11th World Congress on Computational Mechanics (WCCM XI)
5th European Conference on Computational Mechanics (ECCM V)
6th European Conference on Computational Fluid Dynamics (ECFD VI)
E. Oñate, J. Oliver and A. Huerta (Eds)

EFFICIENT AND RELIABLE ERROR CONTROL FOR THE OBSTACLE PROBLEM

Carsten Carstensen[1] **and Karoline Köhler**[1,*]

[1]Institut für Mathematik, Humboldt-Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany, {cc,koehlerk}@mathematik.hu-berlin.de

Key words: Obstacle problem, adaptive algorithm, efficiency, reliability, optimality, Lagrange multiplier.

Abstract. For the obstacle problem as the prototypical example for variational inequalities, the "Other Look" by Braess [2] at Lagrange multipliers has lead to a paradigm with reliable and partly efficient error control. Therein the efficiency is understood in terms of the error $u - v$ in $H_0^1(\Omega)$ for the exact solution u and an approximation v in $H_0^1(\Omega)$ in the primal variable and the error $\lambda - \mu$ for the exact Lagrange multiplier λ and an approximation μ in the dual space $H^{-1}(\Omega)$. These error terms are compared with explicit computable terms, as in the analysis for variational equalities. Reliability and efficiency then leads to the equivalence

$$\|u - v\| + \|\lambda - \mu\|_* \approx \text{computable terms}$$

possibly up to multiplicative generic constants.
This paper presents a reliable and efficient a posteriori error analysis for the conforming finite element method (FEM) from [6]. The reliable error control is even a guaranteed upper bound for the exact error. The paper answers the question of efficiency beyond the aforementioned equivalence. Given the exact Lagrange multiplier λ for which choice of an approximation μ of λ does it hold

$$\|\lambda - \mu\|_* \lesssim \|u - v\| + \text{data oscillations?}$$

It clarifies the role of the Lagrange multiplier and possible choices for suitable approximations. Furthermore reliability and efficiency is viewed as the equivalence

$$\|u - v\| \approx \text{computable terms.}$$

The results of the a posteriori analysis lead to an adaptive algorithm. The optimality for a conforming adaptive FEM is shown by Carstensen und Hu [5].

1 Introduction

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ with polyhedral boundary $\partial\Omega$ the energy (semi-)scalar product $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ on the vector space $H^1(\Omega)$ reads

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x \quad \text{for all } u,v \in H^1(\Omega).$$

It induces the energy norm $|||\cdot||| := a(\cdot,\cdot)^{1/2}$ on the vector space

$$V := H_0^1(\Omega).$$

Given $f \in L^2(\Omega)$ define $F \in V^*$ by

$$F(v) := \int_\Omega fv \mathrm{d}x \quad \text{for all } v \in V.$$

The obstacle $\chi \in W^{1,\infty}(\Omega)$ and the Dirichlet boundary value $u_D \in W^{1,\infty}(\Omega)$ satisfy $\chi \le u_D$ a.e. along Γ_D in order to ensure that the closed and convex subset

$$K := \{v \in \mathcal{A} \mid \chi \le v \text{ a.e. in } \Omega\} \quad \text{of } \mathcal{A} := u_D + V \subseteq H^1(\Omega)$$

is non-empty. The weak formulation of the obstacle problem seeks $u \in K$ with

$$F(v-u) \le a(u, v-u) \quad \text{for all } v \in K. \tag{1}$$

It is well known [9], that a unique weak solution u of (1) exists and $u \in H^2_{loc}(\Omega)$ satisfies the consistency conditions (where $\perp$ abbreviates point wise orthogonality)

$$0 \le u - \chi \perp \lambda := f + \Delta u \le 0 \quad \text{a.e. in } \Omega. \tag{2}$$

In case $u_D \in H^{3/2}(\partial\Omega)$, $\chi \in H^2(\Omega)$ and Ω convex (or $\partial\Omega \in C^{1,1}$), the solution satisfies $u \in H^2(\Omega)$ [13, Corol 2.3, Chap. 5] and the Lagrange multiplier $\lambda \in L^2(\Omega)$ belongs to $L^2(\Omega; (-\infty, 0]) \subseteq V^*$.

Throughout this paper we assume no extra condition on the boundary of the polyhedral Lipschitz domain, except that $\lambda \in L^2(\Omega; (-\infty, 0])$.

