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FINITE GROWTH ON BIOLOGICAL TISSUES

**J. O'CONNOR*, L. M. ALVES BORGES*, F. PEREIRA DUDA*
and M. RODRIGUEZ MADRIGAL†**

* Federal University of Rio de Janeiro (UFRJ) COPPE - PEM
Cidade Universitária, CT, G204. Ilha do Fundão, 21941-972 Rio de Janeiro, Brazil
e-mail: joan@ufrj.br, web page: http://www.mecanica.ufrj.br

†Higher Polytechnic Institute José Antonio Echeverria (ISPJAE)
Mechanical Engineering Faculty. Street 127 S/N. Marianao 15 Havana City, Cuba
e-mail: melchor@mecanica.cujae.edu.cu - web page: http://www.cujae.cu

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Abstract. In this work is presented and reviewed the finite growth theory on biological tissues as an extension of growth theory for thermoelastic materials by meaning of stress induced growth within the context of continuum thermodynamics. Governing equations for this kind of biological materials are also presented, using the basic balance laws: mass balance, linear and angular momentum balance and the mechanical energy balance (which express the first law of thermodynamics) The constitutive equations, which describes the material behavior and the thermodynamical considerations that restricts the constitutive equations in virtue of the second law, are also presented. Studies of biological tissue behavior are also shown in an example of growth and reabsorption in the cardiovascular system due to a ventricular hypertrophy. In this example is described and analyzed how circumferential growth generates a residual stress distribution and how, using theses analysis, it is possible to predict a growth pattern in biological tissues. The model presented can be considered as a tool for medical applications in order to simulate growth and reabsorption processes in biological tissues.

1 INTRODUCTION

In the last years, bioengineering has become in one of the most active research areas. A large amount of research work related with bio-materials, tissue engineering regeneration and implants and prostheses and its adaptation inside the human body, is done and is increasing and in continuous expansion. The development of FEM softwares and CAD tools to generate geometrical bio-forms with an optimal approximation to physical real forms of human organs and tissues, allows, for example; estimate hip prostheses behavior inside femoral bone medular canal, or predict stresses behavior at bone level and its effect in the periodontal ligament when teeth are submitted to loads when using orthodontic appliances, and much more examples of FEM use in applications for hard or soft tissues. It is possible to know the stresses, strains and also displacements for biological systems or systems formed by an organ and implants or prosteses, when submitted to loads with a very good approximation [1], [2] (assuming high level of accuracy of geometrical models without entering in a detailed description), but, up to this point, it is not possible to predict if tissue grows or reabsorb when loaded. So, a powerful tool for medicine is a computational tool that allow to predict, with reliable approximation, if tissue growth or reabsorption will occur, because the success of the implant depends on growth.

The objective of this work is an study and a revision of thermoelastic materials growth theory and its treatment in the continuum mechanics context, starting from bone remodeling model early work [3], the kinematic treatment to describe surface growth based on growth velocities [4], the extension of this work including the effect of incompatible growth using the deformation gradient multiplicative decomposition [5], up to the finite growth theory induced by stress [6] based on classical growth approaches mentioned above and with thermo-mechanics of plasticity [7] and with thermo-mechanics of volumetric growth [8], [9], considered as well established procedures. This work also includes a finite growth adaptive heart growth example due to abnormal loads in a cardiac hypertrophy.

2 BALANCE LAWS

2.1 Continuum growth configurations and Mass Balance

Let X a material point of a body B occupying a region in Euclidean space. $\mathbf{X}$, $\mathbf{x}$ are the position vectors of X relatives to fixed origin 0 in reference and current configurations $\mathcal{B}$ and $\mathcal{B}_t$ Mapping $\mathbf{x} = \chi(\mathbf{X}, t)$ is a bijective function, so invertible, then: $\mathbf{X} = \chi^{-1}(\mathbf{x}, t)$. Deformation gradient is $\mathbf{F} = \nabla\chi$ and the Jacobian is $J(\mathbf{X}, t) = \det\nabla\chi_t(\mathbf{X}) > 0$

Mass in the current and reference configurations are defined as

$$m(P) = \int\limits_{P_t} \rho dv, \quad m(P) = \int\limits_{P} \rho_K dV \tag{1}$$

where $\rho = \rho(\mathbf{x}, t)$, $\rho_K = \rho_K(\mathbf{X}, t)$ are mass densities in $\mathcal{B}$, $\mathcal{B}_t$. Using $dv = JdV$ and localization theorem: $\rho_K = \rho J$. The rate change of mass, which can be non-zero is

$$\frac{d}{dt} m(P) = \frac{d}{dt} \int\limits_{P_t} \rho(\mathbf{x}, t) dv = \int\limits_{P_t} \rho\Gamma dv + \int\limits_{\partial P_t} \mu da \tag{2}$$

being Γ the rate change of mass, μ the mass flux. Using Reynolds transport theorem

$$\int\limits_{P_t} (\dot{\rho} + \rho div\mathbf{v} - \rho\Gamma) dv = \int\limits_{\partial P_t} \mu da \tag{3}$$

