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ANALYSIS OF A DYNAMIC CONTACT PROBLEM INVOLVING A NONLINEAR THERMOVISCOELASTIC BEAM WITH SECOND SOUND

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Abstract. In this paper we consider a dynamic contact problem describing the mechanical and thermal evolution of a damped extensible thermoviscoelastic beam under the Cattaneo law, relying the heat flux to the gradient of the temperature. The beam is rigidly clamped at its left end whereas the right end of the beam is supposed to move vertically between two stops reactive and behaving as a nonlinear spring. Existence and uniqueness of the solution are stated, as well as the exponential decay of the related energy. Then, fully discrete approximations are introduced by using the classical finite element method and the implicit Euler scheme to approximate the spatial variable and to discretize the time derivatives, respectively. An a priori error estimates result is obtained, from which the linear convergence of the algorithm is deduced. Finally, a numerical simulation is presented to demonstrate the behavior of the solution.

1 INTRODUCTION

We investigate here the longtime behavior of a nonlinear dynamic contact problem describing the vibrations of a damped extensible thermoelastic beam of natural length ℓ. The model is derived combining the ideas of Woinowsky-Krieger [1] with the theory of thermoelasticity with second sound (see, e.g., [2]), assuming that the material is viscoelastic of the Kelvin-Voigt type. Denoting by $T > 0$ the final time of interest, the mechanical problem is written as follows:

$$\begin{aligned} &u_{tt}(x,t) + \kappa u_{xxxx}(x,t) - \left[\beta + \int_0^\ell |u_x(x,t)^2|\,dx\right] u_{xx}(x,t) + m\theta_{xx}(x,t) \\ &\qquad\qquad +du_{xxxxt}(x,t) = 0, \\ &\theta_t(x,t) + \alpha q_x(x,t) - m u_{xxt}(x,t) = 0, \\ &\tau q_t(x,t) + q(x,t) + \alpha\theta_x(x,t) = 0, \end{aligned} \tag{1}$$

being $u = u(x,t) : [0,\ell] \times [0,T] \to \mathbb{R}$, $\theta = \theta(x,t) : [0,\ell] \times [0,T] \to \mathbb{R}$ and $q = q(x,t) : [0,\ell] \times [0,T] \to \mathbb{R}$ the unknown variables. Here, u represents the vertical deflection of the beam from its configuration at rest, the "temperature variation" θ actually arises from an approximation of the temperature variation with respect to a reference value (see, e.g., [3]), and the only nonzero component of the heat flux is its x-component q. Throughout the paper, the subscripts x and t indicate partial derivatives, and the coefficients κ, d, α, τ and m represent the scaled elasticity modulus, the viscosity coefficient, the scaled thermal diffusivity, the relaxation time and the coupling coefficient depending on the material properties, respectively, with $\kappa, d, \alpha, \tau \in \mathbb{R}^+$ and $m \in \mathbb{R} \setminus \{0\}$. In particular, the constant β accounts for the axial force acting in the reference configuration: β is positive when the beam is stretched and negative when compressed.

To complete the description of the problem we consider the following initial conditions:

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x),\ q(x,0) = q_0(x) \text{ in } [0,\ell], \tag{2}$$

for some given functions $u_0, u_1, \theta_0, q_0 : (0,\ell) \to \mathbb{R}$.

In order to simplify the analysis, we assume that the beam is rigidly clamped at its left end, that is:

$$u(0,t) = 0, \quad u_x(0,t) = 0 \quad \text{in } [0,T], \tag{3}$$

and also

$$\theta_x(0,t) = 0, \quad q(0,t) = 0 \quad \text{in } [0,T]. \tag{4}$$

For the sake of simplicity, we denote by ζ the triplet $\zeta = (u,\theta,q)$ and by $\sigma(t,\zeta)$ the shear stress at $x = \ell$, *i.e.*

$$\sigma(t,\zeta) := -\kappa u_{xxx}(\ell,t) + \left[\beta + \int_0^\ell |u_x(x,t)|^2\,dx\right] u_x(\ell,t) - m\theta_x(\ell,t) - du_{xxxt}(\ell,t).$$

