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COMBINED DOMAIN DECOMPOSITION AND MODEL ORDER REDUCTION METHODS FOR THE SOLUTION OF COUPLED AND NON-LINEAR PROBLEMS

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Abstract. A new strategy for the efficient solution of highly nonlinear structural problems is proposed in this paper, based on the combined use of Domain Decomposition (DD) and Proper Orthogonal Decomposition (POD) techniques. The formulation here presented is tailored for applications in elasto-plastic structural dynamics. In this context the POD is applied to linear domains and a double strategy to update the reduced basis is adopted. Examples show that a meaningful computational gain of approximately 50% with respect to a monolithic solution can be obtained.

1 INTRODUCTION

Design and reliability assessment of structures like the moving parts of Micro Electro Mechanical Systems (MEMS) are based on realistic simulations of complex multi-physics (e.g. electro–mechanical, thermo–mechanical, magneto–mechanical) and/or highly non-linear and irreversible processes (e.g. elasto-plasticity, damage, fracture), that must be modeled with *sufficient* accuracy. This kind of coupled and non-linear problems lead to numerical models featuring a large number of degrees of freedom, which are prohibitive to solve within a time window compatible with the design workflow if standard finite element strategies are adopted. These applications therefore call for approximation techniques that replace large-scale computational models with simpler ones, still capable of catching the essential features but entailing a (hopefully small) fraction of the initial computational costs.

Recently, we proposed the coupled use of Domain Decomposition (DD, [1]) and Proper Orthogonal Decomposition (POD, [2]) based model order reduction techniques, to reduce the computational burden of the simulations and at the same time efficiently exploit the possibilities offered by large scale computing.

Applications to the electro-mechanical coupled problem for microsystems can be found in [3, 4, 5]. Specifically, in [5] we presented an enhancement of the DD strategy coupled to the model order reduction proposed in [4]: an ad-hoc way to impose the continuity between the de-coupled domains, and some strategies governed by POD to efficiently handle the mechanical problem were investigated.

Here, a further advancement of the methodology offered in [4, 5] is proposed. The strategy, which falls within the realm of model order reduction techniques for highly non-linear structural problems governed by, e.g. plasticity and fracture (see e.g. [6]), exploits the potentialities of the coupled use of DD and POD methods.

To match the structural behavior even beyond the onset of non-linearities, the reduced order model is adapted in two different ways:

- During the training part of the simulation, the reduced space is updated as soon as a new snapshot is collected;
- During the reduced order analysis, an *on-line* adaptation of the reduced space is performed through a *behavior check*.

The POD technique is then adopted only for the elastic parts of the mechanical domain; the solutions relevant to the dynamics of the full non-linear and of the reduced linear regions are advanced in time simultaneously, and glued together through interface relations.

To provide details about the procedure outlined here above, the remainder of the paper is organized as follows. The proposed DD method is formulated in Section 2. Section 3 is instead devoted to the description of the POD methodology, and how it is coupled with the DD strategy (DD-POD). Numerical results concerning the elastic-plastic analysis of a structural frame are discussed in Section 4. Final remarks are eventually gathered in Section 5.

2 DOMAIN DECOMPOSITION STRATEGY WITH ELASTIC INTERFACES

We consider a domain Ω with prescribed displacements on $\partial_u\Omega$ and prescribed tractions on $\partial_f\Omega$. Studying the dynamics of Ω, the finite element discretization of the continuous structural problem at the generic instant t_{n+1} leads to the following system of equations:

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{F}^{int}(\mathbf{U}) = \mathbf{F}^{ext} \tag{1}$$

where: $\mathbf{M}$ is the symmetric, positive-definite mass matrix; $\ddot{\mathbf{U}}$ is the nodal aceleration vector, and $\mathbf{U}$ is the nodal displacement one; $\mathbf{F}^{int}$ is the vector of internal forces, and $\mathbf{F}^{ext}$ is the vector of external loads. The initial conditions ($\mathbf{U}(t=0) = \mathbf{U}_0$, $\dot{\mathbf{U}}(t=0) = \dot{\mathbf{U}}_0$) and the boundary conditions on $\partial_u\Omega$ complete the formulation of the problem.