where $\mathbf{v}$ is the velocity of the body so the local form of the spatial mass balance is

$$\dot{\rho} + \rho div\mathbf{v} = \rho\Gamma - div\mathbf{m} \tag{4}$$

and in the reference configuration

$$\dot{\rho_K} = \rho_K\Gamma - Div\mathbf{M} \tag{5}$$

2.2 Linear and Angular momentum balances

The linear momentum balance law for growth can be written as

$$\frac{d}{dt}\int\limits_{P_t} \rho\mathbf{v}dv = \int\limits_{P_t} \rho\mathbf{b}dv + \int\limits_{\partial P_t} \mathbf{t}da + \int\limits_{P_t} (\rho\Gamma)\tilde{\mathbf{v}}dv - \int\limits_{\partial P_t} (\mathbf{m}\cdot\mathbf{n})\tilde{\mathbf{v}}da \tag{6}$$

The first two right hand side integrals are customary, $\tilde{\mathbf{v}}$ the velocity of the new mass entering the body, third and fourth integrals represents changes due to additional mass entering the body through volumetric sources and mass fluxes. Using $\tilde{\mathbf{v}} = (\tilde{\mathbf{v}} - \mathbf{v}) + \mathbf{v}$ according to [8] and substituting in 6 takes the form

$$\frac{d}{dt}\int\limits_{P_t} \rho\mathbf{v}dv = \int\limits_{P_t} \rho\mathbf{b}dv + \int\limits_{\partial P_t} \mathbf{t}da + \int\limits_{P_t} (\rho\Gamma)\mathbf{v}dv - \int\limits_{\partial P_t} (\mathbf{m}\cdot\mathbf{n})\mathbf{v}da + \int\limits_{P_t} \rho\tilde{\mathbf{b}}dv + \int\limits_{\partial P_t} \tilde{\mathbf{t}}da \tag{7}$$

Third and fourth integrals are the changes due to new mass entering to body with same velocity of the body and the last two terms are the irreversible changes due to volumetric and surface sources, where $\tilde{\mathbf{b}} = \Gamma(\tilde{\mathbf{v}} - \mathbf{v}), \tilde{\mathbf{t}} = (\mathbf{m}\cdot\mathbf{n})(\tilde{\mathbf{v}} - \mathbf{v})$, being $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{t}}$ body force and traction associated with volumetric sources and to fluxes. Local form results in

$$\rho\dot{\mathbf{v}} = \rho\overline{\mathbf{b}} + div\overline{\mathbf{T}} \tag{8}$$

$\overline{\mathbf{b}}, \overline{\mathbf{T}}$ are the effective body forces and Cauchy stress, $\mathbf{T}$ is the Cauchy tensor.

$$\rho_K\dot{\mathbf{v}} = \rho_K\overline{\mathbf{b}} + Div\overline{\mathbf{P}} \tag{9}$$

however, second Piola-Kirchhoff tensor $\overline{\mathbf{S}} = \mathbf{F}^{-1}\overline{\mathbf{P}}$ is used

2.3 Angular momentum balance

Angular momentum balance law for growth leads to the local form

$$\overline{\mathbf{T}} = \overline{\mathbf{T}}^{\mathrm{T}} \tag{10}$$

which express the symmetry of the Cauchy Tensor, and in the reference configuration

$$\overline{\mathbf{P}}\mathbf{F}^{\mathrm{T}} = \mathbf{F}\overline{\mathbf{P}}^{\mathrm{T}} \tag{11}$$

2.4 Energy Balance

The energy balance can be deduced from the conventional power balance postulate [12] in which external expended power is balanced with the summation of internal power and the kinetics energy (last term represents kinetic energy due to added mass),

$$\frac{d}{dt}\int_{P_t} \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}dv + \int_{P_t} \overline{\mathbf{T}}\cdot\mathbf{L}dv = \int_{P_t} \rho\overline{\mathbf{b}}\cdot\mathbf{v}dv + \int_{\partial P_t} \overline{\mathbf{t}}\cdot\mathbf{v}da + \int_{P_t} \frac{1}{2}(\rho\Gamma - div\mathbf{m})\mathbf{v}\cdot\mathbf{v}dv \quad (12)$$

Total energy balance is determined using the thermodynamic first law [3], [12] from balance between internal energy and kinetic energy, the external power and heat transferred,

$$\begin{aligned} \frac{d}{dt}\int_{P_t} \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}dv + \int_{P_t} \rho\dot{u}dv \quad &= \quad \int_{P_t} \rho\overline{\mathbf{b}}\cdot\mathbf{v}dv + \int_{\partial P_t} \overline{\mathbf{t}}\cdot\mathbf{v}da + \int_{P_t} \rho(r+r_i)dv \qquad (13) \\ &- \quad \int_{\partial P_t} (\mathbf{q}+\mathbf{q}_i)\cdot\mathbf{n}da \int_{P_t} \frac{1}{2}(\rho\Gamma - div\mathbf{m})\mathbf{v}\cdot\mathbf{v}dv \\ &+ \quad \int_{P_t} (\rho\Gamma - div\mathbf{m})\tilde{u}dv \end{aligned}$$