The right end of the beam is supposed to move vertically between two stops and the joint is assumed to be asymmetrical, so that $g = g_1 + g_2$, where $g_1 > 0$ and $g_2 > 0$ are, respectively, the upper and lower clearance when the system is at rest. In addition we suppose that the stops are reactive and behave as a nonlinear spring. More precisely, we describe the stress using the so-called normal compliance condition (see, e.g., [4]):

$$\tilde{\sigma}(t) := -\frac{1}{\varepsilon}\left\{[u(\ell,t)-g_1]^+ - [-u(\ell,t)-g_2]^+\right\}, \tag{5}$$

where $[f]^+ := \max\{f,0\}$ denotes the positive part of a function f. The previous law means that when there is no contact, namely $-g_2 < u(\ell,t) < g_1$, the right end is free and the stress vanishes. On the contrary, when the right end exceeds the clearance, namely $u(\ell,t) \geq g_1$ or $u(\ell,t) \leq -g_2$, $\tilde{\sigma}(t)$ is proportional to the interpenetration and is opposite to the displacement. Accordingly, at $x = \ell$ we have the boundary condition

$$\sigma(t,\zeta) = \tilde{\sigma}(t). \tag{6}$$

Finally, we suppose that

$$\kappa u_{xx}(\ell,t) + d u_{xxt}(\ell,t) = 0, \quad \theta(\ell,t) = 0 \quad \text{in } [0,T]. \tag{7}$$

This means that, at this end $x = \ell$, a Dirichlet thermal boundary condition is given and no moment is exerted.

2 EXISTENCE, UNIQUENESS AND ENERGY DECAY RESULTS

Given a Banach space X, let $\|\cdot\|_X$ be the usual norm defined on X. In particular, we denote by $\langle\cdot,\cdot\rangle$, $\|\cdot\|$ the inner product and the norm defined on $L^2(0,\ell)$, respectively. For later convenience, we suppress the dependence on the spatial variable.

Before stating the existence and uniqueness results of problem (1)–(7), we define the following notation for function spaces:

$$\begin{aligned}
\mathcal{V}_1 &= \{\varphi \in H^1(0,\ell); \quad \varphi(0) = 0\},\\
\mathcal{V}_2 &= \{\varphi \in H^2(0,\ell); \quad \varphi(0) = \varphi_x(0) = 0\},\\
\mathcal{U} &= \{\varphi \in \mathcal{V}_2; \quad -g_2 \leq \varphi(\ell) \leq g_1\},\\
\mathcal{H} &= \{\varphi \in H^1(0,\ell); \quad \varphi(\ell) = 0\},
\end{aligned}$$

and we construct the following functionals:

$$\begin{aligned}
&M(t,u) = \beta + \|u_x(t)\|^2,\\
&E(t,\zeta) = \frac{1}{2}\left[\|u_t(t)\|^2 + \kappa\|u_{xx}(t)\|^2 + \|\theta(t)\|^2 + \tau\|q(t)\|^2\right],\\
&\mathcal{J}^\varepsilon(t) = \frac{1}{2\varepsilon}\left\{\left|[u(\ell,t)-g_1]^+\right|^2 + \left|[-u(\ell,t)-g_2]^+\right|^2\right\},\\
&\mathcal{E}(t,\zeta) = E(t,\zeta) + \mathcal{J}^\varepsilon(t).
\end{aligned}$$

By using the Faedo-Galerkin method, we have the following theorem which states the existence of solutions to problem (1)-(7) (see [6]).

Theorem 1. *Let $T > 0$ and initial data $(u_0, u_1, \theta_0, q_0)$ verifying the regularity*

$$u_0 \in H^4(0,\ell), \quad u_1 \in H^4(0,\ell), \quad \theta_0 \in H^2(0,\ell), \quad q_0 \in L^2(0,\ell),$$

and be compatible with boundary conditions (3)–(7) *for $t = 0$. Then, there exists a triplet of functions (u, θ, q) with*

