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AN EXPLICIT DYNAMIC METHOD FOR A DISCRETE ELEMENT MODEL USING THE PRINCIPLE OF HYBRID-TYPE VIRTUAL WORK

T. YAGI*, N. TAKEUCHI†, K. YAMAMURA† AND M. KUSABUKA†

* Hosei University
2-33 Ichigayatamachi, Shinjuku-ku, Tokyo, Japan
e-mail: tadao.yagi.9u@stu.hosei.ac.jp

† Hosei University
2-33 Ichigayatamachi, Shinjuku-ku, Tokyo, Japan
e-mail: takeuchi@hosei.ac.jp

† Nippon Steel & Sumitomo Metal Corporation (NSSMC)
20-12 Shintomi, Futtsu City, Chiba, Japan
email: yamamura.88s.kazuto@jp.nssmc.com

† Subsurface Investigation Office Co., Ltd.
3-22 Ichigayahonmuracho, Shinjuku-ku, Tokyo, Japan

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Abstract. A rigid bodies spring model (RBSM) has been developed as a numerical model to generalize limit analysis in plasticity, in which a structure to be analyzed is idealized as an assemblage of rigid bodies connected by normal and tangential springs. Although contact surfaces are handled differently in RBSM and in the distinct element method (DEM), the number of degrees of freedom is the same in both cases. If each element is formulated by the explicit method, the algorithm for a dynamic analysis is the same. In this paper, we illustrate the formulation of RBSM for each element using the principle of hybrid virtual work. In addition, we propose a new RBSM approach for contact types, namely, edge-to-edge contact. Finally, we examine the applicability of dynamic analysis using RBSM.

1 INTRODUCTION

The explicit method (Belytschko 1984) [1] is widely used for time integration in the numerical analysis of dynamic problems. In this case, the time integration technique is represented by the central difference method, which calculates a solution sequentially. Recently, discontinuous analyses using the distinct element method (DEM) (Cundall 1971) [2] and a combined finite/discrete element method (FDEM) (Munjiza et al. 1995) [3] have attracted considerable attention, and the use of explicit schemes has increased. A combined analysis technique is recognized as being useful in the explicit method, and the same also applies to other numerical algorithms in a discontinuous problem.

A rigid bodies–spring model (RBSM) (Kawai 1977) [4] was developed as a numerical model for generalizing limit analysis in plasticity, in which a structure to be analyzed is idealized as an assemblage of rigid bodies connected by normal and tangential springs. Although contact surfaces are handled differently in RBSM and DEM, the number of degrees of freedom is the same in both cases. If formulation by the explicit method is used for each element, the algorithms for dynamic analyses are identical.

This paper illustrates the formulation of RBSM for each element using the principle of hybrid virtual work. The same discussion is expanded to include DEM, and an explicit dynamic method that combines RBSM and DEM is presented. However, in this case, the contact mechanism is edge-to-edge contact. We numerically verify the stability and accuracy of the solutions obtained using the present method through an example problem.

2 BRIEF DESCRIPTION OF RBSM FOR DYNAMIC PROBLEM

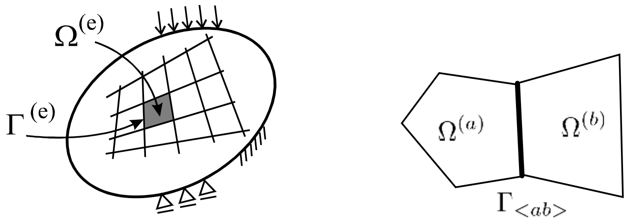
The basic equation of the elastic problem is as follows:

