11th World Congress on Computational Mechanics (WCCM XI)
5th European Conference on Computational Mechanics (ECCM V)
6th European Conference on Computational Fluid Dynamics (ECFD VI)
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2D INCOMPRESSIBLE VISCOUS FLOWS AT MODERATE AND HIGH REYNOLDS NUMBERS: A DIRECT PRIMITIVE VARIABLES APPROACH

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Key words: Asymptotic steady state, time-dependent flows, sub-vortices

Abstract. The unsteady Navier-Stokes equations in primitive variables which model viscous incompressible fluid flow are numerically solved by a simple direct projection method that involves an operator splitting of three steps in the time discretization process. The numerical scheme does not involve any iteration, is independent of the spatial dimension, and its costly part relies on the solution of elliptic problems for which very efficient solvers exist regardless of the spatial discretization. The scheme is tested with the well known two-dimensional lid-driven cavity problem at moderate and high Reynolds numbers Re in the range $400 \leq Re \leq 15000$. For moderate Re the results are validated with previously published results which are supposed to be correct; for these flows the time T_{ss} when the flow converges to the asymptotic steady state is reported. For high Re's, say Re=10000 and 15000, time-dependent flows, results at specific final times T_f are reported. Flows are also reported close from their departure from rest, and it should be noted that for moderate Re's they look different than those for high Re's. For high Re's, transition to 2D turbulence is observed as *time* and/or Re increase: the main structure given by the central vortex remains almost with no change at all whereas the new small structures, sub-vortices or eddies, increment or disappear; to be able to see this, long time computations has to be done, for $Re = 10000$ a result at $t = 5000$ is presented.

1 INTRODUCTION

The main goal of this paper is to present numerical results for moderate and high Reynolds numbers Re in the range of $400 \leq Re \leq 15000$. The results are obtained computationally from the unsteady Navier-Stokes equations in primitive variables on the well known un-regularized (or lid-driven) cavity problem which causes recirculation because of the nonzero velocity boundary condition on the top wall. Each flow is obtained from $t = 0$ regardless of Re, since the numerical scheme has this ability, and not from the solution previously computed for lower Re's.

The numerical scheme was previously reported in Báez and Nicolás [1] for isothermal problems and in Báez and Nicolás [2] for thermal ones, it is based on a direct projection involving an operator splitting of three steps in the time discretization; it is robust enough to handle high Reynolds numbers, which is not an easy task to deal with.

No stabilization process is used; the meshes follow the size implied by the thickness of the boundary layer (of order of $Re^{-\frac{1}{2}}$), no refining on the mesh is required near the boundary. As Re increases the mesh must be refined and this leads to decrease the time step: numerically, by stability and physically, to capture the fast dynamics of the flow.

In earlier works we have mentioned that to get the right iso-vorticity contours, say the ones given by the values in Ghia et al. [3] which are supposed to be correct, is more difficult than to get the right streamlines of the stream function; this is the reason that some published works do no report the iso-vorticity contours, due to oscillations on the top right corner of the cavity

Actually, the mesh size of our results is chosen, at this stage, as the one for which such oscillations are reduced to a minimum, smaller than those in Schreiber and Keller [4].

At moderate Re's, $Re \leq 7500$, the flows approach to an asymptotic steady state as t tends to ∞ (large t in practice). For higher Re, the flow does not seem to be "stationed" somewhere, indicating that the flow is time-dependent; to assure this, long time computations must be done.

In connection with what we have just mentioned above, at high Re the flows exhibit their time-dependent characteristic since they are displayed at significant times t's bigger than those which correspond to flows of moderate Re; they reach their steady state at $t = T_{ss}$, where T_{ss} is defined in Nicolás and Bermúdez [5]. From Báez and Nicolás [1]: $T_{ss} = 23.08$, 31.15, 67, and 121.884 for $Re = 400$, 1000, 3200, and 5000 respectively; all these flows coincides perfectly, streamlines and vorticity, with those in Ghia et al. [3]. The result reported here for $Re = 7500$, at steady state, T_{ss}=195.16.

Flows are reported close from its departure from rest, for moderate Re's they look different than those for high Re's. For the latter Re's, the corresponding flows show physically the congruence in connection with: as *time* and/or Re increase, almost no change is observed in the big structure of the streamlines, that is, the central vortex, but the number of sub-vortexes may increase or disappear; which is likely to occur in the transition to 2D turbulence, Landau and Lifshitz [6] and Foias et al. [7].