$$\begin{aligned}
&u \in W^{2,\infty}\left(0,T; L^2(0,\ell)\right) \cap H^2\left(0,T; H^2(0,\ell)\right) \cap W^{1,\infty}\left(0,T; H^2(0,\ell)\right),\\
&\theta \in W^{1,\infty}\left(0,T; L^2(0,\ell)\right) \cap L^\infty(0,T; H^1(0,\ell)),\\
&q \in W^{1,\infty}\left(0,T; L^2(0,\ell)\right) \cap L^\infty(0,T; H^1(0,\ell)),
\end{aligned} \tag{8}$$

satisfying

$$\begin{aligned}
&\langle u_{tt}(t), w\rangle + \kappa\langle u_{xx}(t), w_{xx}\rangle + \langle M(t,u)u_x(t), w_x\rangle - m\langle\theta_x(t), w_x\rangle\\
&\qquad +d\langle u_{xxt}(t), w_{xx}\rangle = \tilde{\sigma}(t)w(\ell),\\
&\theta_t(t) + \alpha q_x(t) - m u_{xxt}(t) = 0,\\
&\tau q_t(t) + q(t) + \alpha\theta_x(t) = 0,
\end{aligned} \tag{9}$$

for any $w \in \mathcal{V}_2$ and initial and boundary conditions (2)–(7)*.*

In the following theorem, proved in [6], we show that solution $\zeta = (u, \theta, q)$ to problem (9) is unique.

Theorem 2. *For any $T > 0$, the solution $\zeta = (u, \theta, q)$ to problem* (9)*, with initial data satisfying* (2) *and compatible with the boundary conditions* (3)–(7)*, is unique.*

In order to prove that the energy $\mathcal{E}(t, \zeta)$, given in (8) and associated to problem (9), decays exponentially as time goes to infinity, we use some suitable Lyapunov functions equivalent to the related energy and the multiplicative technique that allows us to show explicit relations between the energy of the elastic and thermal components of the system (see again [6] for details).

Theorem 3. *Let (u, θ, q) be the solution provided by Theorem 1. If $\beta > -2\kappa/\ell^2$, then there exist two positive constants R and ω such that*

$$\mathcal{E}(t,\zeta) \le R\,\mathcal{E}(0,\zeta)\,e^{-\omega t}, \quad t \ge 0. \tag{10}$$

3 FULLY DISCRETE APPROXIMATIONS

In this section we describe a fully discrete finite element scheme for numerical approximations of the normal compliance problem (1)–(7).

First, we write problem (1)–(7) in a classical variational form. Multiplying equilibrium equations (1) by adequate test functions $w \in \mathcal{V}_2$, $z \in \mathcal{H}$ and $\xi \in \mathcal{V}_1$, we obtain the following variational formulation of problem (1)–(7) in terms of the velocity field v.

Find $v : [0,T] \to \mathcal{V}_2$, $\theta : [0,T] \to \mathcal{H}$ and $q : [0,T] \to \mathcal{V}_1$ such that $u(0) = u_0$, $v(0) = v_0$, $\theta(0) = \theta_0$ and $q(0) = q_0$, and for a.e. $t \in (0,T)$,

$$\langle v_t(t), w\rangle + \kappa\langle u_{xx}(t), w_{xx}\rangle + \langle M(t,u)u_x(t), w_x\rangle - m\langle\theta_x(t), w_x\rangle + d\langle v_{xx}(t), w_{xx}\rangle + \frac{1}{\varepsilon}\left\{[u(\ell,t)-g_1]^+ - [-u(\ell,t)-g_2]^+\right\} w(\ell) = 0, \tag{11}$$

$$\langle\theta_t(t), z\rangle + \langle\alpha q_x(t) - m v_{xx}(t), z\rangle = 0, \tag{12}$$

$$\langle\tau q_t(t) + q(t) + \alpha\theta_x(t), \xi\rangle = 0, \tag{13}$$

where we recall that the displacement field $u : [0,T] \to \mathcal{V}_2$ is obtained from the velocity field v using the formula:

$$u(t) = u_0 + \int_0^t v(s)\,ds, \tag{14}$$

and we note that we suppressed the dependence on the spatial variable for the sake of clarity.