The remaining part of this paper is organized as follows. Section 2 provides a general error estimate and the efficiency of this error estimate. Section 3 designs a discrete Lagrange multiplier μ for the conforming FEM and states the efficiency of this Lagrange multiplier. Section 4 presents numerical experiments which demonstrate and empirically confirm the theoretic results from the previous sections.
The rather technical results of Section 2 and 3 utilise the Medius analysis and will appear elsewhere

The paper applies standard notation for Lebesgue and Sobolev spaces and their norms $\|\bullet\|_{L^2(\Omega)}$, $|||\bullet||| := \|\nabla\bullet\|_{L^2(\Omega)}$ ($\bullet$ indicates the possible arguments), as well as their local variants $\|\bullet\|_{L^2(\omega)}$ and $|||\bullet|||_\omega$. The operator norm is defined by

$$|||\bullet|||_* := \sup_{\varphi\in H_0^1(\Omega)} \int_\Omega \bullet\varphi \mathrm{d}x/|||\varphi|||.$$

$L^2(\Omega;(-\infty,0])$ denotes the set of L^2 functions with non-positive values. The notation $A \lesssim B$ abbreviates $A \le CB$ for some generic constant C which may depend on the interior angles of the triangulation, i.e., on some constant γ_0, but not on the mesh-size, and $A \approx B$ abbreviates $A \lesssim B \lesssim A$. Throughout the paper the notation $(\bullet)_+ := \max\{\bullet, 0\}$ is employed.

2 General Error Control for the Obstacle Problem

Given any $v \in \mathcal{A}$ and $\mu \in L^2(\Omega;(-\infty,0])$ define some residual $\mathrm{Res} \in V^*$ by

$$\mathrm{Res}(\varphi) := F(\varphi) - \int_\Omega \mu\varphi\mathrm{d}x - a(v,\varphi) \quad \text{for all } \varphi \in V. \tag{3}$$

With $S := \{\varphi \in V |\ |||\varphi||| = 1\}$ set $|||\mathrm{Res}|||_* := \sup\{\mathrm{Res}(\varphi)|\ \varphi \in S\}$.

Without further assumptions, the following general error estimate leads to a guaranteed upper bound with respect to the approximations v of u and μ of λ.

Theorem 2.1. *Suppose $v \in \mathcal{A}$ is some approximation to the exact continuous solution $u \in K \cap H^2_{loc}(\Omega)$ of (1) with the obstacle $\chi \in W^{1,\infty}(\Omega)$ and the source term $f \in L^2(\Omega)$. Suppose $\mu \in L^2(\Omega;(-\infty,0])$ is some non-positive approximation of the Lagrange multiplier $\lambda := f + \Delta u \in L^2(\Omega;(-\infty,0])$ and $v \in \mathcal{A}$ an admissible approximation of the exact solution $u \in K$. Then the error $e := u - v$ and the gap $w := \min\{0, v-\chi\} \in V$ satisfy*

ⓐ $$\int_\Omega (\lambda-\mu)(u-\max\{v,\chi\})\mathrm{d}x + |||e|||^2/2 + |||e+w|||^2/2 = \mathrm{Res}(e+w) + |||w|||^2/2;$$

ⓑ $$0 \le \int_\Omega \mu(\chi-u)\mathrm{d}x - \int_\Omega \lambda(v-\chi)_+\mathrm{d}x = \int_\Omega (\lambda-\mu)(u-\max\{v,\chi\})\mathrm{d}x - \int_\Omega \mu(v-\chi)_+\mathrm{d}x;$$

ⓒ $$\int_\Omega \mu(\chi-u)\mathrm{d}x - \int_\Omega \lambda(v-\chi)_+\mathrm{d}x + |||e|||^2/2 + (1-1/t)|||e+w|||^2/2 \le t|||\mathrm{Res}|||_*^2/2 - \int_\Omega \mu(v-\chi)_+\mathrm{d}x + |||w|||^2/2 \quad \textit{for all} \quad 0<t<\infty;$$

ⓓ $$\Big|\,|||\lambda-\mu|||_* - |||e|||\,\Big| \le |||\mathrm{Res}|||_* \le |||e||| + |||\lambda-\mu|||_*.$$

Recall the abbreviations $e := u - v$ and $w := \min\{0, v - \chi\}$ and define the error terms by

$$\mathrm{Err} := \left(\int_\Omega \mu(\chi - u)\mathrm{d}x\right)^{1/2} + \left(\int_\Omega (-\lambda)(v - \chi)_+\mathrm{d}x\right)^{1/2} \\ + |||e||| + |||e + w||| + |||\lambda - \mu|||_*.$$

Note that the displayed integrands are non-negative (and so the square roots define reals) and are seen as error terms for the consistency condition.

This error Err is controlled by the computable guaranteed upper bound

$$\mathrm{GUB} := |||\mathrm{Res}|||_* + \left(\int_\Omega (-\mu)(v - \chi)_+\mathrm{d}x\right)^{1/2} + |||w|||.$$

The next theorem states the reliability and efficiency of GUB

Theorem 2.2 (reliability and efficiency)**.** *It holds*

$$1/2\mathrm{GUB} \leq \mathrm{Err} \leq (30/7)^{1/2}\mathrm{GUB}.$$

3 Conforming FEM

This section applies the a posteriori error estimates from Section 2 to the conforming Courant FEM (CFEM) as in [1, 2, 3, 4, 7, 10, 11, 12] with homogeneous Dirichlet boundary conditions in two dimensions for simplicity.

