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NUMERICAL ANALYSIS OF THE DAMPED WAVE EQUATION BY "ENERGETIC" WEAK FORMULATIONS

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Abstract. Time-dependent problems modeled by hyperbolic partial differential equations (PDEs) can be reformulated in terms of boundary integral equations (BIEs) and solved via the boundary element method (BEM). In this context, the analysis of damping phenomena that occur in many physics and engineering problems is of particular interest. Starting from a recently developed energetic space-time weak formulation of BIEs related to wave propagation problems, we consider an extension for the damped wave equation and a coupling algorithm is presented, which allows a flexible use of FEM and BEM as local discretization techniques. PDEs associated to BIEs will be weakly reformulated by the energetic approach. This method has shown excellent stability properties, which are crucial in guaranteeing an efficient BEM-FEM coupling. Several numerical results on 1D model problems are presented and discussed.

1 INTRODUCTION

The study of wave propagation modeled by partial differential equations (PDEs) of hyperbolic type and by systems of boundary integral equations (BIEs) is important in many physics and engineering problems. The analysis of damping phenomena that occur, for example, in fluid dynamics, in kinetic theory and semiconductors, is of particular interest: the dissipation is generated in the interaction of the waves with the propagation medium and can be also closely related to the dispersion, as in the interactions between water streams and surface waves or in ferromagnetic materials. For the numerical solution of these problems, one needs consistent approximations and accurate simulations even on large time intervals.

In principle, both frequency-domain and time-domain boundary element method (BEM) can be used for hyperbolic boundary value problems. Most earlier contributions concerned

direct formulations of BEM in the frequency domain, often using Laplace or Fourier transform and addressing wave propagation problems. On the other hand, space-time BEM directly yields the unknown time-dependent quantities. In this last approach, the construction of the BIEs, via representation formula in terms of single and double layer potentials, uses the fundamental solution of the hyperbolic partial differential equation and jump relations (see e.g. [5]).
For the damped wave equation, we consider the extension of the *energetic weak formulation*, introduced for the undamped wave equation ([1, 2]). This weak formulation was presented and applied to retarded BIEs related to the wave equation without damping terms, directly expressed in the space-time domain with proof of important stability properties in time and achieving, in the discretization phase operated by BEM, significant numerical results.
The aim of the present contribution is the analysis of 1D wave propagation problem with damping terms, defined on bounded domains with assigned mixed boundary conditions, reformulated as BIEs written directly in time domain, thus avoiding the use of the Laplace transform and of its inversion. Further, for layered media, the BIE and PDE models are suitably coupled together. Several numerical results have been obtained, starting from energetic weak formulation, by BEM and BEM-FEM coupling in bounded mono- and multi-domains, respectively, and here they will be presented and discussed.

2 DIFFERENTIAL MODEL PROBLEM AND BIEs REFORMULATION

Consider the one-dimensional linear damped wave equation for a rod Ω of finite length L, with vanishing external forces, homogeneous initial data and mixed (Dirichlet - Neumann) boundary conditions, on the bounded time interval $[0, T]$. Having denoted by $u(x, t)$ the longitudinal displacement of the rod, which is a function of space and time, and by $p(x, t) = (\partial u/\partial \mathbf{n}_x)(x, t)$, the differential model problem reads

$$\left(\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\ddot{u} - \frac{2D}{c^2}\dot{u} - \frac{P}{c^2}u\right)(x,t) = 0 \qquad x \in \Omega,\ t \in [0,T], \tag{1}$$

$$u(x,0) = \dot{u}(x,0) = 0 \qquad x \in \Omega, \tag{2}$$

$$u(0,t) = g_D(t) \qquad t \in [0,T], \tag{3}$$

$$p(L,t) = \frac{\partial u}{\partial \mathbf{n}_L}(L,t) = g_N(t) \qquad t \in [0,T], \tag{4}$$

where overhead dots indicate derivatives with respect to time, c the constant wave propagation velocity, D and P represent the viscous damping coefficient and the material damping coefficient, respectively; finally $g_D(t)$ and $g_N(t)$ are given functions.
Let us consider the boundary integral representation of the solution of (1) − (4), for $x \in \Omega$, $t \in [0, T]$:

$$u(x,t) = \sum_{y=0,L} \int_0^t G(x,y;t,\tau)p(y,\tau)d\tau - \sum_{y=0,L} \int_0^t \frac{\partial G}{\partial \mathbf{n}_y}(x,y;t,\tau)u(y,\tau)d\tau \tag{5}$$

where

$$G(x,y;t,\tau) = \frac{c}{2} e^{-D(t-\tau)} I_0\,[x,y;t,\tau]\, H[c(t-\tau) - |x-y|] \tag{6}$$