Now, we obtain the fully discrete approximations of problem (11)–(14). First, we approximate the spatial variable by using a uniform partition of $[0,\ell]$, denoted by $\{I_i\}_{i=1}^{N_{tot}}$, in such a way that $[0,\ell] = \bigcup_{i=1}^{N_{tot}} I_i$. We let h denote the size of the partition, $h = \mathrm{meas}(I_1)$ and let V^h be the finite element space approximating $\mathcal{V}_2$,

$$V^h = \left\{ w^h \in \mathcal{V}_2; \quad w^h_{|_{I_i}} \in P_3(I_i), \quad 1 \le i \le N_{tot} \right\}, \tag{15}$$

where $P_q(I)$ denotes the polynomial space of degree less or equal to q restricted to I. We note that, since $w^h \in \mathcal{V}_2 \subset H^2(0,\ell)$, $w^h \in C^1([0,\ell])$ and then V^h is composed of C^1 piecewise cubic functions.

Similarly, let E^h and K^h be the following finite element spaces approximating $\mathcal{V}_1$ and $\mathcal{H}$, respectively, given by

$$E^h = \left\{ \xi^h \in \mathcal{V}_1; \quad \xi^h_{|_{I_i}} \in P_1(I_i), \quad 1 \le i \le N_{tot} \right\}, \tag{16}$$

$$K^h = \left\{ z^h \in \mathcal{H}; \quad z^h_{|_{I_i}} \in P_1(I_i), \quad 1 \le i \le N_{tot} \right\}. \tag{17}$$

Since ξ^h, $z^h \in H^1(0,\ell)$, then ξ^h, $z^h \in C([0,\ell])$ and so E^h and K^h are composed of continuous and piecewise linear functions.

Secondly, to discretize the time derivatives, let $0 = t_0 < t_1 < \ldots < t_N = T$ be a uniform partition of the time interval $[0,T]$. We denote by $k = t_1 - t_0 = T/N$ the time step size and, for a continuous function $w(t)$, let $w_n = w(t_n)$. Moreover, given a sequence $\{w_n\}_{n=1}^N$, let $\delta w_n = (w_n - w_{n-1})/k$ represent the corresponding divided differences.

Then, using the implicit Euler scheme, problem (11)–(14) is approximated as follows.

Find $v^{hk} = \left\{v_n^{hk}\right\}_{n=0}^N \subset V^h$, $\theta^{hk} = \left\{\theta_n^{hk}\right\}_{n=0}^N \subset K^h$ and $q^{hk} = \left\{q_n^{hk}\right\}_{n=0}^N \subset E^h$ such that $u_0^{hk} = u_0^h$, $v_0^{hk} = v_0^h$, $\theta_0^{hk} = \theta_0^h$ and $q_0^{hk} = q_0^h$, and for all $w^h \in V^h$, $z^h \in K^h$, $\xi^h \in E^h$ and $n = 1, \ldots, N$,

$$\langle \delta v_n^{hk}, w^h\rangle + \kappa\langle (u_n^{hk})_{xx}, w_{xx}^h\rangle + \langle M(t_n, u_n^{hk})(u_n^{hk})_x, w_x^h\rangle - m\langle (\theta_n^{hk})_x, w_x^h\rangle$$
$$+ d\langle (v_n^{hk})_{xx}, w_{xx}^h\rangle + \frac{1}{\varepsilon}\left\{[u_n^{hk}(\ell) - g_1]^+ - [-u_n^{hk}(\ell) - g_2]^+\right\} w^h(\ell) = 0, \tag{18}$$

$$\langle \delta\theta_n^{hk}, z^h\rangle + \langle \alpha(q_n^{hk})_x - m(v_n^{hk})_{xx}, z^h\rangle = 0, \tag{19}$$

$$\langle \tau\delta q_n^{hk} + q_n^{hk} + \alpha(\theta_n^{hk})_x, \xi^h\rangle = 0, \tag{20}$$

where u_0^h, v_0^h, θ_0^h and q_0^h are the approximations of the initial conditions u_0, u_1, θ_0 and q_0 given by

$$u_0^h = P_{V^h}u_0, \qquad v_0^h = P_{V^h}u_1, \qquad \theta_0^h = P_{K^h}\theta_0, \qquad q_0^h = P_{E^h}q_0, \tag{21}$$

where P_{V^h}, P_{K^h} and P_{E^h} denote the standard projection operators over the finite element spaces V^h, K^h and E^h, respectively (see [5]). Moreover, $u^{hk} = \{u_n^{hk}\}_{n=0}^N$ denotes the discrete displacement field defined by

$$u_n^{hk} = u_{n-1}^{hk} + kv_n^{hk}. \tag{22}$$

Proceeding as in the proof of the existence of the continuous solution, it follows that problem (18)–(22) has a unique solution $\left(v^{hk}, \theta^{hk}, q^{hk}\right) \subset V^h \times K^h \times E^h$.