According to the standard methodology proposed in [1] for structural dynamics, extended to electro-mechanical problems in [3, 4], the kinematical fields of the problem are split into two contributions, respectively called free and link, according to:

$$\mathbf{U} = \mathbf{U}^{free} + \mathbf{U}^{link} \tag{2a}$$

$$\ddot{\mathbf{U}} = \ddot{\mathbf{U}}^{free} + \ddot{\mathbf{U}}^{link} \tag{2b}$$

Following this approach, the mechanical problem (1) is decomposed into three stages: the unconstrained problem, denoted as *free*; the *interface problem*; and the constrained problem, denoted as *link*. At time t_{n+1}, in case of two sub-domains ($s = 1, 2$) only and under the assumption of linear-elastic homogeneous behavior of Ω (i.e. $\mathbf{F}^{int}(\mathbf{U}) = \mathbf{KU}$, where $\mathbf{K}$ is the stiffness matrix), such decomposition reads:

$$free \quad \mathbf{M}_s\ddot{\mathbf{U}}_s^{free} + \mathbf{K}_s\mathbf{U}_s^{free} = \mathbf{F}_s^{ext} \tag{3a}$$

$$interface \quad \mathbf{\Lambda} \quad \longrightarrow \quad \mathbb{A}_{el=1}^{n_{el_{int}}}\left[\mathbf{H}_{el}\mathbf{\Lambda}_{el} = \mathbf{K}_{el}\left(\mathbf{C}_2\mathbf{U}_2^{free} - \mathbf{C}_1\mathbf{U}_1^{free}\right)_{el}\right] \tag{3b}$$

$$link \quad \mathbf{M}_s\ddot{\mathbf{U}}_s^{link} + \mathbf{K}_s\mathbf{U}_s^{link} = \mathbf{C}_s^T\mathbf{\Lambda} \tag{3c}$$

Here: $\mathbb{A}$ represents the assemblage operator; $\mathbf{C}_s$, $s = 1, 2$, is a Boolean matrix, which links the degrees of freedom of the whole sub-domain to those belonging to the geometrical interface of the sub-domain. $\mathbf{\Lambda}_{el}$ ($el = 1, \cdots, n_{el_{int}}$) is a vector of Lagrange multipliers for the el-th interface element, representing the tractions acting upon the interface itself, which has to be considered in the local equilibrium of each sub-domain, in order to restore the compatibility. $\mathbf{K}_{el}$ is the stiffness matrix of the el-th interface element, which reads:

$$\mathbf{K}_{el} = \int_{\Gamma_{el_{int}}} \mathbf{N}_{el}^T\boldsymbol{\kappa}\mathbf{N}_{el}\, d\Gamma \tag{4}$$

where: $\mathbf{N}_{el}$ is the matrix of shape functions, which model the displacement jumps across the interface elements; $\boldsymbol{\kappa}$ is the local elastic stiffness linking tractions $\boldsymbol{\tau}$ to displacement jumps $[\mathbf{U}]$ across the interface, through $\boldsymbol{\tau} = \boldsymbol{\kappa}\,[\mathbf{U}]$.

Through a Newmark time stepping technique, featuring parameters β and γ, the interface operator $\mathbf{H}_{el}$ in the Eq. (3b) results to be:

$$\mathbf{H}_{el} = \mathbf{I} + \beta\Delta t^2\mathbf{K}_{el}\left[\mathbf{C}_1\mathbf{M}_1^{-1}\mathbf{C}_1^T + \mathbf{C}_2\mathbf{M}_2^{-1}\mathbf{C}_2^T\right]_{el} \tag{5}$$

In the above approach, the hypothesis of a perfect interface between the sub-domains [1] has been therefore substituted by a linear elastic responses of the interface. While this assumption may lead to (controllable) numerical instabilities in the solution, it appears necessary in view of the reduced order modeling technique to be discussed next.

3 PROPER ORTHOGHONAL DECOMPOSITION AND COUPLING WITH THE DD METHOD

In this section we briefly review the coupled use of POD and DD; readers are referred to [2, 4, 7] for all the details concerning POD.