3.1 Discretisation

Let the bounded polygonal Lipschitz domain $\Omega \subseteq \mathbb{R}^2$ be partitioned in a shape-regular triangulation $\mathcal{T}$ into triangles, with nodes $\mathcal{N}$, interior nodes $\mathcal{N}(\Omega)$, nodes on the boundary $\mathcal{N}(\partial\Omega)$, and with edges $\mathcal{E}$, interior edges $\mathcal{E}(\Omega) := \{E \in \mathcal{E} | E \nsubseteq \partial\Omega\}$, and edges along the boundary $\mathcal{E}(\partial\Omega) := \{E \in \mathcal{E}|\ E \subseteq \partial\Omega\}$. Given any node $z \in \mathcal{N}$, let $\mathcal{T}(z)$ denote the set of all triangles T with $z \in \mathcal{N}(T)$ in the set $\mathcal{N}(T)$ of the three vertices of a triangle T, and let $|\mathcal{T}(z)|$ denote the number of triangles in $\mathcal{T}(z)$; $\overline{\omega}_z := \cup\mathcal{T}(z)$ is the support of the (P_1 conforming) nodal basis function φ_z with interior $\omega_z := \{\varphi_z > 0\}$. The set $\Omega_T := \cup_{z\in\mathcal{N}(T)}\omega_z$ denotes a patch around each triangle $T \in \mathcal{T}$.

The triangulation is shape regular in the sense that there exists a universal constant $\gamma_0 > 0$ such that any interior angle α satisfies $\gamma_0 \leq \alpha$. To avoid unnecessary technicalities with the operators J and J^* in Section 3.2 and in the definition of Osc below, throughout this paper each triangle in the triangulation has at least one vertex in the interior of the domain.

For any $k \in \mathbb{N}_0$ define

$$P_k(T) := \{v_k : T \to \mathbb{R}|\ v_k \text{ is a polynomial of degree } \leq k\},$$
$$P_k(\mathcal{T}) := \{v_k \in L^\infty(\Omega)|\ \forall T \in \mathcal{T}, v_k|_T \in P_k(T)\}.$$

For a given triangulation $\mathcal{T}$, define the piecewise constant mesh size $h_\mathcal{T} \in P_0(\mathcal{T})$ and the L^2 projection $\Pi_0 : L^2(\Omega) \to \Pi_0(\mathcal{T})$ by $h_\mathcal{T}|_T := h_T := \operatorname{diam}(T)$ and $\Pi_0|_T f := \fint_T f \mathrm{d}x := \int_\mathcal{T} f \mathrm{d}x/|T|$ for all $T \in \mathcal{T}$. Furthermore define the oscillations

$$\operatorname{osc}^2(f,T) := \|h_T(f - \Pi_0 f)\|^2_{L^2(T)} \quad \text{and } \operatorname{osc}^2(f,\mathcal{T}) := \sum_{T\in\mathcal{T}} \operatorname{osc}^2(f,T). \tag{3.4}$$

Given a shape-regular triangulation, define the Courant finite element spaces by

$$\begin{aligned} V_\mathrm{C}(\mathcal{T}) &:= C_0(\overline{\Omega}) \cap P_1(\mathcal{T}), \\ K_\mathrm{C}(\mathcal{T}) &:= \{v_\mathrm{C} \in V_\mathrm{C}(\mathcal{T})|\ \forall z \in \mathcal{N}(\Omega), \chi(z) \le v_\mathrm{C}(z)\}. \end{aligned}$$

Define the conforming interpolation operator $\mathrm{I}_\mathrm{C} : L^2(\Omega) \to V_\mathrm{C}$ for any $v \in L^2(\Omega)$ by

$$\mathrm{I}_\mathrm{C}(v) = \sum_{z\in\mathcal{N}} v(z)\varphi_z.$$

The nodal basis function φ_z satisfies $\varphi_z \in V_\mathrm{C}(\mathcal{T})$ and $\varphi_z(y) = \delta_{z,y}$ for $y, z \in \mathcal{N}$. The discrete solution $u_\mathrm{C} \in K_\mathrm{C}(\mathcal{T})$ solves the variational inequality

$$a(u_\mathrm{C}, v_\mathrm{C} - u_\mathrm{C}) \le F(v_\mathrm{C} - u_\mathrm{C}) \quad \text{for all } v_\mathrm{C} \in K_\mathrm{C}(\mathcal{T}).$$