is the forward fundamental solution of the damped wave equation (1), with $H[\cdot]$ the Heaviside distribution and

$$I_0[x,y;t,\tau] = \sum_{n=0}^{\infty} \frac{\left[\frac{(D^2-P)}{c^2}\left(c^2(t-\tau)^2 - (x-y)^2\right)\right]^n}{(2^n n!)^2}$$

the modified Bessel function of order 0.
With a limiting procedure in (5), for x tending to the end-points of the rod, we obtain a first BIE of the form:

$$u(x,t) = \sum_{y=0,L} \int_0^t G(x,y;t,\tau)p(y,\tau)d\tau - \sum_{y=0,L} \int_0^t \frac{\partial G}{\partial \mathbf{n}_y}(x,y;t,\tau)u(y,\tau)d\tau \tag{7}$$

for $x = 0, L$ and $t \in [0,T]$.
The BIE (7) is generally used to solve problems with Dirichlet boundary conditions, but can be used in the presence of mixed boundary conditions, too. However, in this latter case, remembering the definition of $p(x,t)$, from (7) one can obtain a second BIE of the form:

$$p(x,t) = \sum_{y=0,L} \int_0^t \frac{\partial G}{\partial \mathbf{n}_x}(x,y;t,\tau)p(y,\tau)d\tau - \sum_{y=0,L} \int_0^t \frac{\partial^2 G}{\partial \mathbf{n}_x \partial \mathbf{n}_y}(x,y;t,\tau)u(y,\tau)d\tau \tag{8}$$

for $x = 0, L$ and $t \in [0,T]$.
Of course, derivatives in (7), (8) have to be understood in a distributional sense.
Now, writing equations (7) and (8) in $x = 0$ and $x = L$, respectively, using the boundary data, performing an integration by parts involving the kernels $\partial G/\partial \mathbf{n}_x$, $\partial G/\partial \mathbf{n}_y$, $\partial^2 G/\partial \mathbf{n}_x \partial \mathbf{n}_y$ rewritten by means of relations

$$\frac{\partial I_0[x,y;t,\tau]}{\partial x} = \frac{\partial I_0[x,y;t,\tau]}{\partial y} = \frac{\partial I_0[x,y;t,\tau]}{\partial \tau} \frac{x-y}{c^2(t-\tau)}, \tag{9}$$

$$\frac{\partial H[c(t-\tau)-|x-y|]}{\partial x} = -\frac{\partial H[c(t-\tau)-|x-y|]}{\partial y} = -\frac{\partial H[c(t-\tau)-|x-y|]}{\partial \tau} \frac{y-x}{c|x-y|}, \tag{10}$$

after some straightforward calculations, one obtains the following system for $t \in (0,T)$:

$$\begin{bmatrix} \mathcal{V} & -\mathcal{K} \\ -\mathcal{K}' & \mathcal{D} \end{bmatrix} \begin{bmatrix} p^0(t) \\ u^L(t) \end{bmatrix} = \begin{bmatrix} f_D(t) \\ f_N(t) \end{bmatrix}, \tag{11}$$

having set:

$$\begin{aligned}
\mathcal{V}p^0(t) &= c\int_0^t e^{-D(t-\tau)} I_0\left[0,0;t,\tau\right] p^0(\tau)d\tau\,,\\
\mathcal{K}u^L(t) &= -\frac{L}{c}\int_0^{t-L/c} I_0\left[0,L;t,\tau\right]\frac{\partial}{\partial\tau}\left[e^{-D(t-\tau)}\frac{u^L(\tau)}{t-\tau}\right]d\tau\,,\\
\mathcal{K}'p^0(t) &= -\frac{L}{c}\int_0^{t-L/c} I_0\left[0,L;t,\tau\right]\frac{\partial}{\partial\tau}\left[e^{-D(t-\tau)}\frac{p^0(\tau)}{t-\tau}\right]d\tau\,,\\
\mathcal{D}u^L(t) &= \frac{1}{c}\left[Du^L(t)+\dot{u}^L(t)+\int_0^t\frac{\partial}{\partial\tau}\left(I_0\left[L,L;t,\tau\right]\right)e^{-D(t-\tau)}\frac{u^L(\tau)}{t-\tau}d\tau\right]
\end{aligned}\tag{12}$$

and

$$\begin{aligned}
f_D(t) &= g_D(t) - c\int_0^{t-L/c} e^{-D(t-\tau)} I_0\left[0,L;t,\tau\right] g_N(\tau)d\tau\,,\\
f_N(t) &= g_N(t) - \frac{1}{c}\int_0^{t-L/c} I_0\left[L,0;t,\tau\right]\Big[\frac{\partial}{\partial\tau}\big[e^{-D(t-\tau)}g_D(\tau)\big(\frac{1}{t-\tau}+\frac{L^2}{c^2}\frac{1}{(t-\tau)^3}\big)\big]+\\
&\quad\frac{L^2}{c^2}\frac{\partial^2}{\partial\tau^2}\Big(e^{-D(t-\tau)}\frac{g_D(\tau)}{(t-\tau)^2}\Big)\Big]d\tau\,.
\end{aligned}\tag{13}$$