According to the snapshot version of POD [2], the model-specific solution subspace is obtained for each sub-domain by monitoring the time evolution of the displacement vector $\mathbf{U}_s$. Hence, for accuracy reasons POD forces to consider the linear (stiff) response at the interface between the mechanical sub-domains, as stated in Eq. (4). We then write:

$$\mathbf{U}_s = \sum_{i=1}^{N_s} \boldsymbol{\alpha}_{i_s} \boldsymbol{\Xi}_{i_s} \approx \sum_{i=1}^{r_s} \boldsymbol{\alpha}_{i_s} \boldsymbol{\Xi}_{i_{s_r}} = \mathbf{A}_{s_r} \boldsymbol{\Xi}_{s_r} \tag{6}$$

where: N_s is the dimension of vector $\mathbf{U}_s$; $r_s \ll N_s$ is the reduced order of the sought sub-domain model; matrix $\mathbf{A}_{s_r} = [\boldsymbol{\alpha}_{1_s}\ \boldsymbol{\alpha}_{2_s}\ \cdots\ \boldsymbol{\alpha}_{r_s}]$ collects the first r_s orthonormal vector columns of the matrix $\mathbf{A}_{s_r}$, also called proper orthogonal modes (POMs), and vector $\boldsymbol{\Xi}_{s_r}$ gather the relevant combination coefficients. Assuming to have virtually set r_s (which does not need to be necessarily the same in all the sub-domains), we need to define the basis $\boldsymbol{\alpha}_{i_s}$ to guarantee the attainment of the required overall discrepancy between full and reduced representations of the mechanical system.

POD, in its snapshot version, requires an initial *training stage* of the analysis; during this phase, snapshots $\mathbf{U}_{i_s} = \mathbf{U}_s(t_i),\ i = 1, \cdots, n_{snap}$, i.e. responses of the system to the actual excitation, are collected into the matrix $\mathbf{S}_s \in \mathbb{R}^{N_s \times n_{snap}}$, according to:

$$\mathbf{S}_s = [\mathbf{U}_{1_s}\ \mathbf{U}_{2_s} \cdots \mathbf{U}_{n_{snap_s}}] \tag{7}$$

After the training stage, the snaphots matrix $\mathbf{S}_s$ is factorized via a Singular Value Decomposition (SVD) procedure, to give:

$$\mathbf{S}_s = \mathbf{L}_s \boldsymbol{\Lambda}_s \mathbf{R}_s^T \tag{8}$$

where: $\mathbf{L}_s \in \mathbb{R}^{N_s \times n_{snap}}$ and $\mathbf{R}_s \in \mathbb{R}^{n_{snap} \times N_s}$ are orthogonal matrices, that respectively gather the left and right singular vectors; $\boldsymbol{\Lambda}_s \in \mathbb{R}^{n_{snap} \times N_s}$ is a pseudo-diagonal matrix, whose pivotal entries $\boldsymbol{\Lambda}_{ii_s}$ are the relevant singular values. By placing terms $\boldsymbol{\Lambda}_{ii_s}$ in descending order, a method to sort the POMs collected in $\mathbf{L}_s$ is obtained, see [2].

Exploiting the SVD update [5, 8], we are also able to update both POMs and singular values as soon as a new snapshot is collected without setting a priori the duration t_{snap} of the training stage. A rank-1 update of the SVD is needed, to move from $\mathbf{S}_s = \mathbf{L}_s \boldsymbol{\Lambda}_s \mathbf{R}_s^T$ to $\mathbf{S}_s + \mathbf{a}\mathbf{b}^T = \hat{\mathbf{L}}_s\ \hat{\boldsymbol{\Lambda}}_s \hat{\mathbf{R}}_s^T$, where: $\hat{\mathbf{L}}_s$, $\hat{\boldsymbol{\Lambda}}_s$ and $\hat{\mathbf{R}}_s$ are the updated singular value matrices of the s-th sub-domain; vector $\mathbf{a}$ contains the snapshot update; $\mathbf{b}$ is a binary vector, which states e.g. that the last column of $\mathbf{S}_s$ is modified by the newly available snapshot. Additional computational details can be found in [8].