$u, \tilde{u}$ internal energies of existent mass and of added mass, r heat supply, $\mathbf{q}$ heat flux and $r_i, \mathbf{q}_i$ irreversible heat terms. From above equation using $\tilde{u} = (\tilde{u} - u) + u$, local form is

$$\rho\dot{u} = \overline{\mathbf{T}}\cdot\mathbf{L} + \rho(r+\tilde{r}) - div(\mathbf{q}+\tilde{\mathbf{q}}) \qquad (14)$$

and in the reference configuration is

$$\rho_K\dot{u} = \overline{\mathbf{P}}\cdot\dot{\mathbf{F}} + \rho_K(r+\tilde{r}) - Div(\mathbf{q}_K+\tilde{\mathbf{q}}_K) \qquad (15)$$

where $\tilde{r} = (\rho\Gamma - div\mathbf{m})(\tilde{u} - u) + r_i, \tilde{\mathbf{q}} = \mathbf{q}_i$, being $\tilde{r}$ irreversible total heat supply

3 FINITE GROWTH THEORY

The mass, linear and angular momentum, and energy balances provides the equations for the twenty-six unknowns $\{\chi, \rho, \overline{\mathbf{T}}, u, \mathbf{q}, \theta, \Gamma, \mathbf{m}, \tilde{r}, \tilde{\mathbf{q}}\}$ where the first eighteen $\{\chi, \rho, \overline{\mathbf{T}}, u, \mathbf{q}, \theta\}$ are the variables of a conventional thermo-elastic material. Of the other eight, four describe mass sources and mass flow $\{\Gamma, \mathbf{m}\}$ and four describe the irreversible process $\{\tilde{r}, \tilde{\mathbf{q}}\}$, and extend the model to thermoelastic materials with growth. Body force $\mathbf{b}$, heat supply r and temperature θ are known.

3.1 Multiplicative decomposition of the deformation gradient

This multiplicative decomposition is founded in plasticity, was introduced in biological growth by [5] and can be expressed as

$$\mathbf{F} = \mathbf{F}_e \mathbf{F}_g \tag{16}$$

$\mathbf{F}_g$ is the local mapping of material in $\mathcal{B}$ to a local maximally unloaded intermediate configuration. $\mathcal{B}_g$ is the intermediate configuration, $\mathbf{F}_e$ is the elastic mapping from $\mathcal{B}_g$ into $\mathcal{B}_t$ and is the responsible part that restores compatibility of the body B

$\mathbf{F}_g$ and $\mathbf{F}_e$ are incompatible but total deformation gradient is compatible by construction. If the body is in equilibrium in the absence of external loads, the tensional state in $\mathcal{B}_t$ is the residual stress state, so an incompatible growth field gives rise to a residual stress state resulting from elastic deformation required to maintain body continuity.

3.2 Thermoelastic materials with growth and constitutive equations

It is well known that growth and reabsorption occurs as a result of different tension stimulus [3], [2], the theory of finite growth proposed by [6] following an standard procedure established in plasticity by [7], is stated as follows:

Given an arbitrary material point X in a tensional homeostatic state[1] $\overline{\mathbf{S}}_0$ at temperature θ_0 of a body B in the stress-temperature space (7-dimensional), it is defined the existence of three open sets: S_0, S_r and S_g ($S_0 \subseteq S_r \subseteq S_g$), which are assumed simply connected and bounded by the Homeostatic, Reabsorption and Growth hyper-surfaces δS_0, δS_r and δS_g respectively. These hyper-surfaces are analogous to yield surface in plasticity [7].

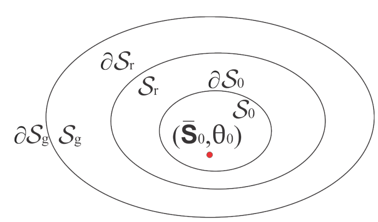

Figure 1: Sets and hyper-surfaces *Adapted from [C. Vignes and P. Papadopoulos 2010]

In S_0, material behaves as a conventional thermoelastic, so internal energy u, stress $\overline{\mathbf{S}}$ and heat flux $\mathbf{q}$ will depend only on the Green deformation tensor $\mathbf{E}$ and temperature θ

$$u = \hat{u}(\mathbf{E}, \theta), \quad \overline{\mathbf{S}} = \hat{\overline{\mathbf{S}}}(\mathbf{E}, \theta), \quad \mathbf{q}_K = \hat{\mathbf{q}}_K(\mathbf{E}, \theta, \nabla\theta) \quad \text{such that} \quad \hat{\mathbf{q}}_K(\mathbf{E}, \theta, 0) = 0 \tag{17}$$

Changes in strain and temperature initiates growth process at the activation surfaces, modifying the stress state and internal energy. Constitutive equations reflects this coupling depending on; $\mathbf{F}_g$, scalar measure of isotropic growth-induced hardening κ, symmetric second order tensor α (kinematic growth-induced hardening), ρ_g, $\mathbf{F}_g$ and θ