In (12) $p^0(\cdot), u^L(\cdot)$ indicates the unknown $p(0,\cdot)$ and $u(L,\cdot)$, respectively; we will use this notation throughout the paper.
The energetic weak formulation of (11) is linked to the energy by this relation (obtained multiplying by $\dot{u}(t,x)$ equation (1) and integrating by parts, on $[0,T]\times\Omega$, the first addendum in space and the second addendum in time):

$$\begin{aligned}
&\sum_{x=0,L}\int_0^T \dot{u}(x,t)p(x,t)\,dt =\\
&=\frac{1}{2}\int_\Omega\left[\left(\frac{\partial u}{\partial x}\right)^2(x,T)+\frac{1}{c^2}\dot{u}^2(x,T)+\frac{P}{c^2}u^2(x,T)+\frac{2D}{c^2}\int_0^T\dot{u}^2(x,t)dt\right]dx=\\
&=\mathcal{E}_{\Omega,T}+\frac{1}{c^2}\int_\Omega\left[\frac{P}{2}u^2(x,T)+D\int_0^T\dot{u}^2(x,t)dt\right]dx\geq 0\,,
\end{aligned}\tag{14}$$

with the natural hypothesis that $P, D\geq 0$ and with $\mathcal{E}_{\Omega,T}$ denoting the total energy at time T in the domain Ω.
Following the same procedure illustrated in [1], we obtain the energetic weak formulation of the BIEs (11) for the damped wave problem:

given $(f_N\,,\,f_D)\in W:=L^2[0,T]\times H^1[0,T]$*, find* $(p^0(t)\,,\,u^L(t))\in W$ *such that*

$$\Big\langle\begin{bmatrix}\dot{\mathcal{V}} & -\dot{\mathcal{K}}\\ -\mathcal{K}' & \mathcal{D}\end{bmatrix}\begin{bmatrix}p^0\\ u^L\end{bmatrix},\begin{bmatrix}\psi\\ \dot{\varphi}\end{bmatrix}\Big\rangle_{L^2[0,T]}=\Big\langle\begin{bmatrix}\dot{f}_D\\ f_N\end{bmatrix},\begin{bmatrix}\psi\\ \dot{\varphi}\end{bmatrix}\Big\rangle_{L^2[0,T]},\qquad\forall(\psi\,,\,\varphi)\in W\,.\tag{15}$$

3 ENERGETIC COUPLING FOR A BI-DOMAIN AND NUMERICAL APPROXIMATION

Let us consider a rod of length L, constituted by two portions of length L_1, $L_2 = L - L_1$, respectively, made up of different materials (see Fig. 1) and therefore presumably characterized by different physical constants c_i, D_i, P_i, $i = 1, 2$. Hence, considering (1)

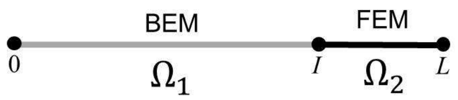

Figure 1: Example of a coupled bidomain.

expressed in the local reference system of each portion, we wish to solve the following differential problem, with assigned boundary conditions of mixed type at the end-points of the rod:

$$\left(\frac{\partial^2 u_i}{\partial x^2} - \frac{1}{c_i^2}\ddot{u}_i - \frac{2D_i}{c_i^2}\dot{u}_i - \frac{P_i}{c_i^2}u_i\right)(x,t) = f_i(x,t) \quad x \in \Omega_i,\ t \in [0,T], \tag{16}$$

$$u_i(x,0) = \dot{u}_i(x,0) = 0 \qquad x \in \Omega_i, \tag{17}$$

$$u_1^0(t) = g_D(t) \qquad t \in [0,T], \tag{18}$$

$$\frac{\partial u_2^L}{\partial \mathbf{n}_L}(t) = p_2^L(t) = g_N(t) \qquad t \in [0,T], \tag{19}$$

for $i = 1, 2$, with compatibility relations at the interface point $x = I$:

$$u_1^I(t) = u_2^I(t)\,, \qquad c_1^2 p_1^I(t) = -c_2^2 p_2^I(t)\,. \tag{20}$$

The problem defined in the domain Ω_1 is reformulated in terms of BIEs and then set in energetic weak form as in (15), considering that at the right end-point of Ω_1 the function $p_1^I(t)$ is unknown, too. Hence, the first BIE has to be considered at $x = 0$, $x = I$, while the second BIE is written at the interface $x = I$:

$$\left\langle \begin{bmatrix} \dot{\mathcal{V}}_1 & \dot{\overline{\mathcal{V}}}_1 & -\dot{\mathcal{K}}_1 \\ \dot{\overline{\mathcal{V}}}_1 & \dot{\mathcal{V}}_1 & -\dot{\mathcal{I}}_1 \\ -\mathcal{K}_1 & -\mathcal{I}_1 & \mathcal{D}_1 \end{bmatrix} \begin{bmatrix} p_1^0 \\ p_1^I \\ u_1^I \end{bmatrix}, \begin{pmatrix} \psi \\ \psi \\ \dot{\varphi} \end{pmatrix} \right\rangle_{L^2[0,T]} = \left\langle \begin{bmatrix} \dot{g_D} \\ \dot{\mathcal{V}}_1[g_D] \\ f_N \end{bmatrix}, \begin{pmatrix} \psi \\ \psi \\ \dot{\varphi} \end{pmatrix} \right\rangle_{L^2[0,T]} \tag{21}$$

where the index 1 next to the operators defined in (12) mean that the physical parameters are those of domain Ω_1, $\mathcal{I}$ is the identity function and

$$\overline{\mathcal{V}}_1[p_1^I] = c_1 \int_0^{t-I/c_1} e^{-D_1(t-\tau)} I_0\,[0, I; t, \tau]\, p_1^I(\tau) d\tau\,.$$

The problem set in the domain Ω_2 is considered in energetic weak form, obtained multiplying equation (1) by the time derivative of test function $\varphi \in H^1([0,T], H^1(\Omega_2))$ and integrating by parts the first addendum in space:

$$\begin{aligned}&\int_0^T \frac{\partial u_2^L}{\partial x}(t)\dot{\varphi}^L(t)dt - \int_0^T \frac{\partial u_2^I}{\partial x}(t)\dot{\varphi}^I(t)dt - \int_I^L \int_0^T p_2(x,t)\frac{\partial \dot{\varphi}}{\partial x}(x,t)dtdx + \\ &- \int_I^L \int_0^T \frac{1}{c_2^2}\left\{\ddot{u}_2 + 2D_2\dot{u}_2 + P_2 u_2\right\}(x,t)\dot{\varphi}(x,t)dtdx = \int_I^L \int_0^T f_2(x,t)\dot{\varphi}(x,t)dtdx\,.\end{aligned} \tag{22}$$

After having multiplied (22) by 2 and applied boundary and coupling conditions we sum it with the third integral equation in (21). The final BEM-FEM energetic formulation for the coupled problem, after suitable integrations by parts in time, is the following:

given $f_N \in L^2[0,T]$ and $f_D \in H^1[0,T]$, find $p_1^0(t), p_1^I(t) \in L^2[0,T]$ and $u_2(x,t) \in H^1([0,T], H^1(\Omega_2))$ such that

$$\begin{cases} \langle \mathcal{A}_{11}[p_1^0], \psi\rangle_{L^2[0,T]} + \langle \mathcal{A}_{12}[p_1^I], \psi\rangle_{L^2[0,T]} + \langle \mathcal{A}_{13}[\dot{u}_2^I], \psi\rangle_{L^2[0,T]} = \mathcal{F}_1(\psi) \\ \langle \mathcal{A}_{21}[p_1^0], \psi\rangle_{L^2[0,T]} + \langle \mathcal{A}_{22}[p_1^I], \psi(\cdot)\rangle_{L^2[0,T]} + \langle \mathcal{A}_{23}[\dot{u}_2^I], \psi\rangle_{L^2[0,T]} = \mathcal{F}_2(\psi) \\ \langle \mathcal{A}_{31}[p_1^0], \dot{\varphi}^I\rangle_{L^2[0,T]} + \langle \mathcal{A}_{32}[p_1^I], \dot{\varphi}^I\rangle_{L^2[0,T]} + \langle \mathcal{A}_{33}[u_2^I], \dot{\varphi}^I\rangle_{L^2[0,T]} + \langle \mathcal{A}_{34}[u_2], \dot{\varphi}\rangle_{L^2[0,T]} = \mathcal{F}_3(\dot{\varphi}) \end{cases} \tag{23}$$