The discrete solution $u_\mathrm{C} \in V_\mathrm{C}(\mathcal{T})$ associates the discrete Lagrange multiplier $\sigma_\mathrm{C} := F - a(u_\mathrm{C}, \bullet) \in V_\mathrm{C}(\mathcal{T})^*$ with the discrete consistency conditions

$$(u_\mathrm{C}(z) - \mathrm{I}_\mathrm{C}\chi(z))\sigma_\mathrm{C}(\varphi_z) = 0 \text{ and } \sigma_\mathrm{C}(\varphi_z) \le 0 \quad \text{for all } z \in \mathcal{N}(\Omega) \tag{3.5}$$

(recall φ_z denotes the nodal basis function to the node $z \in \mathcal{N}(\Omega)$). However, at this point, $\sigma_\mathrm{C} \in V_\mathrm{C}(\mathcal{T})^*$ is neither an L^2 function nor globally non-positive in general. To represent σ_C by some non-positive function as in [2] the first idea may be the Riesz representation λ_C of σ_C in the Hilbert space $V_\mathrm{C}(\mathcal{T})$ endowed with the L^2 scalar product. Notice that $\lambda_\mathrm{C} \in V_\mathrm{C}(\mathcal{T})$ may *not* be non-positive in general and hence λ_C is *not* an immediate choice for some μ_C which could represent μ in Section 2.

3.2 Two Quasi-interpolation Operators and one Discrete Lagrange Multiplier

For any $\varphi \in H^1(\Omega)$ define $J\varphi \in P_1(\mathcal{T}) \cap C(\overline{\Omega})$ as in [1] by

$$J\varphi := \sum_{z\in\mathcal{N}} \left(\int_\Omega \varphi\varphi_z \mathrm{d}x \Big/ \int_\Omega \varphi_z \mathrm{d}x \right) \varphi_z. \tag{3.6}$$

In order to approximate the boundary values of a discrete Lagrange multiplier μ correctly, the above interpolation operator will be modified. For each node $z \in \mathcal{N}(\partial\Omega)$ select some neighbouring free node $\zeta(z)$ in the sense that z and $\zeta(z)$ belong to the same interior edge of the triangulation $\mathcal{T}$ (since z belongs to some triangle T with at least one vertex in the interior there is at least one interior edge $\operatorname{conv}\{z,\zeta(z)\}$). For all free nodes $z \in \mathcal{N}(\Omega)$ set $\zeta(z) := z$,

$$\psi_z := \sum_{y\in\zeta^{-1}(z)} \varphi_y \in P_1(\mathcal{T}) \cap C(\overline{\Omega}), \quad \text{and} \quad \Omega_z := \{\psi_z > 0\}.$$

This is used to define the quasi interpolation operator

$$J^*\varphi := \sum_{z\in\mathcal{N}(\Omega)} \left(\int_\Omega \varphi\psi_z \mathrm{d}x \Big/ \int_\Omega \varphi_z \mathrm{d}x\right) \varphi_z \quad \text{for any } \varphi \in H^1(\Omega)$$

and the oscillations

$$\operatorname{Osc}^2(f, \mathcal{N}(\Omega)) := \sum_{z\in\mathcal{N}(\Omega)} \operatorname{diam}(\Omega_z)^2 \left\| f - ⨍_{\Omega_z} f\, dx \right\|^2_{L^2(\Omega_z)} \text{for any } f \in L^2(\Omega).$$

With $\sigma_\mathrm{C} := F(\bullet) - a(u_\mathrm{C}, \bullet)$ as above define

$$\lambda_\mathrm{CB} := \sum_{z\in\mathcal{N}(\Omega)} \left(\sigma_\mathrm{C}(\varphi_z) \Big/ \int_\Omega \varphi_z \mathrm{d}x\right) \psi_z \in L^2(\Omega; (-\infty, 0]). \tag{3.7}$$

3.3 Reliable and Efficient Error Estimator for Fixed Lagrange Multiplier

This section applies the results of Section 2 to the conforming FEM and shows reliability and efficiency for this method. Suppose u_C is an arbitrary discrete function in K_C and recall the notation $e := u - u_\mathrm{C}$, $w := \min\{0, u_\mathrm{C} - \chi\}$. Given $\mu := \lambda_\mathrm{CB}$ and $v := u_\mathrm{C}$, the residual reads

$$\operatorname{Res}_\mathrm{C}(\varphi) = F(\varphi) - \int_\Omega \lambda_\mathrm{CB}\varphi \mathrm{d}x - a(u_\mathrm{C}, \varphi) \quad \text{for all } \varphi \in V. \tag{3.8}$$