In [6] we proved the following a priori error estimates result assuming the additional regularity conditions:

$$\begin{aligned} &u \in C^1([0,T]; \mathcal{V}_2) \cap C^2([0,T]; L^2(0,\ell)),\\ &\theta \in C^1([0,T]; L^2(0,\ell)) \cap C([0,T]; \mathcal{H}),\\ &q \in C^1([0,T]; L^2(0,\ell)) \cap C([0,T]; \mathcal{V}_1). \end{aligned} \tag{23}$$

Theorem 4. *Under the assumptions of Theorem 1 and the additional regularity (23), if we denote by (u, v, θ, q) and $(u^{hk}, v^{hk}, \theta^{hk}, q^{hk})$ the unique solutions to problems* (11)–(14) *and* (18)–(22)*, respectively, then we have the following error estimates for all $w^h =$*

$\{w_j^h\}_{j=1}^N \subset V^h$, $z^h=\{z_j^h\}_{j=1}^N \subset K^h$ *and* $\xi^h=\{\xi_j^h\}_{j=1}^N \subset E^h$,

$$
\begin{aligned}
&\max_{0\le n\le N} \|v_n-v_n^{hk}\|^2+\max_{0\le n\le N}\|u_n-u_n^{hk}\|^2_{\mathcal{V}_2}+\max_{0\le n\le N}\|\theta_n-\theta_n^{hk}\|^2+\max_{0\le n\le N}\|q_n-q_n^{hk}\|^2\\
&\qquad\qquad + Ck\sum_{j=1}^N\|v_j-v_j^{hk}\|^2_{\mathcal{V}_2}\\
&\quad\le\; Ck\sum_{j=1}^N\Big[\|v_t(t_j)-\delta v_j\|^2+\|\theta_t(t_j)-\delta\theta_j\|^2+\|q_t(t_j)-\delta q_j\|^2+\|v_j-w_j^h\|^2_{\mathcal{V}_2}\\
&\qquad +\|\theta_j-z_j^h\|^2_{H^1(0,\ell)}+\|q_j-\xi_j^h\|^2_{H^1(0,\ell)}\Big]+C\|v_0-v_0^h\|^2+C\|u_0-u_0^h\|^2_{\mathcal{V}_2}\\
&\qquad +C\|\theta_0-\theta_0^h\|^2+C\|q_0-q_0^h\|^2+C\max_{0\le n\le N}\|v_n-w_n^h\|^2\\
&\qquad +C\max_{0\le n\le N}\|\theta_n-z_n^h\|^2+C\max_{0\le n\le N}\|q_n-\xi_n^h\|^2\\
&\qquad +Ck^{-1}\sum_{j=1}^{N-1}\Big[\|v_j-w_j^h-(v_{j+1}-w_{j+1}^h)\|^2+\|\theta_j-z_j^h-(\theta_{j+1}-z_{j+1}^h)\|^2\\
&\qquad\quad +\|q_j-\xi_j^h-(q_{j+1}-\xi_{j+1}^h)\|^2\Big].
\end{aligned}
$$

These estimates are the basis for the convergence analysis of the algorithm. Hence, as an example, assume the following additional regularity conditions on the continuous solution:

$$
\begin{aligned}
&u\in H^3(0,T;L^2(0,\ell))\cap W^{1,\infty}(0,T;H^3(0,\ell)),\\
&\theta\in H^2(0,T;L^2(0,\ell))\cap C([0,T];H^2(0,\ell)),\\
&q\in H^2(0,T;L^2(0,\ell))\cap C([0,T];H^2(0,\ell)).
\end{aligned}
\tag{24}
$$

These regularity conditions imply that

$$
\|q_0-q_0^h\|^2+\|\theta_0-\theta_0^h\|^2+\|v_0-v_0^h\|^2+\|u_0-u_0^h\|^2_{\mathcal{V}_2}\le Ch^2,
$$

and therefore we have the following.