$\forall \psi \in L^2[0,T]$ and $\forall \varphi \in H^1([0,T], H^1(\Omega_2))$,

where

$$\begin{aligned}
\mathcal{A}_{11}[p_1^0] &= c_1 \int_0^t e^{-D_1(t-\tau)} I_0\left[0,0;t,\tau\right] \frac{\partial p^0(\tau)}{\partial \tau} d\tau\,, \\
\mathcal{A}_{12}[p_1^I] &= c_1 \int_0^{t-I/c_1} e^{-D_1(t-\tau)} I_0[0,I;t,\tau] \frac{\partial p_1^I(\tau)}{\partial \tau} d\tau\,, \\
\mathcal{A}_{13}[\frac{\partial u_2^I}{\partial \tau}] &= \frac{I}{c_1}\int_0^{t-I/c_1} I_0[0,I;t,\tau]\frac{\partial}{\partial \tau}\left[\frac{\partial u_2^I(\tau)}{\partial \tau}\frac{e^{-D_1(t-\tau)}}{t-\tau}\right] d\tau\,, \\
A_{21}[p_1^0] &= c_1 \int_0^{t-I/c_1} e^{-D_1(t-\tau)} I_0[0,I;t,\tau]\frac{\partial p_1^0(\tau)}{\partial \tau}\, d\tau\,, \\
A_{22}[p_1^I] &= c_1 \int_0^t e^{-D_1(t-\tau)} I_0[0,0;t,\tau]\frac{\partial p_1^I(\tau)}{\partial \tau}\, d\tau\,, \\
A_{23}[\frac{\partial u_2^I}{\partial \tau}] &= -\dot{u}_2^I(t)\,, \\
A_{31}[p_1^0] &= \frac{I}{c_1}\int_0^{t-I/c_1} I_0[0,I;t,\tau]\frac{\partial}{\partial \tau}\left[\frac{\partial p^0(\tau)}{\partial \tau}\frac{e^{-D_1(t-\tau)}}{t-\tau}\right]\dot{\varphi}^I(t)\, d\tau\,, \\
A_{32}[p_1^I] &= -p_1^I(t)\,, \\
A_{33}[u_2^I] &= -\Big(\frac{D_1}{c_1}u_2^I(t) + \frac{1}{c_1}\dot{u}_2^I(t)\Big)\dot{\varphi}^I(t) - \int_0^t \frac{\partial I_0[I,I;t,\tau]}{\partial \tau}\frac{e^{-D_1(t-\tau)}}{c_1(t-\tau)}\, d\tau\,, \\
A_{34}[u_2] &= -2\int_I^L \Big(\frac{\partial u_2}{\partial x}\frac{\partial \dot{\varphi}}{\partial x} + \Big[\frac{1}{c_2^2}\ddot{u}_2 + 2\frac{D_2}{c_2^2}\dot{u}_2 + \frac{P_2}{c_2^2}\Big]\dot{\varphi}\Big)(x,t)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}_1(\psi) &= \int_0^T \dot{g}_D(t)\psi(t)dt\,,\\
\mathcal{F}_2(\psi) &= \int_0^T \int_0^{t-\frac{I}{c_1}} \frac{I}{c_1} I_0\left[I,0;t,\tau\right] \frac{\partial}{\partial\tau}\left(\dot{g}_D(\tau)\frac{e^{-D_1(t-\tau)}}{\tau-t}\right)\psi(t)d\tau dt\,,\\
\mathcal{F}_3(\dot{\varphi}) &= -\int_0^T \int_0^{t-\frac{I}{c_1}} I_0\left[I,0;t,\tau\right]\left\{\frac{I^2}{c_1^3}\frac{\partial}{\partial\tau}\left(g_D(t)\frac{e^{-D_1(t-\tau)}}{(\tau-t)^3}\right)+\,,\right.\\
&\quad\left.+\frac{I^2}{c_1^3}\frac{\partial^2}{\partial\tau^2}\left(g_D(t)\frac{e^{-D_1(t-\tau)}}{(\tau-t)^2}\right)+\frac{1}{c_1}\frac{\partial}{\partial\tau}\left(g_D(t)\frac{e^{-D_1(t-\tau)}}{\tau-t}\right)\right\}\dot{\varphi}^I(t)d\tau dt+\\
&\quad -2\int_0^T g_N(t)\dot{\varphi}^L(t)dt + 2\int_I^L\int_0^T f_2(x,t)\dot{\varphi}(x,t)dtdx\,.
\end{aligned}$$

