\omega - \frac{1}{Re}\nabla^2\omega + \mathbf{u}\cdot\nabla\omega = f_\omega \ , \qquad \omega|_\Gamma = \omega_{bc}, \tag{8b}$$

where $\propto = \frac{3}{2\Delta t}$, and $f_\omega = \frac{4\omega^{n}-\omega^{n-1}}{2\Delta t}$. In the first time step, to obtain (ψ^1, ω^1), any second order strategy using a combination of one step can be applied. The equation (8b) is a transport type equation.

To solve (8a)-(8b) two strategies were used in this work: first we used a fixed point iterative method, described in [1].

Denoting

$$R_\omega(\omega,\psi) = \alpha\omega - \frac{1}{Re}\nabla^2\omega + \mathbf{u}\cdot\nabla\omega - f_\omega, \text{ in } \Omega, \tag{9}$$

system (8a)-(8b) is equivalent to

$$\nabla^2 \psi = -\omega, \qquad \psi|_\Gamma = 0, \tag{10a}$$

$$R_\omega(\omega,\psi) = 0, \qquad \omega|_\Gamma = \omega_{bc}. \tag{10b}$$

So then (10a)-(10b), at time level n+1, is solved via the following iterative process [1]

Given $\omega^0 = \omega^n$, and $\psi^0 = \psi^n$, solve until convergence in ω and Ψ:

$$\nabla^2 \psi^{m+1} = -\omega^m \ , \qquad \text{in } \Omega, \qquad \psi^{m+1}|_\Gamma = 0, \tag{11a}$$

$$\left(\alpha I - \frac{1}{Re}\nabla^2\right)\omega^{m+1} = \omega^m - \rho R_\omega(\omega^m, \psi^{m+1}), \text{ in } \Omega, \tag{11b}$$

$\omega^{m+1} = \omega_{bc}^{m+1}$, on Γ, ρ>0;

and then take $(\omega^{n+1}, \psi^{n+1}) = (\omega^{m+1}, \psi^{m+1})$.

In order to reduce computing time we worked on solving (8a)-(8b) by the following method at each time step:

$$\nabla^2\psi^{n+1} = -\omega^n, \tag{12a}$$

$$\left(\alpha I - \frac{\epsilon}{h^2}A\right)\omega^{n+1} + \frac{1}{2h}B\omega^{n+1} = f_\omega. \tag{12b}$$

Equation (12b) is solved using Gauss-Seidel.

4 NUMERICAL EXPERIMENTS

With respect to the driven cavity problem, Ω=(0,1)×(0,1). The top wall is moving with a nonzero velocity given by (1,0) and for the other three walls the velocity is given by (0,0). A translation of the boundary condition in terms of the velocity (primitive variable) has to be used by Taylor expansion of equation (6) and we get

$$\omega(0, y, t) = -\frac{1}{2h_x^2}[8\psi(h_x, y, t) - \psi(2h_x, y, t)] + O(h_x^2), \tag{13a}$$

$$\omega(a, y, t) = -\frac{1}{2h_x^2}[8\psi(a - h_x, y, t) - \psi(a - 2h_x, y, t)] + O(h_x^2), \tag{13b}$$

$$\omega(x, 0, t) == -\frac{1}{2h_y^2}\left[8\psi\left(x, h_y, t\right) - \psi\left(x, 2h_y, t\right)\right] + O\left(h_y^2\right), \tag{13c}$$

$$\omega(x, b, t) = -\frac{1}{2h_y^2}\left[8\psi\left(x, b - h_y, t\right) - \psi\left(x, b - h_y, t\right)\right] - \frac{3}{h_y} + O\left(h_y^2\right), \tag{13d}$$

with h_x, h_y space steps.