$$\begin{gathered}\operatorname{div}\boldsymbol{\sigma} + \boldsymbol{f} + \boldsymbol{f}_{\alpha} = 0 \quad \text{in} \quad \Omega \\ \boldsymbol{\sigma} = \boldsymbol{D} : \boldsymbol{\varepsilon} \quad \boldsymbol{\varepsilon} = \boldsymbol{\nabla}^{s}\boldsymbol{u} \overset{\text{def.}}{=} \frac{1}{2}\left[\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^{t}\right] \\ \boldsymbol{u}|_{\Gamma_u} = \hat{\boldsymbol{u}} \ \text{(given)} \quad \boldsymbol{\sigma}|_{\Gamma_\sigma}\hat{\boldsymbol{n}} = \hat{\boldsymbol{t}} \ \text{(given)}\end{gathered} \tag{1}$$

where Ω is the reference configuration of a continuum body with a smooth boundary $\Gamma := \partial\Omega$; $\Gamma_u := \partial_u\Omega \subset \partial\Omega$, the geometric boundary; $\Gamma_\sigma := \partial_\sigma\Omega \subset \partial\Omega$, the kinetic boundary; $\boldsymbol{\sigma}$ is the Cauchy stress tensor; $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor; $\boldsymbol{f}$ is the body force per unit volume; $\boldsymbol{\nabla}$ is the differential vector operator; and $\boldsymbol{\nabla}^s$ is the symmetric part of $\boldsymbol{\nabla}$. When the displacement field in $\boldsymbol{x} \in \Omega$ is expressed as $\boldsymbol{u}$ and the density is expressed as ρ, the inertia force $\boldsymbol{f}_\alpha$ of equation (1) is expressed as follows:

$$\boldsymbol{f}_{\alpha} = -\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2} \tag{2}$$

Let Ω consist of M subdomains $\Omega^{(e)} \subset \Omega$ with a closed boundary $\Gamma^{(e)} := \partial\Omega^{(e)}$, as shown in Figure 1(a). In other words, $\Omega = \bigcup_{e=1}^{M}\Omega^{(e)}$. Here, $\Omega^{(r)} \cap \Omega^{(q)} = 0 \quad (r \neq q)$.



(a) subdomain $\Omega^{(e)}$ (b) common boundary $\Gamma_{<ab>}$

Figure 1. Subdomain and its common boundary

We use $\Gamma_{<ab>}$, defined as $\Gamma_{<ab>} \overset{\text{def.}}{=} \Gamma^{(a)} \cap \Gamma^{(b)}$, as the common boundary for two subdomains $\Omega^{(a)}$ and $\Omega^{(b)}$, adjoined as shown in Figure 1(b). The relation for the displacement $\tilde{\boldsymbol{u}}^{(e)}$ on $\Gamma_{<ab>}$, which is the intersection boundary between $\Omega^{(a)}$ and $\Omega^{(b)}$, is as follows:

$$\tilde{\boldsymbol{u}}^{(a)} = \tilde{\boldsymbol{u}}^{(b)} \quad \text{on} \quad \Gamma_{<ab>} \tag{3}$$

The following hybrid-type virtual work equation is obtained by introducing this subsidiary condition into a virtual work equation using Lagrange multipliers $\boldsymbol{\lambda}$:

$$\sum_{e=1}^{M}\left(\int_{\Omega^{(e)}} \boldsymbol{\sigma}:\mathrm{grad}(\delta\boldsymbol{u})dV - \int_{\Omega^{(e)}} \boldsymbol{f}\cdot\delta\boldsymbol{u}\,dV - \int_{\Omega^{(e)}} \boldsymbol{f}_{\alpha}\cdot\delta\boldsymbol{u}\,dV\right) - \sum_{s=1}^{N}\left(\delta\int_{\Gamma_{<s>}} \boldsymbol{\lambda}\cdot(\tilde{\boldsymbol{u}}^{(a)} - \tilde{\boldsymbol{u}}^{(b)})\,dS\right) - \int_{\Gamma_{\sigma}} \hat{\boldsymbol{t}}\cdot\delta\boldsymbol{u}\,dS = 0 \qquad \forall\delta\boldsymbol{u}\in\mathbb{V} \tag{4}$$