Let $z_\mathrm{C} \in V_\mathrm{C}(\mathcal{T})$ denote the Riesz representation of $\operatorname{Res}_\mathrm{C}|_{V_C(\mathcal{T})} \in V_C(\mathcal{T})$ from (3.8) in the Hilbert space $(V_\mathrm{C}(\mathcal{T}), a)$, and define $\operatorname{Err}_\mathrm{C}$ and $\operatorname{GUB}_\mathrm{C}$ by

$$\begin{aligned}\operatorname{Err}_\mathrm{C} :=& \left(\int_\Omega \lambda_\mathrm{CB}(\chi - u)\mathrm{d}x\right)^{1/2} + \left(-\int_\Omega \lambda(u_\mathrm{C} - \chi)_+ \mathrm{d}x\right)^{1/2} \\ &+ |||e||| + |||e + w||| + |||\lambda - \lambda_\mathrm{CB}|||_*; \\ \operatorname{GUB}_\mathrm{C} :=& \left(\|h_\mathcal{T}(f - \lambda_\mathrm{BC})\|^2_{L^2(\Omega)} + \sum_{E\in\mathcal{E}(\Omega)} h_E \, \|[\nabla(u_\mathrm{C} + z_\mathrm{C}) \cdot \nu_E]\|^2_{L^2(E)} + |||z_\mathrm{C}|||^2 \right)^{1/2} \\ &+ \left(-\int_\Omega \lambda_\mathrm{BC}(u_\mathrm{C} - \chi)_+ \mathrm{d}x\right)^{1/2} + |||w|||/2.\end{aligned}$$

With these error terms $\operatorname{Err}_\mathrm{C}$, the global upper bound $\operatorname{GUB}_\mathrm{C}$ provides a reliable and efficient a posteriori error estimate for the conforming FEM.

Theorem 3.1. *The Courant FEM satisfies*

$$\mathrm{Err}_{\mathrm{C}} \lesssim \mathrm{GUB}_{\mathrm{C}} \lesssim \mathrm{Err}_{\mathrm{C}} + \mathrm{osc}(f, \mathcal{T}).$$

3.4 Efficiency of the Discrete Lagrange Multiplier

This section is devoted to the efficiency of the discrete Lagrange multiplier λ_{CB} in the terms $|||\lambda - \lambda_{\mathrm{CB}}|||_*$ and $\int_\Omega (-\lambda_{\mathrm{CB}})(u_C - \chi)\mathrm{d}x$. For convex obstacles, both terms can be bounded by the error $|||u - u_C|||$ plus additional data oscillation terms. The theorem below employs the two subsets $\mathcal{T}_{\mathrm{C}}$ and $\mathcal{T}_{\mathrm{I}}$ of the triangulation $\mathcal{T}$

$$\mathcal{T}_{\mathrm{C}} := \{T \in \mathcal{T} |\ u_{\mathrm{C}} = \chi \text{ on } T\}; \tag{3.9}$$

$$\mathcal{T}_{\mathrm{I}} := \{T \in \mathcal{T} |\ \exists y_T, z_T \in \mathcal{N}(\overline{\Omega}_T), \chi(y_T) < u_{\mathrm{C}}(y_T) \text{ and } \chi(z_T) = u_{\mathrm{C}}(z_T)\} \tag{3.10}$$

which represent the triangles $T \in \mathcal{T}_{\mathrm{C}}$ with full contact and the intermediate triangles $T \in \mathcal{T}_{\mathrm{I}}$ which lie in the discrete interface of triangles where at least one node is in contact and one node is not. The estimates in this section depend strongly on the specific choice of Lagrange multiplier.

Theorem 3.2 (Efficiency)**.** *Consider the exact discrete solution* u_{C} *and the exact discrete Lagrange multiplier* λ_{CB}*. Then it holds*

$$ⓐ\ |||\lambda - \lambda_{\mathrm{CB}}|||_* \lesssim |||u - u_{\mathrm{C}}||| + \mathrm{Osc}(\lambda, \mathcal{N}(\Omega)).$$

Provided $\chi \in P_1(\Omega)$ *it also holds*

$$\begin{aligned} ⓑ\ \int_\Omega (-\lambda_{\mathrm{CB}})(u_{\mathrm{C}} - \chi)\mathrm{d}x &\lesssim |||u - u_{\mathrm{C}}|||^2 + |||\lambda - \lambda_{\mathrm{CB}}|||_*^2 + \mathrm{osc}(f, \mathcal{T}_{\mathrm{I}} \setminus \mathcal{T}_{\mathrm{C}})^2 \\ &\lesssim |||u - u_{\mathrm{C}}|||^2 + \mathrm{Osc}(f, \mathcal{T}_{\mathrm{I}} \setminus \mathcal{T}_{\mathrm{C}}) + \mathrm{Osc}(\lambda, \mathcal{N}(\Omega)). \end{aligned}$$

For globally affine obstacle $\chi \in P_1(\Omega)$ an immediate consequence of Theorems 3.1 and 3.2 is the following corollary. It states that reliability and efficiency do not only hold for $\mathrm{Err}_{\mathrm{C}}$ but also for $|||u - u_{\mathrm{C}}|||$.