For time discretization, we consider a uniform decomposition of the time interval $[0,T]$ with time step $\Delta\,t = T/N_{\Delta t}\,, N_{\Delta t}\in\mathbb{N}^+$, generated by the $N_{\Delta t}+1$ instants:

$$t_k = k\,\Delta\,t,\quad k=0,\cdots,N_{\Delta t}$$

and we choose temporally piecewise constant shape functions for the approximation of p_1 and piecewise linear shape functions for the approximation of u_2, although higher degree shape functions can be used. Note that, for this particular choice, temporal shape functions, for $k=0,\cdots,N_{\Delta t}-1$, will be defined as

$$\psi_k(t) = H[t-t_k] - H[t-t_{k+1}]$$

for the approximation of p_1, hence

$$p_1^0(t) \cong \sum_{k=0}^{N_{\Delta t}-1}\alpha_k\psi_k(t),\qquad p_1^I(t)\cong\sum_{k=0}^{N_{\Delta t}-1}\gamma_k\psi_k(t)\,, \tag{24}$$

or as

$$\varphi_k(t) = R(t-t_k) - R(t-t_{k+1})$$

for the approximation of u_2, where $R(t-t_k) = \frac{t-t_k}{\Delta\,t}\,H[t-t_k]$ is the ramp function.
For the space discretization, we consider a uniform decomposition of the space interval $[I,L]$ with space step $\Delta\,x = |L-I|/N_{\Delta x}\,, N_{\Delta x}\in\mathbb{N}^+$, generated by the $N_{\Delta x}+1$ nodes $\{x_0,\cdots,x_{N_{\Delta x}}\}$, so that in the FEM domain Ω_2, u_2 is approximated as

$$u_2(x,t)\cong\sum_{k=0}^{N_{\Delta t}-1}\sum_{j=0}^{N_{\Delta x}}\beta_{kj}\varphi_k(t)\widehat{\varphi}_j(x) \tag{25}$$

with $\widehat{\varphi}_j(x)$ piecewise linear hat-function such that $\widehat{\varphi}_j(x_h)=1$ only if $j=h$, 0 otherwise.
The discretization of the energetic weak formulation (23) produces the linear system

$$\mathrm{M}\,\mathrm{x} = \mathrm{y}\,, \tag{26}$$

where matrix M has a block lower triangular Toeplitz structure, as frequently pointed out for the energetic BEM applied to undamped wave propagation problems (see [2, 3, 4]):

$$\begin{pmatrix} \mathrm{M}^{(0)} & 0 & 0 & \cdots & 0 \\ \mathrm{M}^{(1)} & \mathrm{M}^{(0)} & 0 & \dots & 0 \\ \mathrm{M}^{(2)} & \mathrm{M}^{(1)} & \mathrm{M}^{(0)} & \dots & 0 \\ \dots & \dots & \dots & \ddots & \vdots \\ \mathrm{M}^{(N_{\Delta t}-1)} & \mathrm{M}^{(N_{\Delta t}-2)} & \mathrm{M}^{(N_{\Delta t}-3)} & \dots & \mathrm{M}^{(0)} \end{pmatrix} \begin{pmatrix} \mathrm{x}^{(0)} \\ \mathrm{x}^{(1)} \\ \mathrm{x}^{(2)} \\ \vdots \\ \mathrm{x}^{(N_{\Delta t}-1)} \end{pmatrix} = \begin{pmatrix} \mathrm{y}^{(0)} \\ \mathrm{y}^{(1)} \\ \mathrm{y}^{(2)} \\ \vdots \\ \mathrm{y}^{(N_{\Delta t}-1)} \end{pmatrix} \quad (27)$$

and the unknowns are organized as follows:

$$\mathrm{x}^{(\ell)} = \big(\alpha_\ell,\, \gamma_\ell,\, \beta_{\ell 0},\, \beta_{\ell 1}, \dots, \beta_{\ell N_{\Delta x}}\big)^\top \quad \ell = 0, \dots, N_{\Delta t} - 1\,.$$

Note that, with this choice of basis, each block is symmetric, highly sparse and it has the following structure:

$$\mathrm{M}^{(\ell)} = \begin{pmatrix} \mathrm{M}^{(\ell)}_{BEM} & \mathrm{M}^{(\ell)}_{BEM,I} & 0 \\ \mathrm{M}^{(\ell)}_{I,BEM} & \mathrm{M}^{(\ell)}_{I} & \mathrm{M}^{(\ell)}_{I,FEM} \\ 0 & \mathrm{M}^{(\ell)}_{FEM,I} & \mathrm{M}^{(\ell)}_{FEM} \end{pmatrix}$$

where diagonal sub-blocks have dimensions 1, 2, $N_{\Delta x}$, respectively, and whose elements have been evaluated numerically.

4 NUMERICAL RESULTS

• As an application of energetic BEM, let us consider the following example, taken from paper [6], which includes the damping in one-dimensional wave propagation using a fully boundary element formulation. In that paper, as here, the authors consider the free-space Green's function G given in (6) incorporating both viscous and material damping, but their formulation involves an implicit time discretization scheme.