[1]stable state that when submitted to small perturbations a body is capable to return to it by intrinsic biological regulation

$$u = \hat{u}(\mathbf{F}, \theta, \mathbf{F}_g, \kappa, \alpha, \rho_g), \overline{\mathbf{S}} = \hat{\overline{\mathbf{S}}}(\mathbf{F}, \theta, \mathbf{F}_g, \kappa, \alpha, \rho_g), \mathbf{q}_K = \hat{\mathbf{q}_K}(\mathbf{F}, \theta, \nabla\theta, \mathbf{F}_g, \kappa, \alpha, \rho_g) \quad (18)$$

using the invariance observer principle [12], the response function of $\mathbf{S}$ results in

$$\overline{\mathbf{S}} = \hat{\overline{\mathbf{S}}}(\mathbf{E}, \theta, \mathrm{G}) \quad (19)$$

where $\mathrm{G} = (\mathbf{E}_g, \kappa, \alpha, \rho_g)$ is the set of growth variables and $\mathbf{E}_g = \frac{1}{2}(\mathbf{F}_g^T \mathbf{F}_g - \mathbf{I})$. Assuming $\overline{\mathbf{S}} = \hat{\overline{\mathbf{S}}}(\mathbf{E}, \theta, \mathrm{G})$ is invertible for fixed θ and G, deformation tensor is

$$\mathbf{E} = \hat{\mathbf{E}}(\overline{\mathbf{S}}, \theta, \mathrm{G}) \quad (20)$$

3.3 Thermodynamics second law for thermoelastic materials with growth

Thermodynamics assumptions add restrictions to constitutive equations. For elastic-thermo-plastic materials, an entropy function can be constructed and use the second law to obtain those restrictions [7], [6]. For a thermo-elastic material with growth submitted to an homothermal process for fixed G, the energy balance reduces to

$$\rho_K \dot{u} = \overline{\mathbf{S}} \cdot \dot{\mathbf{E}} + \rho_K r \quad (21)$$

Clausius-Duhem integral as a consequence of the second law is

$$\int_{t_0}^{t} \frac{r}{\theta} dt = \int_{t_0}^{t} \frac{1}{\theta}(\dot{u} - \frac{\overline{\mathbf{S}} \cdot \dot{\mathbf{E}}}{\rho_K}) dt \quad (22)$$

Defining a potencial $\eta = \hat{\eta}(\mathbf{E}, \theta, \mathrm{G})$, the entropy function is

$$\dot{\eta} = \frac{r}{\theta} = \frac{1}{\theta}(\dot{u} - \frac{\overline{\mathbf{S}} \cdot \dot{\mathbf{E}}}{\rho_K}) \quad (23)$$

For all homothermal process with fixed G, Helmholtz free energy ψ is introduced and Gibbs equation is obtained from Eq. 21

$$\psi = \hat{\psi}(\mathbf{E}, \theta, \mathrm{G}) = u - \eta\theta \qquad \rho_K \dot{\psi} = \overline{\mathbf{S}} \cdot \dot{\mathbf{E}} - \rho_K \eta \dot{\theta} \quad (24)$$

expanding the material derivatives

$$\rho_K(\frac{\partial \hat{\psi}}{\partial \theta} + \eta)\dot{\theta} + (\rho_K \frac{\partial \hat{\psi}}{\partial \mathbf{E}} - \overline{\mathbf{S}}) \cdot \dot{\mathbf{E}} = 0 \quad (25)$$

valid for all values of $\dot{\mathbf{E}}$ and $\dot{\theta}$, which are rate-independent, so the Gibbs relations are

$$\eta = \hat{\eta}(\mathbf{E}, \theta, \mathrm{G}) = -\frac{\partial \hat{\psi}}{\partial \theta}, \qquad \overline{\mathbf{S}} = \hat{\overline{\mathbf{S}}}(\mathbf{E}, \theta, \mathrm{G}) = \rho_K \frac{\partial \hat{\psi}}{\partial \mathbf{E}} \quad (26)$$

Energy balance in terms of Helmholtz free energy and entropy using Gibbs relations is

$$\rho_K \dot{\eta}\theta = \rho_K(r + \tilde{r}) - Div(\mathbf{q}_K + \tilde{\mathbf{q}}_K) - \rho_K(\frac{\partial \hat{\psi}}{\partial \mathbf{E}_g} \cdot \dot{\mathbf{E}}_g + \frac{\partial \hat{\psi}}{\partial \kappa}\dot{\kappa} + \frac{\partial \hat{\psi}}{\partial \alpha} \cdot \dot{\alpha} + \frac{\partial \hat{\psi}}{\partial \rho_g}\dot{\rho}_g) \qquad (27)$$