2 MATHEMATICAL MODEL AND NUMERICAL METHOD

Let $\Omega \subset R^N$ $(N = 2, 3)$ be the region of an unsteady viscous incompressible fluid flow, and let Γ its boundary. This kind of flows are modeled by the dimensionless unsteady Navier-Stokes equations, in Q=$\Omega \times (0, T), T > 0$, with boundary $\Sigma = \Gamma \times [0, T]$, given by

$$\begin{array}{lll} \mathbf{u}_t + \nabla p + (\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{Re}\nabla^2\mathbf{u} + \mathbf{f} & & (a) \\ \nabla \cdot \mathbf{u} \qquad\qquad\qquad\quad = 0, & & (b) \end{array} \tag{1}$$

where $\mathbf{u}$ and p represent the velocity and pressure of the fluid; $\mathbf{f}$ is a external force. The parameter Re is the Reynolds number. The system must be complemented with initial and boundary conditions; for instance, an initial condition for $\mathbf{u}$ given by $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$ in Ω and boundary condition $\mathbf{u}(\mathbf{x}, t) = \mathbf{g}(s, t)$ on Σ. In Báez and Nicolás [1] it has already pointed out the difficulties we must face when trying to solve this system.

To solve system (1), the time derivative $\mathbf{u}_t$ is approximated by the second order scheme

$$\mathbf{u}_t(\mathbf{x}, (n + 1)\Delta t) \approx \frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t}, n \geq 1 \tag{2}$$

with Δt the time step and $\mathbf{u}^k = \mathbf{u}(\mathbf{x}, k\Delta t)$.

Then, at each time level $t = (n + 1)\Delta t$ a semi-discrete stationary system is obtained from (1), in Ω, that reads

$$\begin{array}{r} \frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} + \ (\mathbf{u}^{n+1} \cdot \nabla)\mathbf{u}^{n+1} + \nabla p^{n+1} = \frac{1}{Re}\nabla^2\mathbf{u} + \mathbf{f}^{n+1} \\ \nabla \cdot \mathbf{u}^{n+1} = 0. \end{array} \tag{3}$$

The convective nonlinear term

$$(\mathbf{u}^{n+1} \cdot \nabla)\mathbf{u}^{n+1}$$

is approximated explicitly, with a linear extrapolation from the value $\mathbf{u}$, at time levels $n\Delta t$ and $(n - 1)\Delta t$.

Thus, at each time level the scheme obtains p and $\mathbf{u}$, which are calculated through a projection method of three steps to solve the momentum and continuity equations. These steps are explained in detail in Báez and Nicolás [1].

A key step of the projection method is a Poisson pressure equation supplemented, see Badalassi et al. [8] and the references there in, by taking the normal component of a semi-discrete momentum equation, where the Laplacian of the velocity is split into a solenoidal and an irotational parts

$$\nabla^2\mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times \nabla \times \mathbf{u}. \tag{4}$$

Since (3) holds for $\mathbf{u}$, (6) leads to

$$-\nabla^2\mathbf{u} = \nabla \times \nabla \times \mathbf{u}, \tag{5}$$

which is approximated by a linear extrapolation, using the known values at time levels n and $n-1$. The expensive part of the process relies on solving four elliptic problems: one for p^{n+1} and two for $\mathbf{u}^{n+1}$. Efficient solvers exist for this purpose regardless of the spatial discretization; details can be found in Báez and Nicolás [1], there is also indicated how to obtain p^1 and $\mathbf{u}^1$ that are required in the first time level.

Once the final velocity $\mathbf{u} = (u_1, u_2)$ is obtained, the vorticity ω and stream function ψ can be computed by

$$\omega = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \text{ and } \nabla^2 \psi = -\omega, \quad in\ \Omega.$$

3 NUMERICAL RESULTS AND DISCUSSION

The numerical results take place in the two dimensional cavity Ω=$(0,1) \times (0,1)$, in connection with the lid-driven cavity problem; then, the boundary condition for the velocity $\mathbf{u}$ is given by $\mathbf{u} = (1,0)$ on the moving wall $(x, y = 1)$ and $\mathbf{u} = (0,0)$ elsewhere. Considering that the fluid is initially a rest, the initial condition is $\mathbf{u}_0(\mathbf{x}) = (0,0)$ in Ω and since no external force is considered, $\mathbf{f} = \mathbf{0}$ in $(1a)$ and thereafter. As mentioned in the Introduction the range of Reynolds numbers Re that is considered is $400 \leq Re \leq 15000$. The results are presented through streamlines of the stream function and the iso-contours of the vorticity. The mesh size is denoted by h and the time step Δt and it will be indicated in each case under study. The time T_{ss} where the flow reaches its asymptotic steady state is defined by the point-wise discrete L_∞ absolute criterion in $\overline{\Omega}$

$$\psi: \ ||\psi_h^{n+1} - \psi_h^n||_\infty < tol, \quad \omega: ||\omega_h^{n+1} - \psi_h^n||_\infty < tol$$

and it was introduced for the first time in [5].