In this paper, the POD technique is coupled with the DD one to solve the structural mechanical problems. The convergence of the updating procedure in the s-th sub-domain is assumed to be attained when the estimates of the number of POMs to be retained in the reduced order model, and of the relevant oriented energy content are not increased by the new snapshot collected at time t_{snap+1}, that is:

$$r_{t_{snap+1_s}} = r_{t_{snap_s}} \tag{9a}$$

$$\left(\frac{\sum_{i=1}^{r_{t_{snap+1_s}}} \Lambda_{ii_s}^2}{\sum_{i=1}^{N_s} \Lambda_{ii_s}^2} \right)_{t_{snap+1_s}} \leq \left(\frac{\sum_{i=1}^{r_{t_{snap_s}}} \Lambda_{ii_s}^2}{\sum_{i=1}^{N_s} \Lambda_{ii_s}^2} \right)_{t_{snap_s}} \tag{9b}$$

Once the order reduction matrix $\mathbf{A}_{s_r}$ is built, the dynamics of the elastic sub-domain is projected onto the obtained sub-space spanned by the POMs of the sub-domain itself. Within a Galerkin frame, we finally arrive at the following version of the semi-discretized local equilibrium equations in the s-th sub-domain:

$$\mathbf{M}_{s_r} \ddot{\mathbf{\Xi}}_{s_r} + \mathbf{K}_{s_r} \mathbf{\Xi}_{s_r} = \mathbf{F}_{s_r}^{ext} + \mathbf{\Lambda}_r \tag{10}$$

where: $\ddot{\mathbf{\Xi}}_{s_r}$ and $\mathbf{\Xi}_{s_r}$ are the reduced model acceleration and displacement vectors; $\mathbf{M}_{s_r} = \mathbf{A}_{s_r}^T \mathbf{M}_s \mathbf{A}_{s_r}$ and $\mathbf{K}_{s_r} = \mathbf{A}_{s_r}^T \mathbf{K}_s \mathbf{A}_{s_r}$ are the reduced mass and stiffness matrices of the sub-domain, respectively; $\mathbf{F}_{s_r}^{ext} = \mathbf{A}_{s_r}^T \mathbf{F}_s^{ext}$ is the reduced vector of external loads; and $\mathbf{\Lambda}_r = \mathbf{A}_{s_r}^T \mathbf{C}_s^T \mathbf{\Lambda}$ is the reduced vector of Lagrange multipliers.

According to the DD method with elastic interface law presented in Section (2), we can split the reduced kinematical solutions of each sub-domain into two terms, respectively denoted as reduced *free* and reduced *link*. The high order problems are thus projected onto the reduced space spanned by POMs gathered in the matrix $\mathbf{A}_{s_r}$.

During the reduced order analysis, an *on-line* adaptation technique of the whole reduced model is performed, through a behavior check in each sub-domain. In those sub-domains of the structure where the non-linear phenomena are incepted the POD reduced analysis is stopped, and, through a zoom-in strategy, the non-linear modeling is performed.

Property	Symbol	Value	Units
Young's modulus	E	2100	GPa
Mass density	ρ	7800	kg/m^3
Poisson's ratio	ν	0.3	–
Interface stiffness	κ_{ii}	10^{12}	N/m^3
Yield stress	σ_y	200	MPa

Table 1: Mechanical properties of the steel.

4 NUMERICAL EXAMPLE: A STRUCTURAL FRAME

We consider the two-dimensional structural frame shown in Fig. (1), as a simple model of a single-story steel building. The frame comprises two columns of height 6 m and width 0.25 m, anchored to the foundations and to an horizontal beam of span 5.5 m and width 0.5

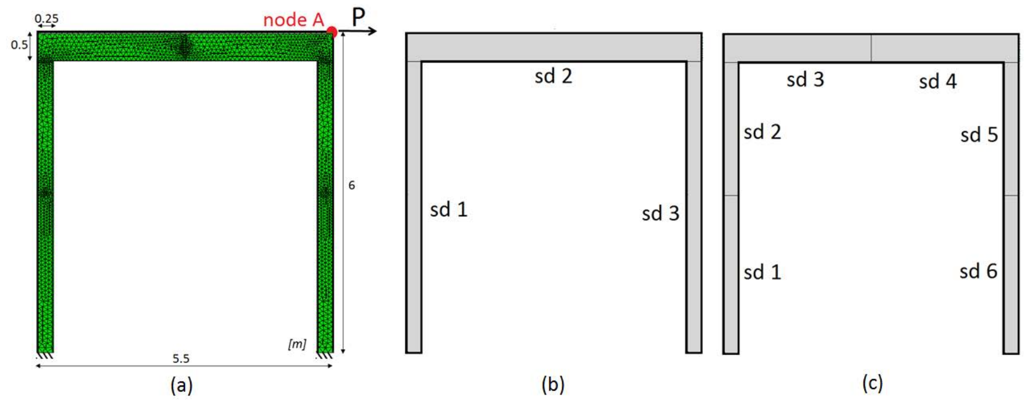

Figure 1: Structural frame: (a) system geometry, space discretization and domain decomposition in (b) three, and (c) six sub-domains.