Integrating the above expression, and on the other hand assuming Clausius-Duhem inequality as a valid form of second law including additional entropy entering the body with new mass, manipulating and localizing yields the inequality

$$\rho_K \tilde{r} - Div\tilde{\mathbf{q}}_K - \rho_K(\frac{\partial \hat{\psi}}{\partial \mathbf{E}_g} \cdot \dot{\mathbf{E}}_g + \frac{\partial \hat{\psi}}{\partial \kappa}\dot{\kappa} + \frac{\partial \hat{\psi}}{\partial \alpha} \cdot \dot{\alpha} + \frac{\partial \hat{\psi}}{\partial \rho_g}\dot{\rho}_g) - \frac{\mathbf{q}_K \cdot \nabla\theta}{\theta} \geq 0 \qquad (28)$$

Considering an arbitrary homothermal process without irreversible heat terms

$$\frac{\partial \hat{\psi}}{\partial \mathbf{E}_g} \cdot \dot{\mathbf{E}}_g + \frac{\partial \hat{\psi}}{\partial \kappa}\dot{\kappa} + \frac{\partial \hat{\psi}}{\partial \alpha} \cdot \dot{\alpha} + \frac{\partial \hat{\psi}}{\partial \rho_g}\dot{\rho}_g \leq 0 \qquad (29)$$

the inequality is valid for all process without irreversible heat because all terms are independent of $\nabla\theta$, not irreversible heat leads to the standard heat conduction equation.

4 STRESS INDUCED FINITE GROWTH IN HUMAN TISSUES

4.1 Example of growth in cardiovascular system

In this section is developed an example that is associated with adaptive heart growth due to abnormal loads, know as cardiac hypertrophy [5], [11], [15]

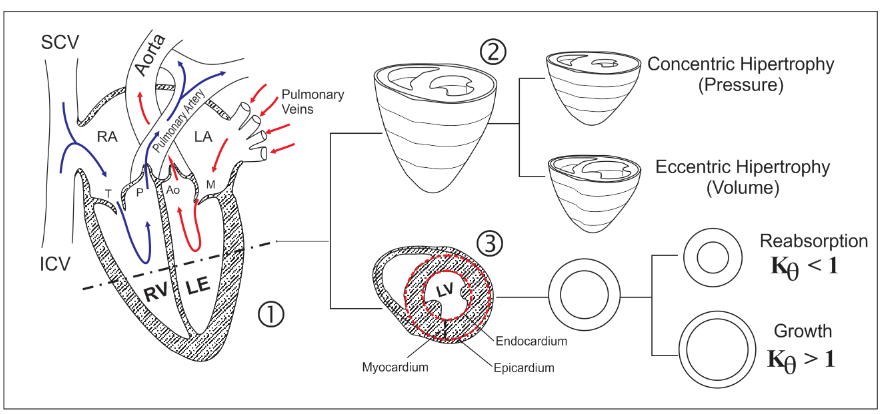

Figure 2: Heart Scheme 1) Frontal view 2) Isometric, ventricular enlargement due to volume overload or high filling pressure 3) Left ventricle and [K_θ] describing growth and reabsorption *(1) Adapted from [Taber L A 1995], ICV, SCV inferior, superior cava veins; RA, LA Right, Left Atriums; RV, LV Right, Left Ventricles; T, P, Ao and M are tricuspid, pulmonary, aortic and mitral valves

In general terms, blood deoxygenated comes from the body through cava veins and fills the right atrium, which contracts sending blood to the right ventricle which contracts and sends blood to the lungs through pulmonary artery. Blood oxygenated back to heart through pulmonary veins filling left atrium which contracts and sends blood to left ventricle that when contracts, sends blood to the circulatory system through the aorta as schematically represented in Fig. 2.1, more details can be founded in [15].

A growth displacement field is specified in an unloaded cylindrical tube ([LV] considered incompressible, elastic and isotropic) and residual stress resulting from growth fields show how circumferential growth give rise to residual stresses that would cause the cylinder to change shape when cut as the opening angle experimental tests [10], [5]. The point P in $\mathcal{B}$ has coordinates $(\mathrm{R}, \Theta, \mathrm{Z})$, $\mathbf{F}_g$ maps $\mathcal{B}$ in $\mathcal{B}_g$ where P has coordinates (ρ, φ, ξ) as shown in Fig. 3.a. The term $\mathrm{K}_\theta(\mathrm{R})$ is the circumferential growth stretch ratio which depends on the radius and is assumed constant, when $\mathrm{K}_\theta > 1$ growth occurs and when $\mathrm{K}_\theta < 1$ reabsorption occurs (Fig. 3. b, c)