Here, N denotes the number of common boundaries of the subdomain, and $\delta\boldsymbol{u}$ denotes the virtual displacement. An independent displacement field in each subdomain is assumed as follows:

$$\boldsymbol{u}^{(e)} = \boldsymbol{N}^{(e)}\boldsymbol{U}^{(e)} \tag{5}$$

$$\boldsymbol{U}^{(e)} = \lfloor \boldsymbol{d}^{(e)}, \boldsymbol{\varepsilon}^{(e)} \rfloor^{t}, \quad \boldsymbol{N}^{(e)} = \lfloor \boldsymbol{N}_{d}^{(e)}, \boldsymbol{N}_{\varepsilon}^{(e)} \rfloor$$

Here, $\boldsymbol{d}^{(e)}$ denotes the rigid displacement and the rigid rotation at point P in the subdomain (e), and $\boldsymbol{\varepsilon}^{(e)}$ denotes a constant strain in the subdomain (e). Equation (4) implies that the Lagrange multiplier $\boldsymbol{\lambda}$ is the surface force on the boundary $\Gamma_{<ab>}$ in subdomains $\Omega^{(a)}$ and $\Omega^{(b)}$. Hence, the surface force is defined as follows:

$$\boldsymbol{\lambda}_{<ab>} = \boldsymbol{k}\cdot\boldsymbol{\delta}_{<ab>} \tag{6}$$

Here, $\boldsymbol{\delta}_{<ab>}$ denotes the relative displacement at the boundary $\Gamma_{<ab>}$, and $\boldsymbol{k}$ denotes the penalty function. The equation of motion, discretized with respect to space by substituting the abovementioned relations in equation (4), is obtained as follows:

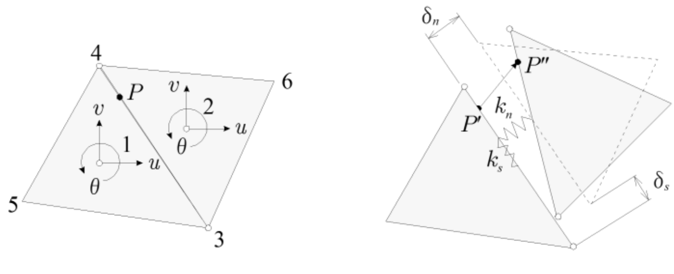
$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} = \boldsymbol{P} \tag{7}$$

$$\boldsymbol{M} = \sum_{e=1}^{M}\boldsymbol{M}^{(e)} \quad \boldsymbol{K} = \sum_{e=1}^{M}\boldsymbol{K}^{(e)} + \sum_{s=1}^{N}\boldsymbol{K}_{<s>}$$

3 CONTACT FORCES

3.1 Contact interface between elements

The formulation of an RBSM is advanced by evaluating the energy stored in the springs between adjoining elements, as shown in Figure 2.



(a) before deformation (b) after deformation

Figure 2. Model of the contact interface in the RBSM

The equation of motion (7) is simplified by $\boldsymbol{d}$ and $\boldsymbol{\varepsilon}$. Moreover, it is expressed in the global coordinate system as follows:

$$\begin{bmatrix} \boldsymbol{M}_{dd} & \boldsymbol{M}_{d\varepsilon} \\ \boldsymbol{M}_{\varepsilon d} & \boldsymbol{K}_{\varepsilon\varepsilon} \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{d}} \\ \ddot{\boldsymbol{\varepsilon}} \end{Bmatrix} + \begin{bmatrix} \boldsymbol{K}_{dd} & \boldsymbol{K}_{d\varepsilon} \\ \boldsymbol{K}_{\varepsilon d} & \boldsymbol{K}_{\varepsilon\varepsilon} + \boldsymbol{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{d} \\ \boldsymbol{\varepsilon} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{P}_d \\ \boldsymbol{P}_\varepsilon \end{Bmatrix} \tag{8}$$

Here, the linear displacement field of equation (5), assuming a rigid displacement field and a mass matrix that contains only diagonal elements, is given as follows:

$$\boldsymbol{M}_{dd} = \rho \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & I_p \end{bmatrix} \tag{9}$$