For the driven cavity problem we show results for Re=5000 and Re=7500, with $h_x = h_y$= 1/64 for Re=5000, and $h_x = h_y$=1/128 for Re=7500. For both Reynolds numbers, results are for t=100. With the two methods described we get the same graphs. The streamlines and the isovorticity contours are shown in Figures 1 and 2. Upwind is used to handle such Reynolds numbers, and results are reported for t=100, since for these Reynolds numbers, there is no steady state. In Table 1 we show a table comparing the times for both methods: the Fixed Point Iterative

Method (F.P.I. method) and the second one, working with both matrixes, A and B, resulting from the discretization of the laplacian and the advective term, respectively.

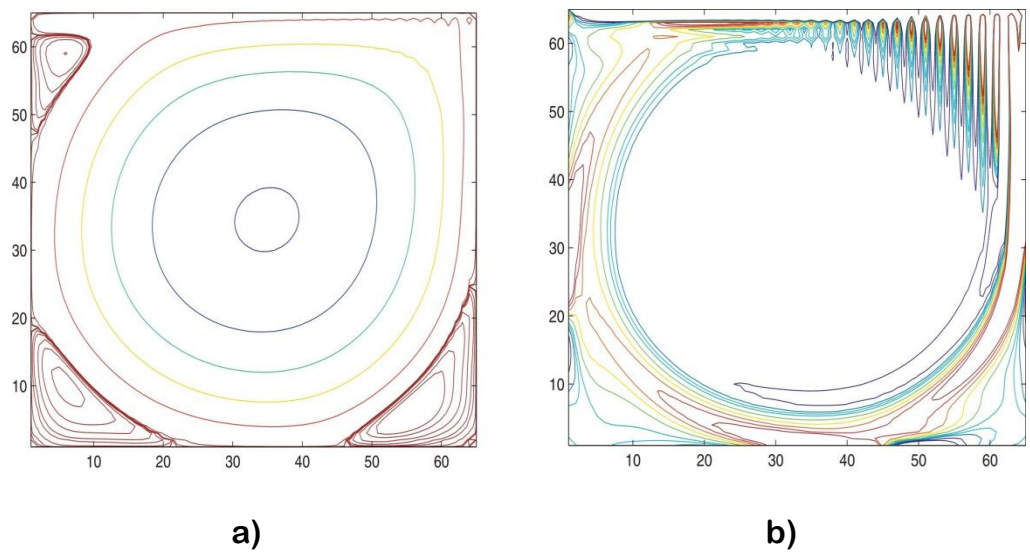

Figure 1: Streamlines (left) and isovorticity contours (right) for Re=5000, $h_x = h_y = 1/64$ and t=100.

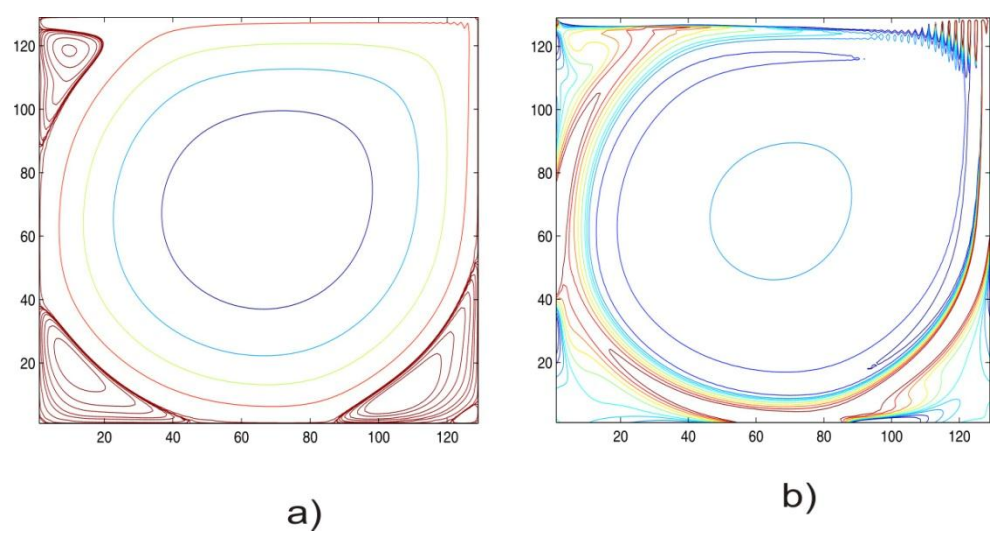

Figure 2: streamlines (left) and isovorticity contours (right) for Re=7500, $h_x = h_y = 1/128$ and t=100.