Corollary 3.3. For the discrete solution $u_{\mathrm{C}} \in V_{\mathrm{C}}(\mathcal{T})$ and $\chi \in P_1(\Omega)$ with $\chi \leq u_{\mathrm{C}}$ it holds

$$|||u - u_{\mathrm{C}}||| \lesssim \mathrm{GUB}_{\mathrm{C}} \lesssim |||u - u_{\mathrm{C}}||| + |||e + w||| + \mathrm{osc}(f, \mathcal{T}) + \mathrm{osc}(\lambda, \mathcal{T}). \qquad \square$$

4 Numerical Experiments

This section is devoted to the performance of the global upper bound established in Section 3. This bound is tested on adaptive and uniform meshes.

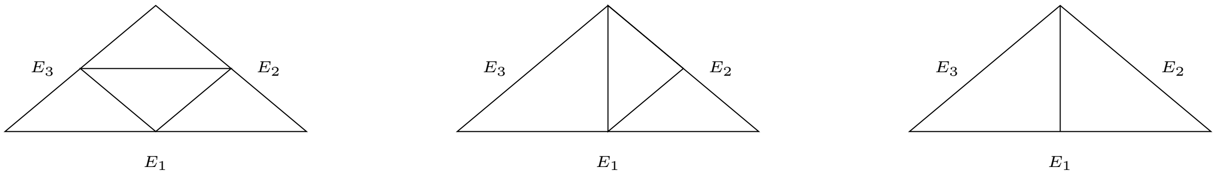

Figure 1: Red-, blue-, and green-refinement of a triangle

4.1 Numerical Realisation

The outputs of the following algorithm are the values of the guaranteed upper error bound $\mathrm{GUB_C}$ as well as the respective efficiency indices $\mathrm{GUB_C}/\mathrm{Err_C}$.

Algorithm The INPUT is an initial mesh $\mathcal{T}_0$, and a constant $0 < \Theta < 1$
LOOP $\forall \ell = 0, 1, 2...$ until termination `do`

COMPUTE (with the Matlab routine `quadprog`) the discrete solutions to the conforming finite element discretisation, u_C, on the mesh $\mathcal{T}_\ell$ with `ndofC` many unknowns.

ESTIMATE the error $\mathrm{Err_C} \lesssim \mathrm{GUB_C}$.

MARK a minimal subset $\mathcal{M}_\ell \subset \mathcal{T}_\ell$ with respect to the refinement indicator η_C described below, such that $\mathcal{M}_\ell$ satisfies

$$\Theta \sum_{T \in \mathcal{T}_\ell} \eta(T)^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta(T)^2.$$

The respective refinement indicator reads

$$\eta_\mathrm{C}^2(T) := \Bigg(\bigg(\|h_\mathcal{T}(f - \lambda_\mathrm{BC})\|_{L^2(\Omega)} + \sum_{E \in \mathcal{E}(\Omega)} h_E \,\|[\nabla(u_\mathrm{C} + z_\mathrm{C}) \cdot \nu_E]\|_{L^2(E)} \\ + |||z_\mathrm{C}|||^2 \bigg)^{1/2} + \Big(- \int_\Omega \lambda_\mathrm{BC}(v - \chi)_+ \mathrm{d}x \Big)^{1/2} + |||w|||/2 \Bigg)^2.$$

REFINE the triangles in $\mathcal{M}_\ell$ by red-refinement and perform *red-green-blue*-refinement (see Figure 1) on further triangles to avoid hanging nodes.

4.2 L-shaped Domain

For vanishing obstacle and Dirichlet data $u_D \equiv \chi \equiv 0$ on the L-shaped domain $\Omega := (-2,\ 2)^2 \backslash ([0,\ 2] \times [-2,\ 0])$, the source term

$$f(r, \varphi) := -r^{2/3} \sin(2\varphi/3)(7/3(\partial g/\partial r)(r)/r + (\partial^2 g/\partial r^2)(r)) - H(r - 5/4)$$
$$g(r) := \max\{0, \min\{1, -6s^2 + 15s^4 - 10s^3 + 1\}\} \text{ for } s := 2(r - 1/4)$$