The substitution of the abovementioned relations into equation (8) yields the following:

$$\boldsymbol{M}_{dd}\ddot{\boldsymbol{d}} = \boldsymbol{P}_d - \boldsymbol{K}_{dd}\boldsymbol{d} \tag{10}$$

In equation (6), $\boldsymbol{k}$ is the spring constant, and is the following are assumed:

$$\begin{Bmatrix} \lambda_n \\ \lambda_s \end{Bmatrix} = \begin{bmatrix} k_n & 0 \\ 0 & k_s \end{bmatrix} \begin{Bmatrix} \delta_n \\ \delta_s \end{Bmatrix} \tag{11}$$

Here, under plane stress conditions, k_n and k_s are expressed as follows:

$$\left.\begin{aligned} \boldsymbol{k}_n &= \frac{E}{(1-\nu^2)(h1+h2)} \\ \boldsymbol{k}_s &= \frac{E}{(1+\nu)(h1+h2)} \end{aligned}\right\} \tag{12}$$

where E is Young's modulus, ν is Poisson's ratio, and h is the length of the vertical line from the centroid of each subdomain to the boundary edge.

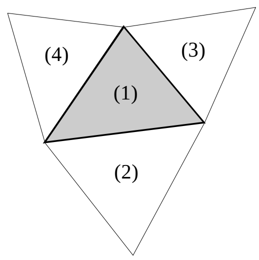


Figure 3. Element (1) and adjoining element

As Figure 3 shows, we expand equation (11) as an example to element (1) and the adjoining element. In this case, the integration at the boundary edge, with a focus on element (1), is only relevant to elements (2)–(4). Therefore, the other elements are not relevant simultaneous equations. We represent a portion of equation (10) for this example as follows:

$$\begin{Bmatrix} \vdots \\ \boldsymbol{M}^{(1)} \\ \boldsymbol{M}^{(2)} \\ \boldsymbol{M}^{(3)} \\ \boldsymbol{M}^{(4)} \\ \vdots \end{Bmatrix} \begin{Bmatrix} \vdots \\ \ddot{\boldsymbol{d}}^{(1)} \\ \ddot{\boldsymbol{d}}^{(2)} \\ \ddot{\boldsymbol{d}}^{(3)} \\ \ddot{\boldsymbol{d}}^{(4)} \\ \vdots \end{Bmatrix} = \begin{Bmatrix} \vdots \\ \boldsymbol{P}_d^{(1)} \\ \boldsymbol{P}_d^{(2)} \\ \boldsymbol{P}_d^{(3)} \\ \boldsymbol{P}_d^{(4)} \\ \vdots \end{Bmatrix} - \begin{bmatrix} \ddots & & & & & \\ 0 & 3\boldsymbol{k}_{dd}^{(1,1)} & \boldsymbol{k}_{dd}^{(1,2)} & \boldsymbol{k}_{dd}^{(1,3)} & \boldsymbol{k}_{dd}^{(1,4)} & 0 \\ & \boldsymbol{k}_{dd}^{(1,2)} & \boldsymbol{k}_{dd}^{(2,2)} & & & \\ & \boldsymbol{k}_{dd}^{(1,3)} & & \boldsymbol{k}_{dd}^{(3,3)} & & \\ & \boldsymbol{k}_{dd}^{(1,4)} & & & \boldsymbol{k}_{dd}^{(4,4)} & \\ & & & & & \ddots \end{bmatrix} \begin{Bmatrix} \vdots \\ \boldsymbol{d}^{(1)} \\ \boldsymbol{d}^{(2)} \\ \boldsymbol{d}^{(3)} \\ \boldsymbol{d}^{(4)} \\ \vdots \end{Bmatrix} \tag{13}$$

Because $M^{(e)}$ is independent of each element, when focusing on element (1), the following relations are obtained.