m, through moment-resisting connections, so as they can carry bending forces. Resistance to lateral and vertical actions is provided by the rigidity of the connections and by the bending stiffness of the members.

	degrees of freedom						elements					
Monolithic	3282						2805					
DD (3sd)	852		1614		852		685		1435		685	
DD (6sd)	464	406	824	826	406	464	367	318	716	719	318	367

Table 2: Number of degrees of freedom and elements corresponding to each sub-domain subdivision

The mechanical properties of the steel are listed in Tab. 1: a non-linear elastic-perfectly plastic constitutive model, with yield stress equal to 200 MPa, is considered. Fig. 1 shows the adopted finite element mesh of constant strain triangles; plane strain conditions are assumed to hold.

The structure is either divided into three or six sub-domains. Tab. 2 gives the number of degrees of freedom and elements corresponding to each sub-domain division.

In POD simulations we have adopted $k \geq 0.999$ (see [4]) in each sub-domain, to ensure high accuracy of the solutions. When the standard SVD has been used in the training stage, 300 snapshots were collected in each sub-domain.

	Tot time	Error w.r.t. M	Gain w.r.t. M
Monolithic (M)	<u>51074</u>	$--$	$--$
POD ($t_{snap} = 0.1$)	30887	$2.56 \cdot 10^{-3}$	-39.5
DD(3sd)-POD ($t_{snap} = 0.1$)	29400	$1.02 \cdot 10^{-2}$	-42.4
DD(6sd)-POD ($t_{snap} = 0.1$)	28908	$1.2 \cdot 10^{-2}$	-43.3

Table 3: Run times ($t_{tot} = 0.5$ s).

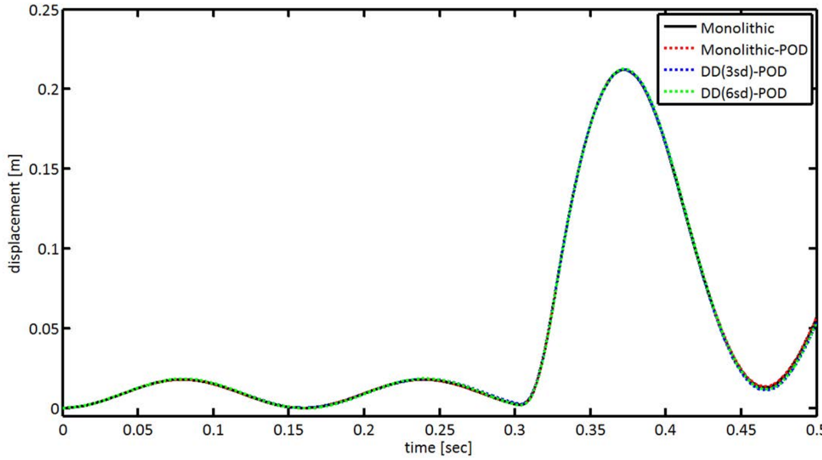

Figure 2: Horizontal displacement of node A: comparison between the reference monolithic solution and outcomes of the proposed methodology.

Focusing on node A of Fig. 1 where a time-varying load P is applied, the time evolution of its horizontal displacement is reported in Fig. 2, as obtained with the non-linear monolithic code and with the proposed procedure at varying sub-domain division. A noteworthy good agreement among all the responses can be observed.

<table>
<tr><th></th><th colspan="6">number of POMs</th></tr>
<tr><td>POD ($t_{snap} = 0.1$)</td><td colspan="6">5</td></tr>
<tr><td>DD(3sd)-POD ($t_{snap} = 0.1$)</td><td colspan="2">5</td><td colspan="2">3</td><td colspan="2">5</td></tr>
<tr><td>DD(6sd)-POD ($t_{snap} = 0.1$)</td><td>10</td><td>6</td><td>5</td><td>6</td><td>6</td><td>11</td></tr>
</table>

Table 4: Number of POMs retained in the reduced order models.