Corollary 1. *Let the assumptions of Theorem 4 and additional regularity conditions* (24) *still hold. Then, the numerical approximation of problem* (11)–(14) *by problem* (18)–(22) *is linearly convergent; that is, there exists a positive constant* $C>0$*, independent of the discretization parameters* h *and* k*, such that*

$$
\max_{0\le n\le N}\|v_n-v_n^{hk}\|+\max_{0\le n\le N}\|u_n-u_n^{hk}\|_{\mathcal{V}_2}+\max_{0\le n\le N}\|\theta_n-\theta_n^{hk}\|+\max_{0\le n\le N}\|q_n-q_n^{hk}\|\le C(h+k).
$$

4 NUMERICAL RESULTS

We present in this section the results of some numerical experiments. Since problem (18)–(20) is nonlinear, to solve it, we use an iterative algorithm that we now describe. Assuming that v_{n-1}^{hk}, u_{n-1}^{hk}, θ_{n-1}^{hk}, q_{n-1}^{hk} are known, we set

$$v_{n,0}^{hk}=v_{n-1}^{hk},\quad u_{n,0}^{hk}=u_{n-1}^{hk},\quad \theta_{n,0}^{hk}=\theta_{n-1}^{hk},\quad q_{n,0}^{hk}=q_{n-1}^{hk},$$

and we look for a solution to the problem: *find* $v_{n,j}^{hk}$, $\theta_{n,j}^{hk}$ *and* $q_{n,j}^{hk}$ *satisfying*

$$\begin{aligned}
&\langle\theta_{n,j}^{hk},z^h\rangle-k\langle\alpha q_{n,j}^{hk},(z^h)_x\rangle=\langle\theta_{n-1}^{hk},z^h\rangle-k\langle m(v_{n,j-1}^{hk})_x,z_x^h\rangle,\\
&\langle\tau q_{n,j}^{hk}+kq_{n,j}^{hk},\xi^h\rangle+k\langle\alpha(\theta_n^{hk})_x,\xi^h\rangle=\langle\tau q_{n-1}^{hk}\xi^h\rangle,\\
&\langle v_{n,j}^{hk},w^h\rangle+k\langle\kappa(u_{n,j}^{hk})_{xx},w_{xx}^h\rangle+k\langle M(t_n,u_{n,j-1}^{hk})(u_{n,j}^{hk})_x,w_x^h\rangle+\langle d(v_{n,j}^{hk})_{xx},w_{xx}^h\rangle\\
&\qquad=k\langle m(\theta_{n,j}^{hk})_x,w_x^h\rangle-\frac{k}{\varepsilon}\left\{[u_{n,j-1}^{hk}(\ell)-g_1]^+-[-u_{n,j-1}^{hk}(\ell)-g_2]^+\right\}w^h(\ell),
\end{aligned}$$

for all $z^h\in K^h$, $\xi^h\in E^h$ and $w^h\in V^h$, with $u_{n,j}^{hk}=u_{n-1}^{hk}+kv_{n,j}^{hk}$, $j=1,2,\ldots$. We remark that, at each iteration j, first we solve a linear system of algebraic equations to find $\theta_{n,j}^{hk}$ and $q_{n,j}^{hk}$ and then, a second linear system is solved to obtain $v_{n,j}^{hk}$.

In the simulations we take $g_1=g_2=0.02$, $\ell=\kappa=m=d=\alpha=1$ and $\varepsilon=0.01$. A tolerance of $TOL=10^{-7}$ is used to stop the iterations.

In this experiment we investigate the influence of β and τ. We choose

$$\theta_0(x)=40x^2(x-1)^2,\quad q_0(x)=0,\quad u_0(x)=0,\quad u_1(x)=2x,$$

and discretization parameters $h=1/100$, $k=10^{-4}$. As soon as we start the simulation, the beam gets in contact with the upper stop, then contact is lost and the beam moves in the direction of the lower stop and a large penetration in both stops is seen. During some time it stays in contact with the lower stop. As the time increases, the system approaches the steady-state $\theta=q=u=0$ and the discrete energy $\mathcal{E}_n$ decays very fast to zero as shown in Figures 1 and 2. The simulations performed indicate that the evolution towards the stationary solution is faster when $\beta=2$ and $\tau=0.1$.