The test example states that, in problem (1)-(4), the rod is fixed in the left end-point $x = 0$ ($g_D(t) = 0$), while on the right end-point $x = L = 1$ a traction $g_N(t)$ is applied. For this kind of configuration the analytical solution, useful for comparison with numerical results, is known:

$$u(x,t) = \sum_{n=1}^{+\infty} (-1)^{n-1} \int_0^{+\infty} 2\big[G\big(x, (2n-1)L, t, \tau, \big) - G\big(x, -(2n-1)L, t, \tau\big)\big] g_N(\tau) d\tau\,. \quad (28)$$

In order to investigate the effect of material damping in structures, a single pulse traction

$$g_N(t) = H[t] - H[t - 1/4]$$

is applied; in Fig. 2, $u(L,t)$ has been computed starting from energetic boundary element formulation (15) and using $\Delta t = 0.01$, and it overlaps the analytical solution. Furthermore, Table 1 shows the convergence towards the analytical solution (28) in $L^2([0,1])$-norm, refining the discretization parameter Δt.

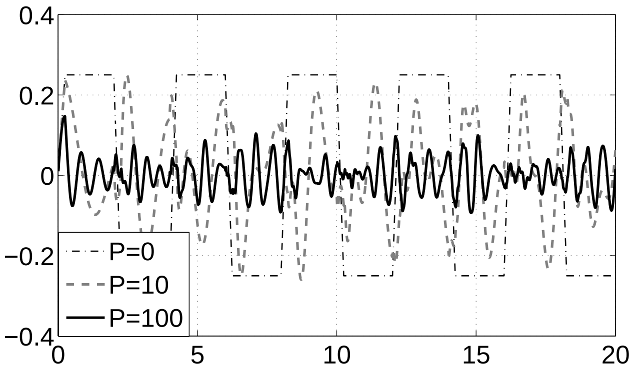

Figure 2: $u(L,t)$ when $g_N(t) = H[t] - H[t-1/4]$, $c=1$, $D=0$, varying P.

In Fig. 3, we emphasize convergence, for the case $P = 10$ in the time interval $[0,4]$.

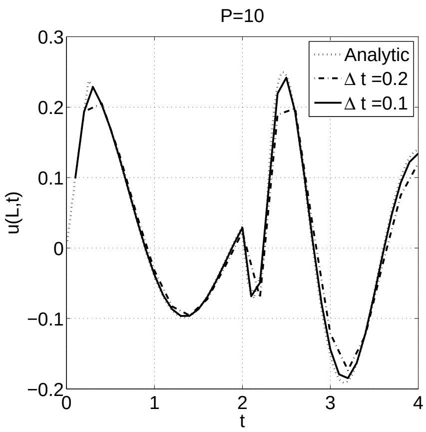

Figure 3: Focus of solution $u(L,t)$ in the interval $[0,4]$, as in [6], emphasizing convergence.

	$P = 10$
$\Delta t = 0.2$	3.07E-01
$\Delta t = 0.1$	1.14E-01
$\Delta t = 0.05$	3.11E-02
$\Delta t = 0.025$	8.47E-03
$\Delta t = 0.01$	3.27E-03

Table 1: Table of errors in L^2-norm in $x = L$ diminishing the discretization parameter Δt.

In order to investigate the diffusive nature of viscous damping in structures, the repetitive pulse traction

$$g_N(t) = \sum_{n=0}^{\infty} H[t-n] - H[t-n-1/4] \tag{29}$$

represented in Fig. 4 is now applied.

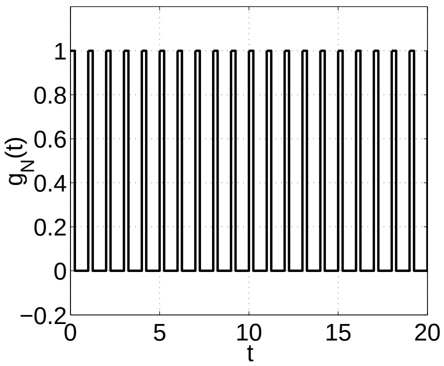

Figure 4: Repetitive pulse traction.

The approximate solution $u(L, t)$, depicted in Fig. 5 reproduces the analytical one that can be obtained substituting (29) in (28).

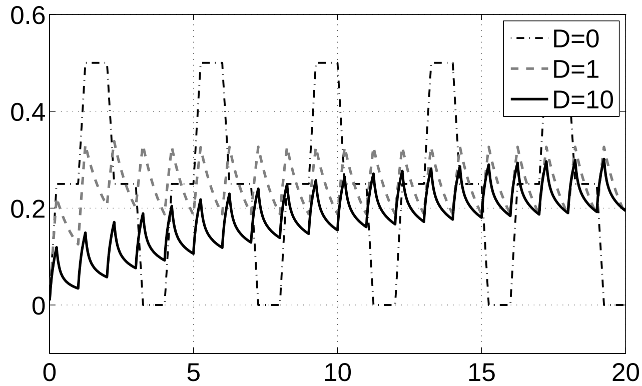

Figure 5: $u(L, t)$ when $g_N(t)$ is given by (29), $c = 1$, $P = 0$, varying D.