All the reduced simulations considered in this example start by adopting the POD technique in all the sub-domains. During the reduced order analysis, a *behavior check* is implemented to control if the linear elastic hypothesis in each sub-domain keeps valid. The response of each elastic sub-domain is integrated in time through the Newmark average acceleration scheme ($\gamma = 1/2$ and $\beta = 1/4$), whereas the response of the sub-domains switched to plastic modeling is integrated in time through an explicit Newmark average acceleration scheme ($\gamma = 1/2$ and $\beta = 0$).

These results show the capability of the method to handle the material non-linearities. Exploiting the DD-POD algorithm, we have been able to reduce the number of non-linear time steps, attaining a computational gain of up to 43%, with a marginal effect in comparison with the monolithic POD case (see Tab. 3). This happens even if the total number of POMs in the DD-POD simulations is slightly higher than in the case not featuring the subdivision of the frame, see Tab. 4.

To finally check the accuracy of DD-POD methodology in providing the space evolution of the von Mises stress and of the equivalent plastic strain, results of the simulations are

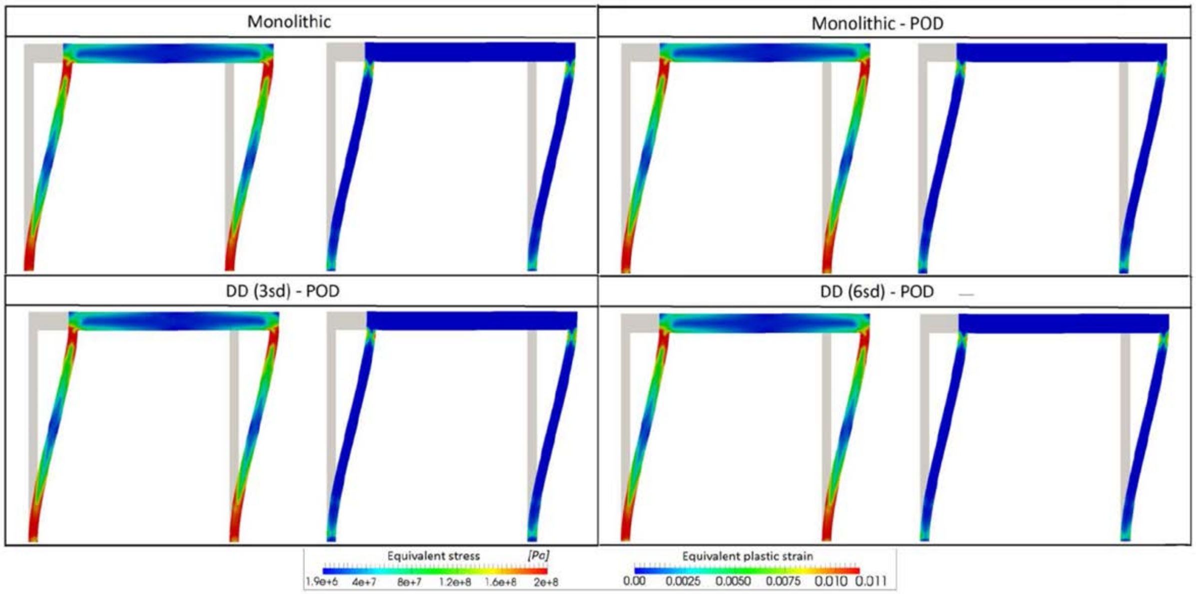

Figure 3: Von Mises stress and equivalent plastic strain on the deformed configuration (with an amplification factor equal to 5): comparison between the reference monolithic solution and the outcomes of the proposed methodology.

compared in Fig. 3 in terms of amplitude of the aforementioned fields at $t = 0.36$ s; a similar comparison can be obviously obtained for any other time instant. The good agreement among all the simulations is to be noted here.

5 CONCLUSION

In this paper, an advancement is offered with respect to what proposed in [4, 5], so as to handle in reduced order analyses diffused or localized nonlinearities in mechanical domains due to, e.g. plasticity. The proposed technique has been framed within the general DD-POD approach, which allows the simulation of multi-physics and/or highly non-linear coupled problems.

A critical discussion of the algorithm performances has been provided for an example concerning the elastic-plastic, dynamic response of a structural frame subject to time-varying loading. The coupled use of POD and DD has allowed to attain a computational gain of up to about 50%, without affecting much the accuracy of the results.

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