Table 1: Time (in seconds) for the two Reynolds numbers given and the two methods described and the driven cavity problem.

Reynolds number	F. P. I Method	Working with A and B
5000	612	480
7500	3204	2441

For the Taylor Vortex problem we show results for the same values of the Reynolds number, Re =5000 and Re=7500, $h = 2\pi/64$, and t=10. For Re=5000 we show the exact stream function and vorticity.

For this problem we use $\Omega = [0,2\pi] \times [0,2\pi]$. The exact solution is given by the following equations:

$$u\,(x,y,t) = -\,Cos(x)\,Sen(y)\,e^{\frac{-2t}{Re}},$$
$$v\,(x,y,t) = Sen(x)\,Cos(y)\,e^{\frac{-2t}{Re}}. \tag{14}$$

In the primitive variable formulation we have, as initial conditions:

$$u\,(x,y,0) = -\,Cos(x)\,Sen(y),$$
$$v\,(x,y,0) = Sen(x)\,Cos(y). \tag{15}$$

The stream function ψ and vorticity ω are defined by:

$$u = \psi_y, \quad v = -\psi_{x,} \quad \omega = u_y - v_x. \tag{16}$$

The initial conditions for the stream function and the vorticity are obtained from equations (14) - (16)

$$\psi(x,y,0) = Cos(x)Cos(y), \tag{17}$$

$$\omega(x,y,0) = -2\,Cos(x)\,Cos(y). \tag{18}$$

The boundary conditions for the stream function and the vorticity are obtained from equations (14) and (16). For the stream function, these boundary conditions are:

$$\psi(x,0,t) = \psi(x,2\pi,t) = Cos(x)e^{\frac{-2t}{Re}},$$
$$\psi(0,y,t) = \psi(2\pi,y,t) = Cos(y)e^{\frac{-2t}{Re}}. \tag{19}$$

For the vorticity, the boundary conditions are:

$$\omega(x,0,t) = \omega(x,2\pi,t) = 2Cos(x)e^{\frac{-2t}{Re}},$$
$$\omega(0,y,t) = \omega(2\pi,y,t) = 2Cos(y)e^{\frac{-2t}{Re}}. \tag{20}$$

Next, in Figure 3 we show streamlines and isovorticity contours for Re=5000 and t=10 with h=2π/64. In Figure 4 we show the graphs of the stream function and the vorticity in 3-D for Re=5000 at t=10. In Figure 5 we show the exact solution for Re=5000 and t=10 and they look the same. In Figure 6 we show the streamlines and the isovorticity contours for Re=7500 and t=10 with h=2π/64, respectively. Again, with both methods, we get the same results. We show the graph in 3D of the stream function and vorticity in order to see the difference in scales at different times and for the different Reynolds numbers mentioned. In Table 2 we also show the times comparing the two methods.