Figures 1 and 2 picture the flows at $T_f = 30$ for $Re = 7500$ and $Re = 10000$ respectively; Figures 3 and 4 display the flows at $T_{ss} = 195.16$ and at $T_f = 3000$ for $Re = 7500$ and $Re = 10000$ respectively; in the same order, Figures 5 and 6 show the flows at $T_f = 3000$ and at $T_f = 5000$ for $Re = 15000$ and $Re = 10000$.

What do we observe?

1) As expected $Re = 7500$ reaches its steady state but $Re = 10000$ and 15000 do not.

2) For $Re = 7500$ and 10000, since the beginning at $T_f = 30$, Figures 1 and 2 on the left, it is noted that the main vortex (or central), moving clockwise, in the streamlines remains almost the same (except that for $Re = 10000$ the fifth circle is smaller than the one for $Re = 7500$); concerning the secondary sub-vortexes, moving counterclockwise, for $Re = 7500$ there are 3 whereas for $Re = 10000$ there are 5 (one very small to the right of the one on the left bottom corner, which is more noticeable with a zoom). For $Re = 7500$ there are 2 tertiary sub-vortexes, moving clockwise, below the secondary ones on the bottom left and right corners whereas for $Re = 10000$ there are also 2 tertiary sub-

vortexes in the same places; in both cases, they are very small but they are noticeable with a big zoom.

3) Concerning the vorticity, it behaves something like what it has already happened for Re = 4000 and 10000 at T_f = 25, Báez and Nicolás [1]: the iso-vorticity contours that arise in the center of the cavity tend towards the walls whereas for $Re = 10000$ they spread almost over all the cavity, Figures 1 and 2 on the right.

4) The flow at steady state for Re = 7500, T_{ss} = 195.16, Figure 3, streamlines and iso-vorticity contours, agree perfectly with the ones in Ghia et al. [3].

5) For Re = 10000 and 15000, the central vortex remains the same: 9 circles like for Re = 7500 at T_{ss}, Figure 3 and Figures 4-6 on the left, which agree with what Landau and Lifshitz [6] say, turbulence Chapter.

6) The iso-vorticity contours, surprisingly, except by the central circle, most of the structure is almost like for $Re = 7500$, same Figures as in 3) but on the right; for instance, the big fang that starts on the top right corner and goes around the cavity has three small ones inside, one behind and inside the other; this situation appears in Ghia et al. [3] since $Re = 3200$.

7) The number of secondary sub-vortexes for Re = 10000, Figure 4, left, and 15000, Figure 5, left, at T_f = 3000 there are 5 and 4 respectively whereas there are 2 tertiary sub-vortexes for Re = 10000, 1 very small (it is noted with a zoom) behind the one on the left bottom corner and other below the secondary one on the right bottom corner, and there are also 2 for Re = 15000, 1 behind the secondary one on the top left corner (it is also noted with a zoom) and 1 below the secondary one below the secondary one on right corner bottom .

8) For Re = 10000 at T_f=5000, Figure 6, left, there are 4 secondary sub-vortexes and 1 tertiary one.

Remark. We are not claiming the last word, more investigation is required. For instance, despite the fact that for all the results, the iso-vorticity contours on the top right corner do not show oscillations which mean that not mesh refining is needed, Schreiber and Keller [4], for our early experiences with the stream function-vorticity formulation using an iterative method, Nicolás and Bermúdez [9], the circle around the center of the cavity in the iso-vorticity contours, Figures 4-6 (on the right), might disappear. In Nicolás and Bermúdez [9] just mentioned, a result is displayed for Re = 10000 at T_f = 1500, which is closed to the one here at T_f = 3000, the mesh size is also 1/256 and the time step $\Delta t = 0.0025$ too. Surprisingly such circle does not appear but a notorious small oscillation appear on the top right corner; concerning the streamlines, the central vortex has also 9 circles but the 2 innermost are much smaller than the ones here; there are 4 secondary sub-vortexes and 1 tertiary one. As a matter of fact, with the stream function-vorticity formulation a result has also been computed for Re = 15000 at T_f = 1600 with mesh size finer than here, $h = 1/768$, and time step, $\Delta t = 0.0025$, showing 6 secondary sub-vortexes, 1 tertiary sub-vortex, and 1 *quaternary* sub-vortex, moving counterclockwise; no oscillation in the iso-vorticity contours on the top right corner is observed.