$$\boldsymbol{M}^{(1)}\ddot{\boldsymbol{d}}^{(1)} = \boldsymbol{P}_d^{(1)} - \left(3\boldsymbol{k}_{dd}^{(1,1)}\boldsymbol{d}^{(1)} + \boldsymbol{k}_{dd}^{(1,2)}\boldsymbol{d}^{(2)} + \boldsymbol{k}_{dd}^{(1,3)}\boldsymbol{d}^{(3)} + \boldsymbol{k}_{dd}^{(1,4)}\boldsymbol{d}^{(4)}\right) \tag{14}$$

From the above relationship, the stress element is obtained using the surface forces of the element boundary, which can be expressed as follows:

$$\boldsymbol{M}^{(e)}\ddot{\boldsymbol{U}}^{(e)} = \boldsymbol{P}_d^{(e)} - \oint_{\Gamma^{(e)}} \boldsymbol{N}_d^{(e)}\boldsymbol{t}^{(e)}\, d\Gamma \tag{15}$$

Thus, the equation of motion becomes computable for each element. As shown in Figure 4, the element acceleration is obtained by the resultant of the contact forces in the case of discontinuous bodies.

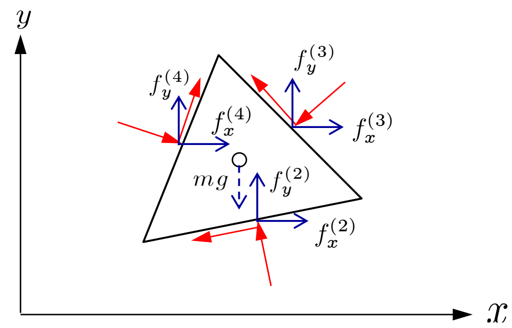


Figure 4. Resultant of contact forces

$$\begin{aligned}
\ddot{\boldsymbol{u}}^{(e)} &= \sum_{e=1}^{3} fx^{(e)}/m^{(e)} \\
\ddot{\boldsymbol{v}}^{(e)} &= \sum_{e=1}^{3} fy^{(e)}/m^{(e)} \\
\ddot{\boldsymbol{\theta}}^{(e)} &= \sum_{e=1}^{3} -(y-y_G)fx^{(e)} + (x-x_G)fy^{(e)}/I^{(e)}
\end{aligned} \tag{16}$$

3.2 Contact mechanism

The approach represented by equation (16) is basically the same as in the DEM. However, in the DEM, the contact mechanism is based on point contact, whereas in the RBSM, the contact mechanism is surface contact. Therefore, it is necessary to characterize the nature of the contact. We propose a new RBSM approach for contact types, namely, edge-to-edge contact, as shown in Figure 5.

In the contact state, the amounts of penetration in the normal and shear directions, δn and δs, respectively, are defined as follows:

$$\begin{aligned}
\delta_n &= (y_m - y_c)\frac{x_{b_2} - x_{b_1}}{L_b} - (x_m - x_c)\frac{y_{b_2} - y_{b_1}}{L_b} \\
\delta_s &= (x_m - x_c)\frac{x_{b_2} - x_{b_1}}{L_b} + (y_m - y_c)\frac{y_{b_2} - y_{b_1}}{L_b}
\end{aligned} \tag{17}$$

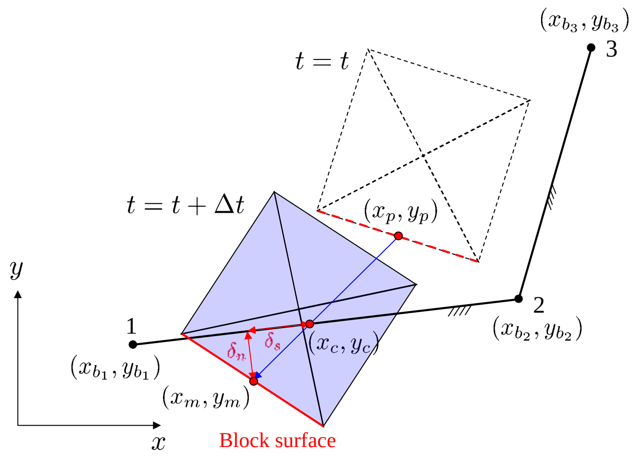


Figure 5. Edge-to-edge contact

The intersection coordinates will be determined from the contact line segment intersection determination that connects the points before and after the update of the element position, as shown in Figure 6.