In Fig. 6 and Table 2, we emphasize convergence, for the case $D = 10$, in the time interval $[0, 4]$.

In all cases observe that no instabilities appear and the approximate solutions are in agreement with the exact ones also considering time intervals longer than those investigated in [6].

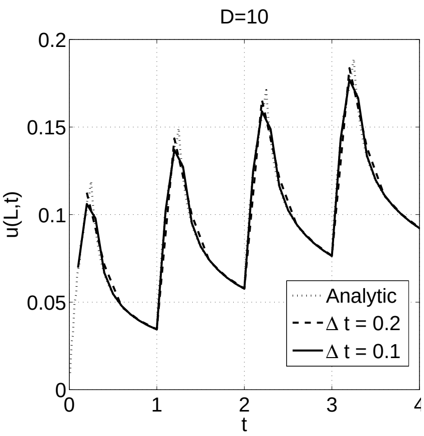

Figure 6: Focus of solution $u(L,t)$ in the interval $[0,4]$, as in [6], emphasizing convergence.

	$D = 10$
$\Delta t = 0.2$	2.35E-02
$\Delta t = 0.1$	1.59E-02
$\Delta t = 0.05$	2.97E-03
$\Delta t = 0.025$	2.23E-03
$\Delta t = 0.01$	2.20E-03

Table 2: Table of errors in L^2-norm in $x = L$ diminishing the discretization parameter Δt.

• As an application of energetic BEM-FEM, the following example takes into consideration the domain Ω split in 2 sub-domains of equal length (Fig. 7), with $g_D(t) = 0$ assigned in $x = 0$ and the traction $g_N(t) = H[t]$ applied in $x = L = 1$.

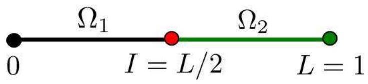

Figure 7: Domain.

$$\begin{cases} c_1 = c_2 = 1;\; D_1 = D_2 = 0;\; P_2 = 0 \\ f_2(x,t) = 0 \end{cases}$$

We fix the discretization parameters $\Delta x = 0.05$ and $\Delta t = 0.01$ and we increase the material damping P_1 in the BEM sub-domain Ω_1, observing the evolution of the behavior of $u(L,t)$ and $u(I,t)$.

If $P_1 = 0$, as in FEM sub-domain Ω_2, the approximated solution behaves like the analytical solution of the wave equation without damping, both in $x = L$ and $x = I$; in the limit $P_1 \to +\infty$, the wave tends to be completely reflected by the BEM sub-domain: in fact the bold lines in Fig. 8 represent an approximation of the analytical solution of the wave equation without damping for a rod of length $L = 0.5$ with boundary conditions $g_D(t) = 0$ in $x = I$ and $g_N(t) = H[t]$ in $x = L$.

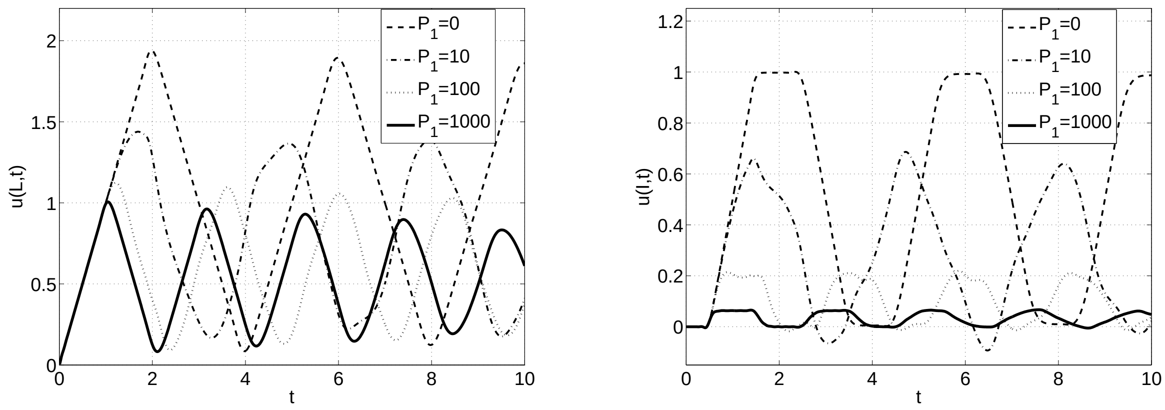

Figure 8: $u(L,t)$ on the left and $u(I,t)$ on the right, varying P_1.

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