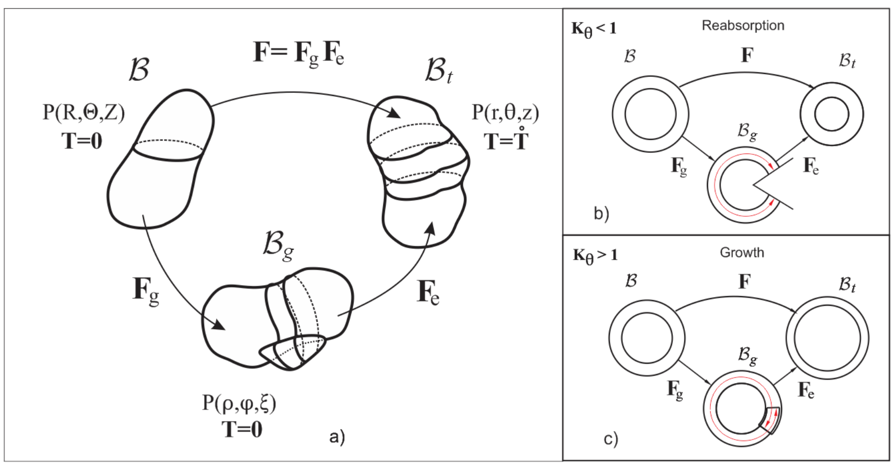

Figure 3: a) Multiplicative decomposition of growth deformation gradient b), c) Cylindrical models of LV after uniform circumferential growth. *b), c) Adapted from [Rodriguez *et. al* 1994]

To obtain growth deformation gradient the following displacement field is prescribed:

$$\rho = \mathrm{R}, \quad \varphi = \mathrm{K}_\theta(\mathrm{R})\Theta, \quad \xi = \mathrm{Z} \tag{30}$$

and the field that maps $\mathcal{B}_g$ in $\mathcal{B}_t$ where P has coordinates $(\mathrm{r}, \theta, \mathrm{z})$ is

$$\mathrm{r} = \mathrm{r}(\rho), \quad \theta = \eta_\theta(\rho)\varphi, \quad \mathrm{z} = \epsilon \mathrm{z} \tag{31}$$

then, $\mathbf{F}_g$ and $\mathbf{F}_e$ in cylindrical coordinates results in

$$\mathbf{F}_g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{\rho}{\mathrm{R}}\mathrm{K}_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{F}_e = \begin{bmatrix} \dfrac{d\mathrm{r}(\rho)}{d\rho} & 0 & 0 \\ 0 & \dfrac{\mathrm{r}}{\rho}\eta_\theta & 0 \\ 0 & 0 & \epsilon \end{bmatrix} \tag{32}$$

$\eta_\theta(\rho)$ allows total deformation gradient to be compatible. For K_θ constant, $\eta_\theta(\rho) = 1/\mathrm{K}_\theta$ according to [5]. Incompressibility constraint is only applied $\mathbf{F}_e$, so the third invariant of right Cauchy-Green tensor is unity:

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \begin{bmatrix} \left(\dfrac{d\mathrm{r}(\rho)}{d\rho}\right)^2 & 0 & 0 \\ 0 & \left(\dfrac{\mathrm{r}}{\rho}\eta_\theta\right)^2 & 0 \\ 0 & 0 & (\epsilon)^2 \end{bmatrix} \Leftrightarrow \mathbf{I}_3 = \left[\frac{d\mathrm{r}(\rho)}{d\rho}\frac{\mathrm{r}}{\rho}\eta_\theta\epsilon\right]^2 = 1 \tag{33}$$

which can be integrated to obtain an expression for the growth radius r resulting in

$$\mathrm{r} = \sqrt{\frac{\mathrm{R}^2\mathrm{K}_\theta}{\epsilon} + \mathrm{C}_2} \qquad \text{where} \quad \mathrm{C}_2 \quad \text{is an integration constant} \tag{34}$$

Green deformation tensor components referred to $\mathcal{B}_g$ coordinates are calculated from

$$\mathbf{E} = \frac{1}{2}\left[\mathbf{F}_e{}^{\mathrm{T}}\mathbf{F}_e - \mathrm{I}\right] = \begin{bmatrix} \left(\dfrac{d\mathrm{r}(\rho)}{d\rho}\right)^2 - 1 & 0 & 0 \\ 0 & \left(\dfrac{\mathrm{r}}{\rho}\eta_\theta\right)^2 - 1 & 0 \\ 0 & 0 & (\epsilon)^2 - 1 \end{bmatrix} \tag{35}$$

then

$$\mathbf{E}_{\rho\rho} = \frac{1}{2}\left[\left(\frac{\mathrm{RK}_\theta}{\mathrm{r}\epsilon}\right)^2 - 1\right]; \mathbf{E}_{\varphi\varphi} = \frac{1}{2}\left[\left(\frac{\mathrm{r}}{\mathrm{RK}_\theta}\right)^2 - 1\right]; \mathbf{E}_{\xi\xi} = \frac{1}{2}\left[(\epsilon)^2 - 1\right] \tag{36}$$