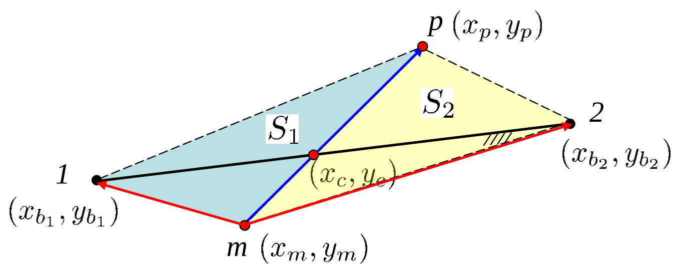


Figure 6. Intersection coordinates

Here, when the segment configured is present on both sides of the straight line at two points before and after the update of the element position at a given point of the element on the boundary side, the straight boundary line 1–2 crosses the boundary. The equations used to check that each point exists on either side are as follows:

$$\left.\begin{aligned} t_p &= (x_{b_1} - x_{b_2})(y_p - y_{b_1}) - (y_{b_1} - y_{b_2})(x_p - x_{b_1}) \\ t_m &= (x_{b_1} - x_{b_2})(y_m - y_{b_1}) - (y_{b_1} - y_{b_2})(x_m - x_{b_1}) \\ t_{b_1} &= (x_p - x_m)(y_{b_1} - y_p) - (y_p - y_m)(x_{b_1} - x_p) \\ t_{b_2} &= (x_p - x_m)(y_{b_2} - y_p) - (y_p - y_m)(x_{b_2} - x_p) \end{aligned}\right\} \tag{18}$$

When the following conditions are satisfied, they define the contact state.

$$\left.\begin{aligned} t_p \cdot t_m &< 0 \\ t_{b_1} \cdot t_{b_2} &< 0 \end{aligned}\right\} \tag{19}$$

For the cross coordinates, we obtain the following:

$$\left.\begin{aligned} x_c &= x_{b_1} + (x_{b_2} - x_{b_1}) \cdot \frac{S_1}{S_1 + S_2} \\ y_c &= y_{b_1} + (y_{b_2} - y_{b_1}) \cdot \frac{S_1}{S_1 + S_2} \end{aligned}\right\} \tag{20}$$

Thus, we determine the contact force per unit length applied in Equation (11).

3.3 Computation algorithm

We can update the position of each element by looping Δt in these calculations. The flow chart of the computation algorithm is shown in Figure 8.