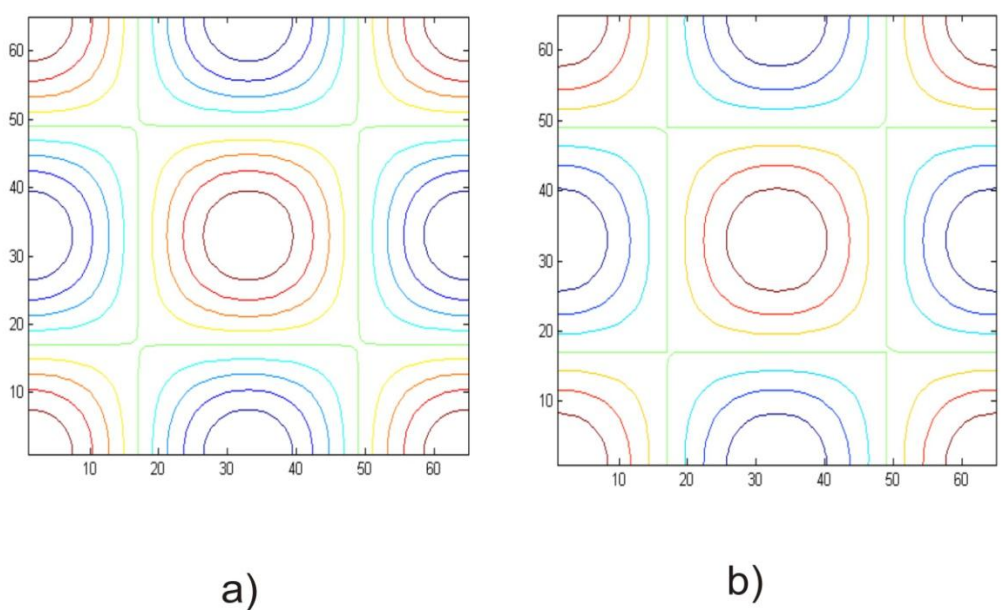

Figure 3: Streamlines and vorticity contours for Re=5000 and t=10.

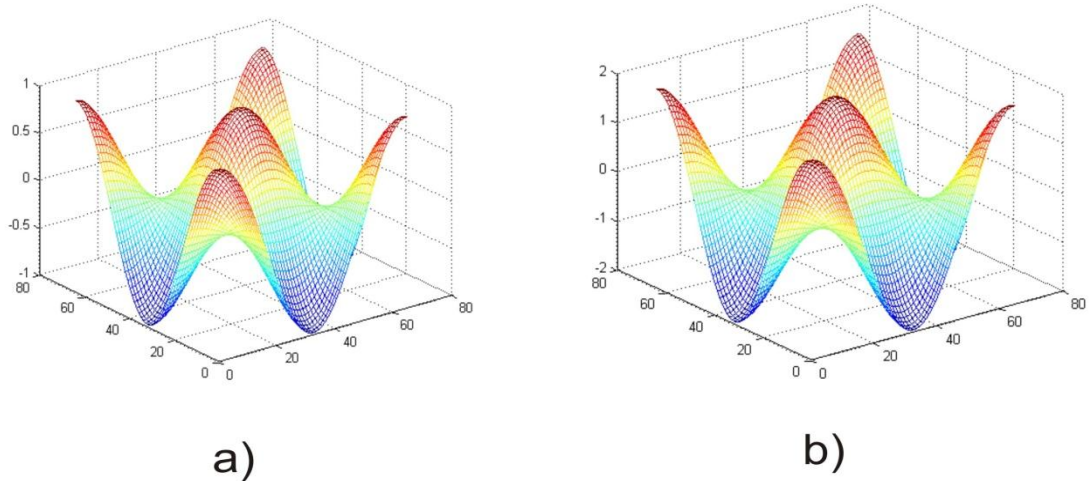

Figure 4: Stream function and vorticity for Re=5000 and t=10.