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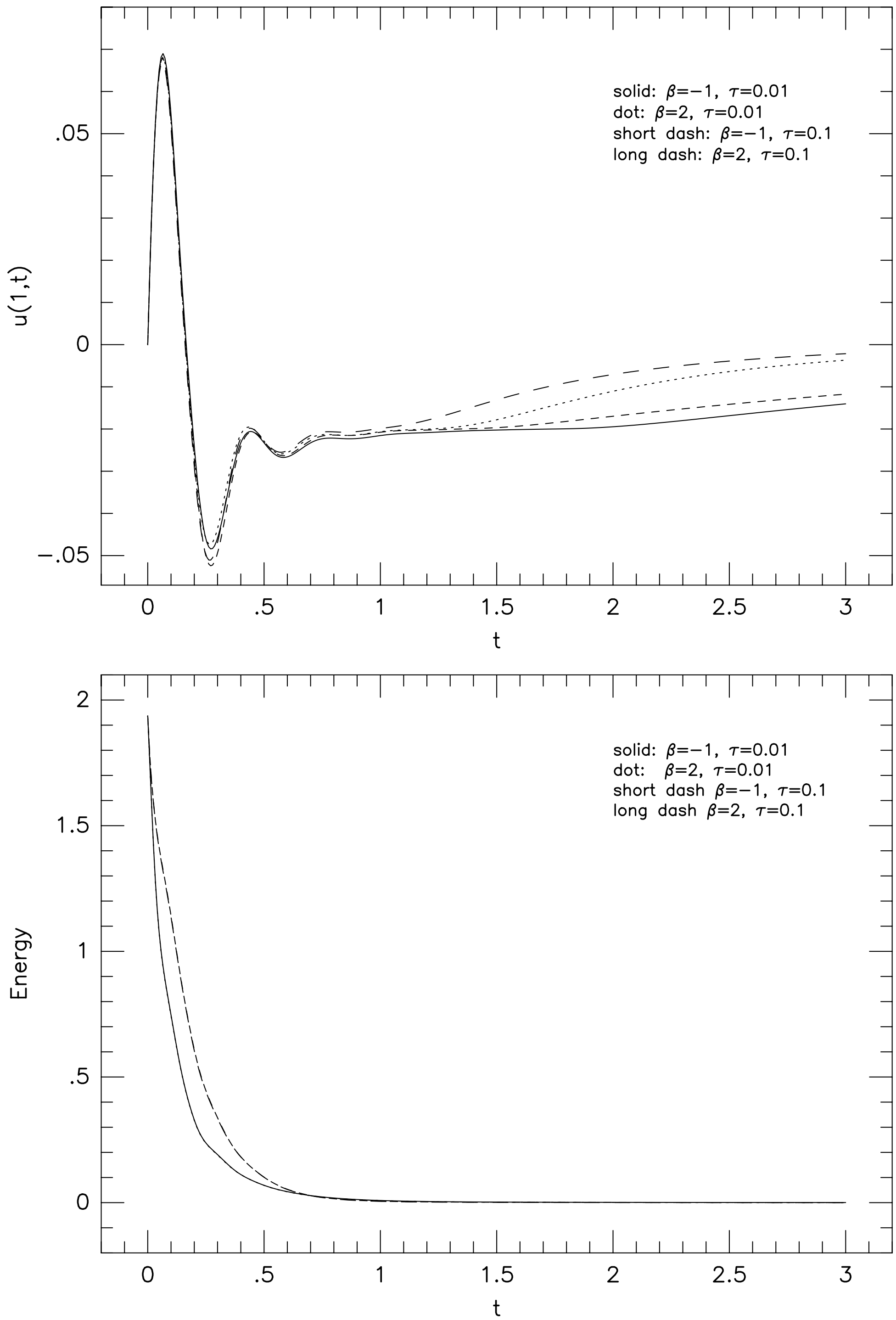

Figure 1: The time evolution of the displacement at the point of contact and energy.