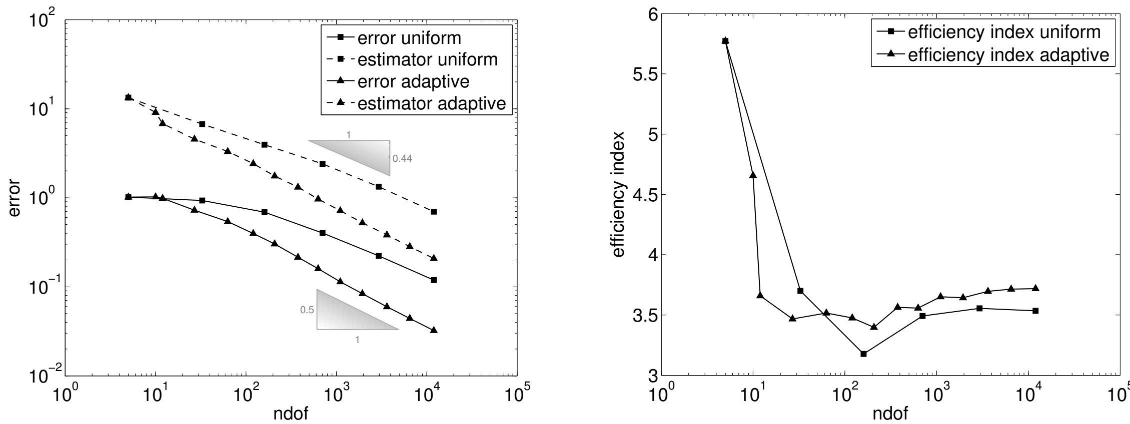

Figure 2: Values of exact error (left) and efficiency indices (right) for the conforming and non-conforming FEM with respect to ndof from Subsection 4.2

with the Heaviside function H leads to the known exact solution

$$u(r, \varphi) = r^{2/3} g(r) \sin(2\varphi/3)$$

from [1] which has a typical corner singularity at the re-entrant corner. Figure 2 presents the convergence rates of the exact error and the guaranteed global upper bound and the efficiency indices of the global upper bounds for the conforming CFEM on adaptive and uniform meshes. Figure 2 left shows that the global upper bound as well as the error terms converge with the same convergence rate of -0.44 on uniform meshes and with the optimal convergence rate of -0.5 on adaptive meshes. This observation leads to the almost constant efficiency indices, both for the uniform and the adaptive algorithm, cf. Figure 2 right, with values around 3.5.

4.3 Square Domain

This example from [10] considers the constant obstacle $\chi \equiv 0$ on the square domain $\Omega := (0,\ 1)^2$ with the source term

$$f(r, \varphi) := \begin{cases} -16r^2 + 3.92 & \text{for } r > 0.7 \\ -5.8408 + 3.92r^2 & \text{for } r \leq 0.7 \end{cases}$$

and the Dirichlet boundary conditions $u_D(r, \varphi) := r^2 - 0.49$. The exact solution reads

$$u(r, \varphi) = \max\{0, r^2 - 0.49\}^2.$$

Figure 3 shows optimal convergence rates both for the error estimator and the exact error. Undocumented experiments confirm that the overall error is dominated by the error terms which concern the Lagrange multipliers.

The efficiency indices both for the adaptive and uniform mesh refinement lie between 3.8 and 4.

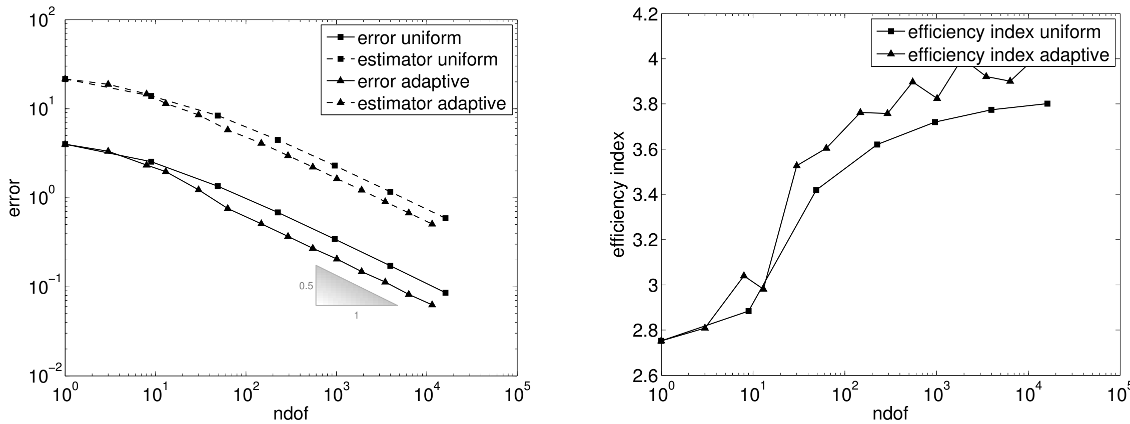

Figure 3: Values of the Global Upper Bounds GUB, and the error terms Err (left) and efficiency indices (right) for the conforming and non-conforming FEM with respect to ndof from Subsection 4.3