4 CONCLUSIONS

Numerical results have been presented from the unsteady Navier-Stokes equations in primitive variables, using a direct projection method which involves an operator splitting of three steps in the time discretization. Moderate and high Reynolds numbers Re are studied. The flows show different characteristics: for moderate Re's the steady state is reached whereas for high Re's do not. Various properties follow for high Re's, show the physical transition to 2D turbulence: through the streamlines, the big structure, focussed on the central vortex, remain almost the same as well as the appearing or disappearing of small sub-vortexes or eddies as *time* and/or Re increase. The results also indicate that more research is necessary to have conclusive characteristics of the flows.

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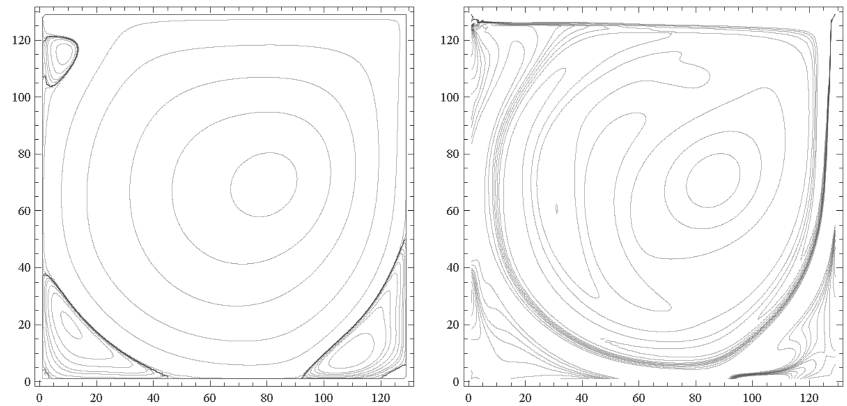

Figure 1: $Re = 7500$, ψ (left), ω (right): $T_f = 30.002$; $h = 1/128$, $\Delta t = 0.0035$

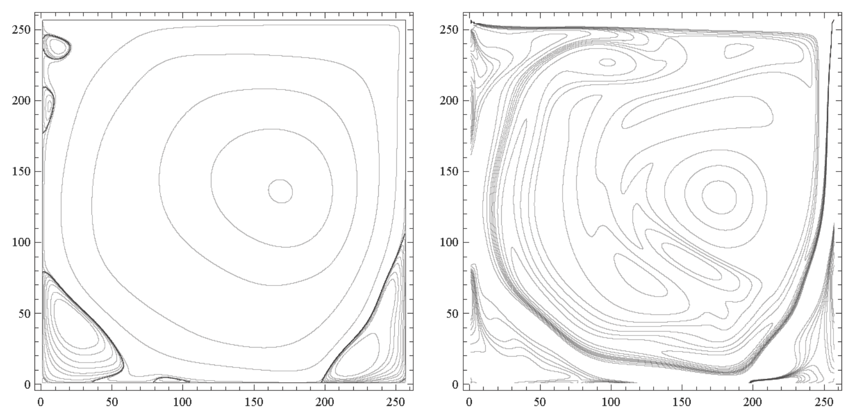

Figure 2: $Re = 10000$, ψ (left), ω (right): $T_f = 30$; $h = 1/256$, $\Delta t = 0.0025$

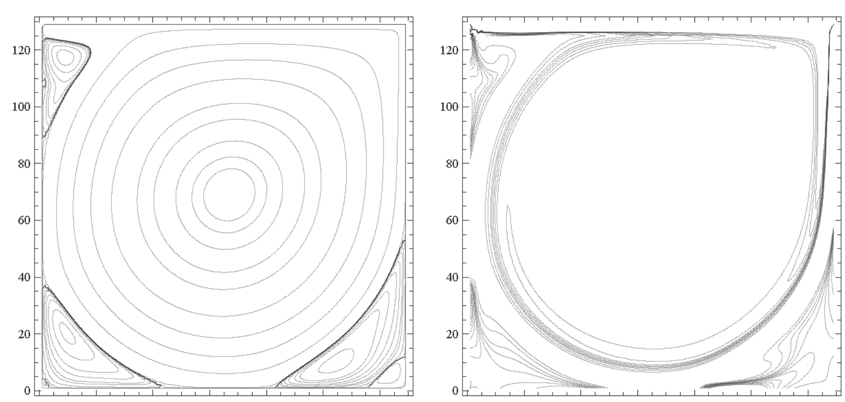

Figure 3: $Re = 7500$, ψ (left), ω (right): $T_{ss} = 195.16$, $h = 1/128$, $\Delta t = 0.0035$