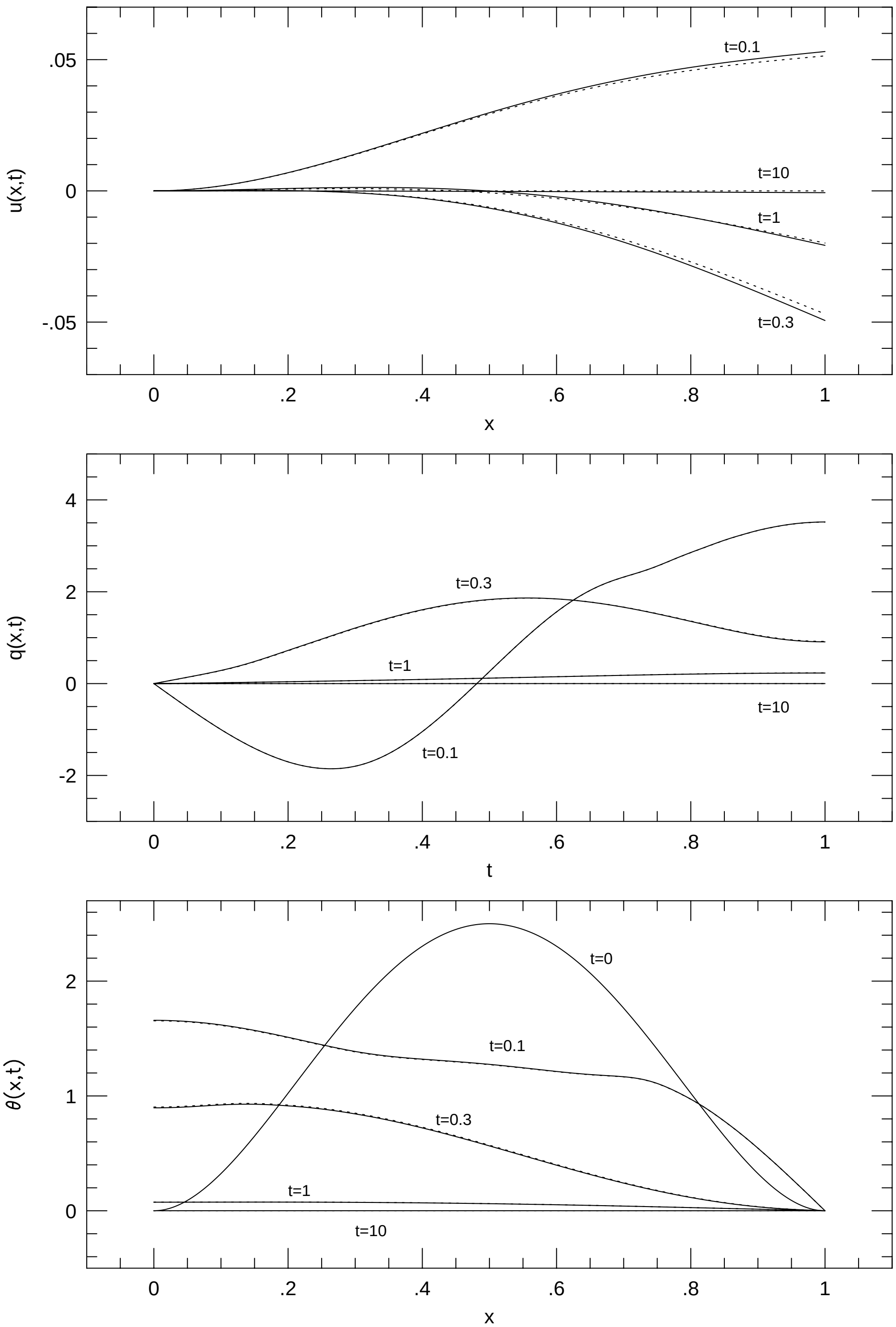

Figure 2: The time evolution of the temperature, heat flux and displacement when $\tau = 0.1$, for $\beta = -1$ (solid) and $\beta = 2$ (dot).

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