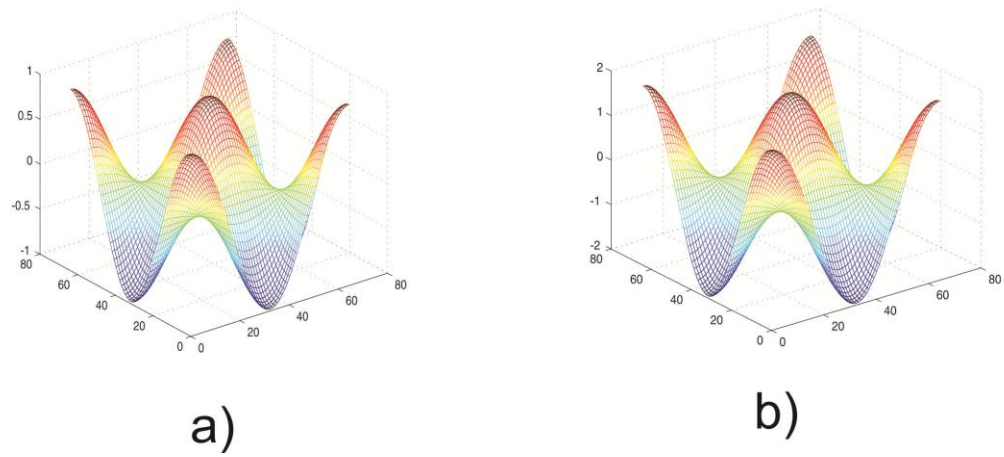

Figure 5:Exact stream function and vorticity for Re=5000 and t=10.

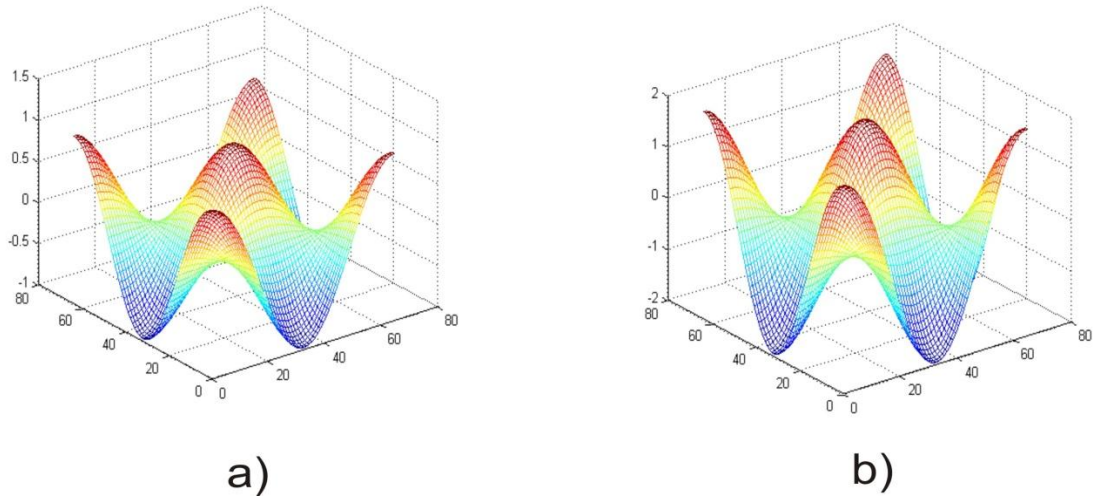

Figure 6: Stream function and vorticity for Re=7500 and t=10.

Table 2: Time (in seconds) for the two Reynolds numbers given and the two methods described and the Taylor vortex problem.

Reynolds number	F. P. I Method	Working with A and B
5000	62	51
7500	62	51

5 CONCLUSIONS

- For the driven cavity problem, results agree very well with those reported in the literature [3-6], [8,9] and with the second method, introduced here, we were able to reduce processing time for about 30% to 35%. It can be seen in Figures 1 and 2, oscillations occur because the Reynolds number is very large, so it is necessary to use smaller values of h [12], numerically for stability and physically to capture the fast dynamics of the flow. For high Reynolds numbers and small values of h the computational work takes some days, so reducing the time at least in 30% to 35% is very

important.

- For the Taylor vortex problem [7], [10], we were also able to reduce processing time for about 30% to 35%.
- We are still trying to reduce processing time. Till now we are using Fishpack [11] for solving equation (12a), which is an elliptic equation. We are working on solving this equation using Gauss-Seidel or SOR methods instead of using Fishpack, since the equation we are solving is a very simple one and Fishpack is used for solving a more general kind of equations.

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