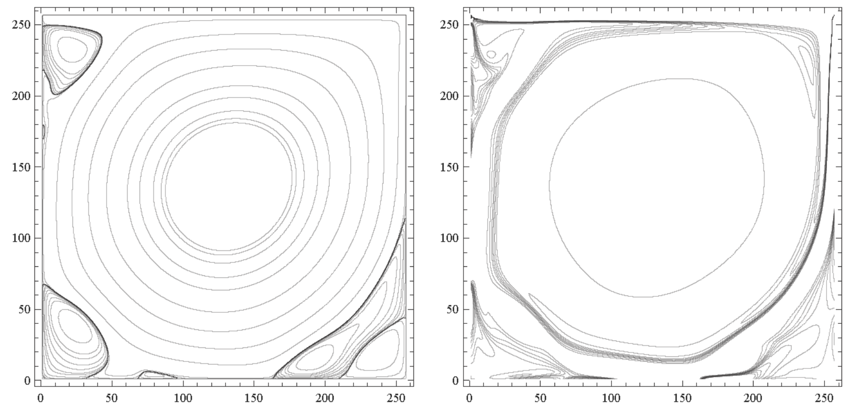

Figure 4: $Re = 10000$, ψ (left), ω (right): $T_f = 3000$; $h = 1/256$, $\Delta t = 0.0025$

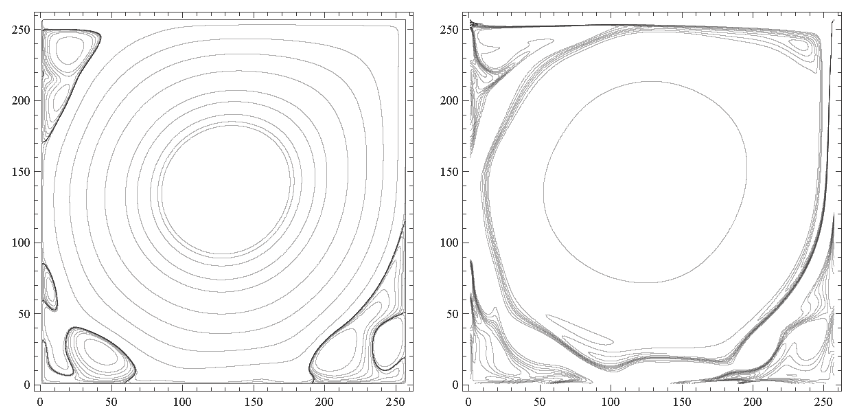

Figure 5: $Re = 15000$, ψ (left), ω (right): $T_f = 3000$; $h = 1/256$, $\Delta t = 0.0025$

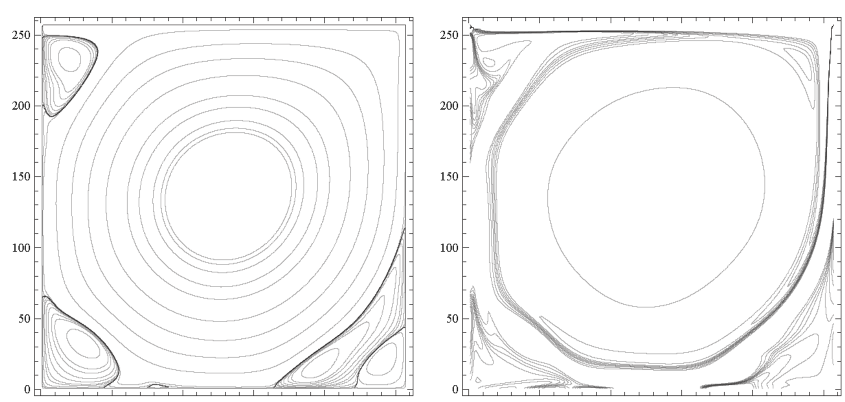

Figure 6: $Re = 10000$, ψ (left), ω (right): $T_f = 5000$; $h = 1/256$, $\Delta t = 0.0025$