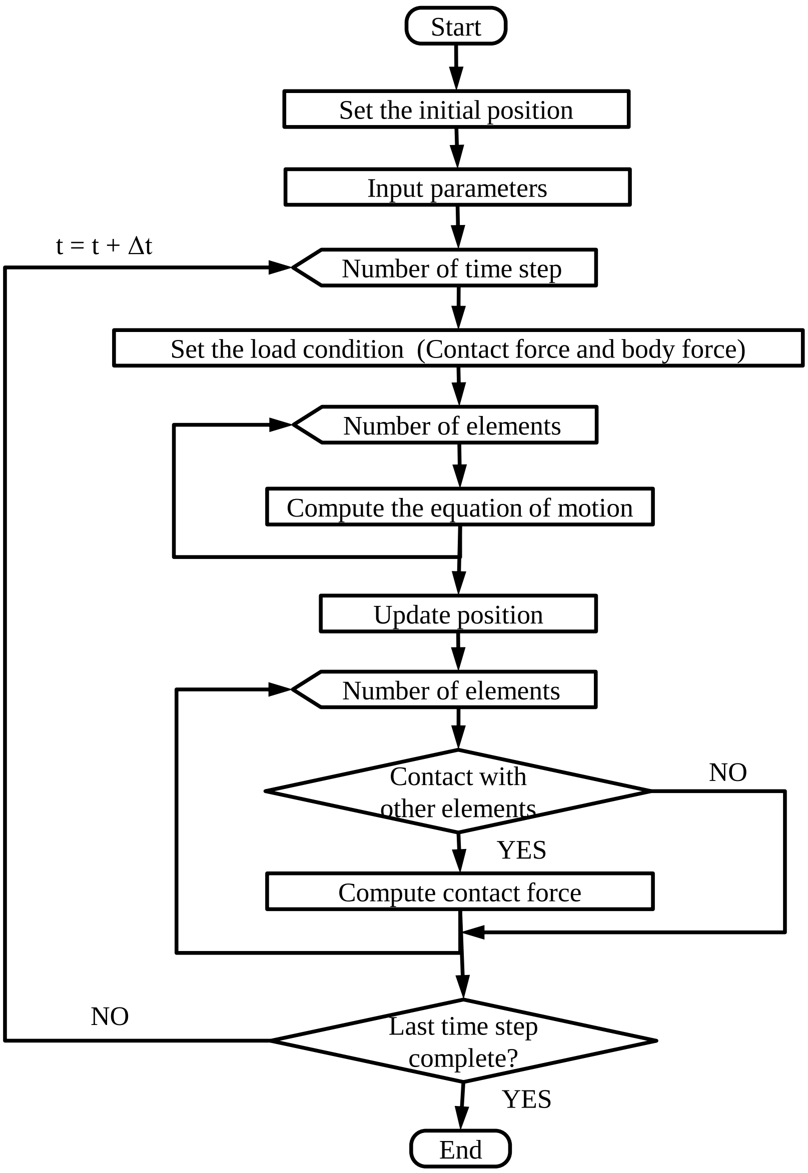
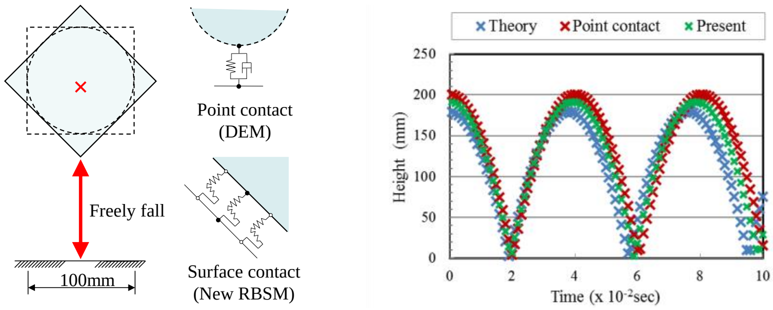


Figure 8. Flow chart of the modified RBSM for an explicit dynamic algorithm

4 NUMERICAL EXAMPLE

We present a simple problem as a numerical example. We consider a problem involving a hopping movement in which a block is allowed to freely fall and collide with the ground, as shown in Figure 9(a). The gravitational acceleration is 9.80665 m/s^2. The material constants of the block are as follows: Young's modulus, 5,127 N/mm^2; Poisson's ratio, 0.112; and density, 1,850 kg/m^3. In this case, we assume that no energy loss from the block occurs as a result of rebounding from the ground after the collision.



(a) model (b) update position

Figure 9. Case of hopping movement

Figure 9(b) shows the status of a center point when a brick block falls and hits the ground. The position is updated by repeated calculations when the length of ground penetration on the boundary side is larger than the allowable spring stiffness length, which verifies the effect of the contact force. The results are similar to the theoretical solution in the present method than point contact.

5 CONCLUSIONS

This paper describes the formulation of RBSM for each element using the principle of hybrid virtual work. The same discussion is expanded to include DEM, and an explicit dynamic method that combines RBSM and DEM is presented. We consider as a numerical example a collision problem involving a falling block, and we use an explicit dynamic method for RBSM and assess the effectiveness of the way in which the surface contact is modeled.

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