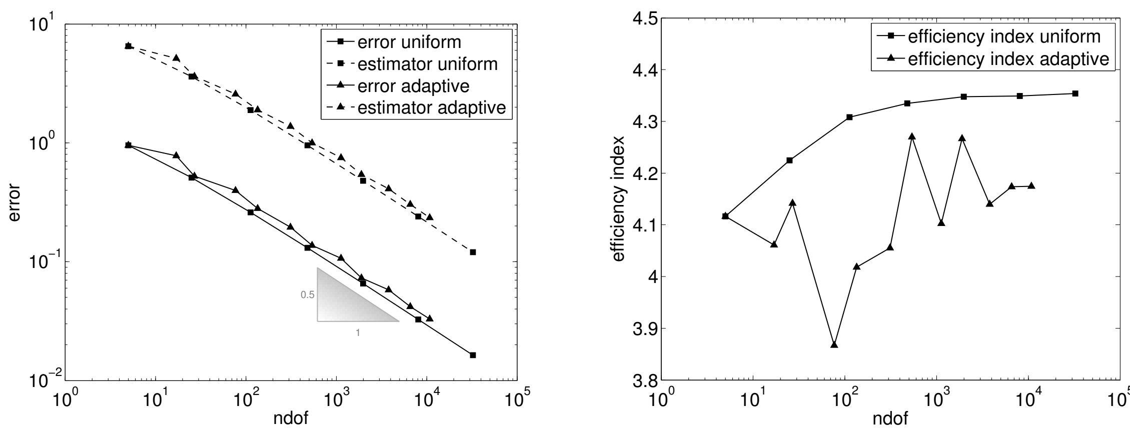

Figure 4: Exact errors (left) and efficiency indices (right) for the conforming and non-conforming FEM with respect to ndof from Subsection 4.4

4.4 Smooth Obstacle

The last example from [8] concerns the smooth obstacle $\chi(x, y) := -(x^2 - 1)(y^2 - 1)$ on the square domain $\Omega := (0,\ 1)^2$, with $f := \Delta\chi$ and the exact solution $u = \chi$. Both for the adaptive and the uniform algorithm, the error estimator and the exact error demonstrate the optimal convergence rate of -0.5. Since Ω is convex an improvement of the convergence rate by adaptive mesh refinement cannot be expected in this example. As for the examples on the L-shaped domain (Subsection 4.2) and from Subsection 4.3, the exact overall error in this case is dominated by the error terms which concern the Lagrange multipliers. The efficiency indices lie at approximately 4.4.

4.5 Conclusion

The numerical experiments confirm that the general approach presented in this paper indeed leads to reliable and efficient error estimation for the conforming Courant finite element method at hand. Further undocumented experiments show that the fairly large overestimation of the Courant FEM is due to an error estimator employed to estimate the error of an auxiliary Poisson model problem which arises in Theorem 3.1.

REFERENCES

[1] Bartels, S. and Carstensen, C. Averaging techniques yield reliable a posteriori finite element error control for obstacle problems. *Numer. Math.*, Vol. **99**,2004

[2] Braess, D. A posteriori error estimators for obstacle problems–another look. *Numer. Math.*, Vol. **3**, 2005

[3] Braess, D., and Carstensen, C. and Hoppe, R.H.W. Convergence analysis of a conforming adaptive finite element method for an obstacle problem. *Numer. Math.*, Vol. **107**, 2007

[4] Braess, D., Hoppe,R.H.W. and Schöberl, J. A posteriori estimators for obstacle problems by the hypercircle method. *Computing and Visualization in Science*, Vol. **11**, 2008

[5] Carstensen, C. and Hu, J. Optimality of a conforming adaptive finite element method for an affine obstacle problem. (in preparation)

[6] Carstensen, C. and Koehler, K. Discrete Lagrange multipliers in numerical simulations for the obstacle problem (work in progress)

[7] Carstensen, C. and Merdon, C. A posteriori error estimator competition for conforming obstacle problems. *Numer. Methods Partial Differential Eq*, Vol. **29**, 2013

[8] Gräser, C. and Kornhuber, R. Multigrid methods for obstacle problems. *J. Comput. Math.*, Vol. **27**, 2009

[9] Kinderlehrer, D. and Stampacchia, G. *An Introduction to Variational Inequalities and Their Applications, Harcourt Brace Jovanovich, 1980*

[10] Nochetto,R.H., Siebert, K.G. and Veeser, A. Pointwise a posteriori error control for elliptic obstacle problems. *Numer. Math.* , Vol. **95**, 2003

[11] Nochetto,R.H., Siebert, K.G. and Veeser, A. Fully localized a posteriori error estimators and barrier sets for contact problems. *SIAM J. Numer. Anal.*, Vol. **42**, 2005

[12] Veeser, A. Efficient and reliable a posteriori error estimators for elliptic obstacle problems. *SIAM J. Numer. Anal.*, Vol. **39**, 2001

[13] Rodrigues, J.F. *Obstacle Problems in mathematical Physics*, Elsevier Science Publishers B.V., 1987