It is assumed the stress-strain relationship used in [5] and [11]

$$\mathbf{T}_{ij} = \frac{1}{2}\mathbf{F}_{iS}\mathbf{F}_{jT}\left(\frac{\partial \mathrm{W}}{\partial \mathbf{E}_{ST}} + \frac{\partial \mathrm{W}}{\partial \mathbf{E}_{TS}}\right) - \mathrm{p}\delta_{ij} \tag{37}$$

which takes into account the strain energy function proposed by [10]:

$$\mathrm{W} = \frac{\mathrm{C}}{2}\left(\mathrm{e}^{\mathrm{Q}} - 1\right) \qquad \mathrm{Q} = 2\mathrm{b}_1\left(\mathbf{E}_{\rho\rho} + \mathbf{E}_{\varphi\varphi} + \mathbf{E}_{\xi\xi}\right) \tag{38}$$

being C material constant, Q a function of principal deformation components that define material symmetry (isotropic case), b_1 material constant (from [11], b_1=4.24 and C=0.765 kPa), p hydrostatic pressure. Then, from 32.2, 34, 36 and 38 results in

$$\mathbf{T}_{rr} = \frac{\left(\mathrm{r}^2 - \mathrm{C}_2\right)\mathrm{K}_\theta}{2\mathrm{r}^2}\left[\mathrm{Cb}_1\mathrm{e}^{\mathrm{b}_1\left[\frac{\left(\mathrm{r}^2-\mathrm{C}_2\right)\mathrm{K}_\theta}{\mathrm{r}^2} + \frac{\mathrm{r}^2}{\left(\mathrm{r}^2-\mathrm{C}_2\right)\mathrm{K}_\theta} - 2\right]}\right] - \mathrm{p}(r) \tag{39}$$

$$\mathbf{T}_{\theta\theta} = \frac{\mathrm{r}^2}{2\left(\mathrm{r}^2 - \mathrm{C}_2\right)\mathrm{K}_\theta}\left[\mathrm{Cb}_1 \mathrm{e}^{\mathrm{b}_1\left[\frac{\left(\mathrm{r}^2-\mathrm{C}_2\right)\mathrm{K}_\theta}{\mathrm{r}^2} + \frac{\mathrm{r}^2}{\left(\mathrm{r}^2-\mathrm{C}_2\right)\mathrm{K}_\theta} - 2\right]}\right] - \mathrm{p}(r) \tag{40}$$

considering: $f = \mathrm{K}_\theta\left(1 - \frac{\mathrm{C}_2}{\mathrm{r}^2}\right)$ gives

$$\mathbf{T}_{rr} = \frac{f}{2}\left[\mathrm{Cb}_1 \mathrm{e}^{\mathrm{b}_1\left[\frac{(f-1)^2}{f}\right]}\right] - \mathrm{p}(r); \quad \mathbf{T}_{\theta\theta} = \frac{1}{2f}\left[\mathrm{Cb}_1 \mathrm{e}^{\mathrm{b}_1\left[\frac{(f-1)^2}{f}\right]}\right] - \mathrm{p}(r) \tag{41}$$

from equilibrium equations

$$\frac{d\mathbf{T}_{rr}}{d\mathrm{r}} + \frac{\mathbf{T}_{rr} - \mathbf{T}_{\theta\theta}}{\mathrm{r}} = 0; \quad \frac{d\mathbf{T}_{r\theta}}{d\mathrm{r}} + 2\frac{\mathbf{T}_{r\theta}}{\mathrm{r}} = 0; \quad \frac{d\mathbf{T}_{rz}}{d\mathrm{r}} + \frac{\mathbf{T}_{rz}}{\mathrm{r}} = 0 \tag{42}$$

Since zero-stress boundary conditions are assumed on the inner and outer walls, just the first of the above equations has to be solved. Integrating and using 41,

$$\mathbf{T}_{rr} = \int_{r_2}^{r} \frac{\mathbf{T}_{\theta\theta} - \mathbf{T}_{rr}}{\mathrm{r}} d\mathrm{r} + \mathbf{T}_{rr}|_{r=r_2} = \int_{r_2}^{r} \frac{1}{2}\left(\frac{1}{f} - f\right)\left[\mathrm{Cb}_1 \mathrm{e}^{\mathrm{b}_1\left[\frac{(f-1)^2}{f}\right]}\right] d\mathrm{r} \tag{43}$$

$\mathbf{T}_{rr}$ in $r = r_2$ is radial stress at outer grown wall specified as zero, internal pressure is $-\mathbf{T}_{rr}$ in $r = r_1$. Model was solved numerically using MathCAD version 15.0 because not has analytical solution, by specifying growth outer radius r_2 and solving for final inner radius value that gives a zero transmural pressure. Processing scheme: an initial value of external radius r_2 is specified. With Eq. 34, C_2 and r_1 are obtained. Then, roots of $\mathbf{T}_{rr}$ are calculated to obtain new values of r_2 and r_1, which enter again in the scheme to calculate new values of C_2, r_2 and r_1, the scheme loops until the difference $(r_{2i+1} - r_{2i})$ approaches zero (tolerance 10^{-12}) Results for different values of K_θ shown on Fig. 4.

Considering $\epsilon = 1$ and initial values of internal and external radius of 2 cm and 3 cm respectively, three values of K_θ for reabsorption were analyzed: 0.9, 0.85 and 0.8. The radius values, internal and external, satisfying the equilibrium were: 1.755 and 2.753 cm, 1.633 and 2.63 cm and 1.512 and 2.507 cm (Fig. 4.a) Residual stress shown zero values of radial stress $\mathbf{T}_{rr}$ at inner (endocardium) and outer (epicardium) walls since the cylinder was unloaded. Circunferencial stress $\mathbf{T}_{\theta\theta}$ shows nonlinear behavior from compression at the endocardium to tension at the epicardium.

Also, three values of K_θ for growth were analyzed: 1.1, 1.15 and 1.2 where the grown radius values, internals and externals, satisfying the equilibrium were: 2.247 and 3.248 cm, 2.37 and 3.372 cm and 2.49 and 3.496 cm (Fig. 4.b) As in reabsorption, residual stress shown zero values of radial stress $\mathbf{T}_{rr}$ at inner and outer walls since the cylinder was unloaded, but stress gradient were reversed from those of reabsorption and circunferencial stress $\mathbf{T}_{\theta\theta}$ were compressive at the epicardium and tensile at the endocardium. Finally, with this study and motivated by previous works of the authors in hip prostheses biomechanics and in dental biomechanics [13], [14] this work can be considered as a

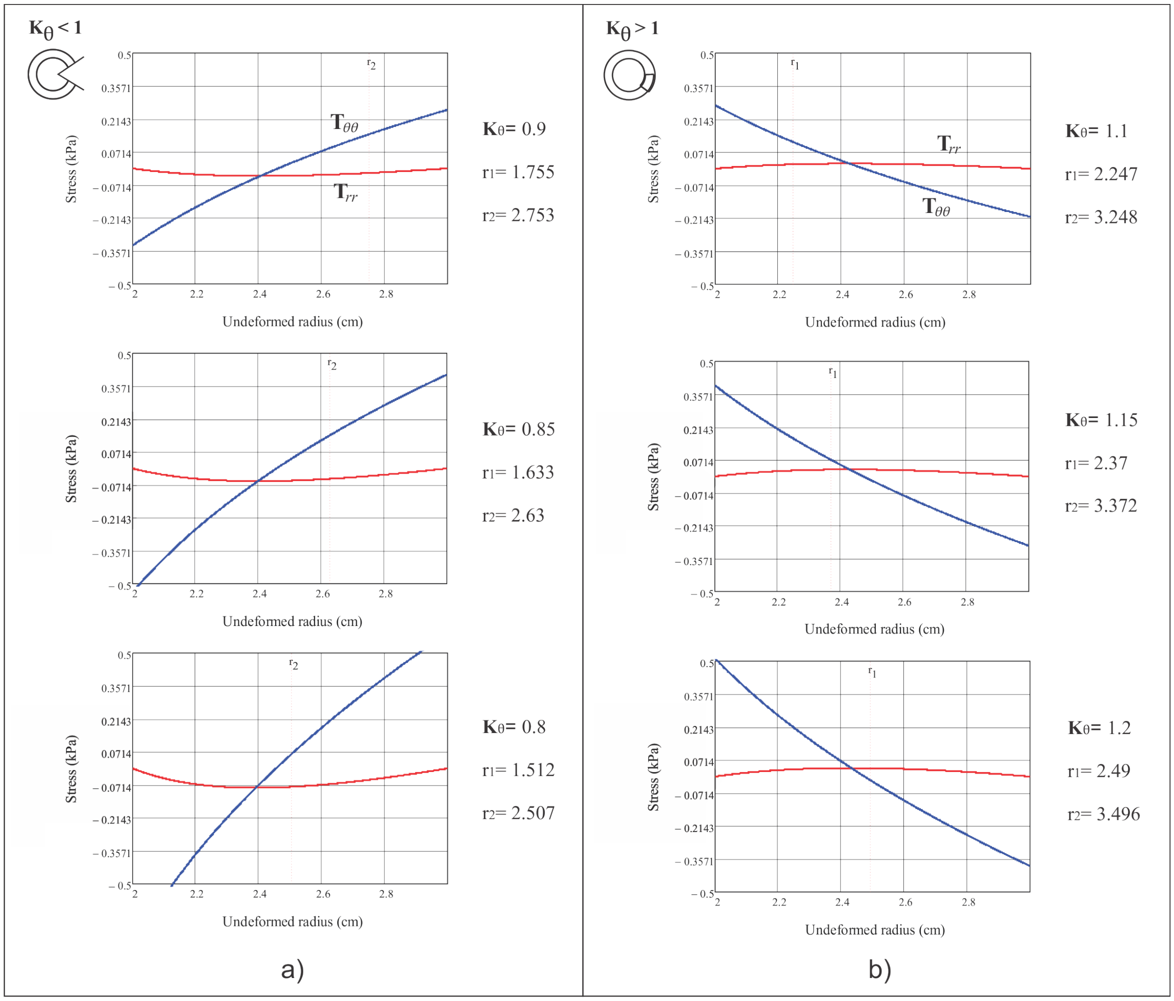

Figure 4: Radial and circunferencial stresses for K_θ values; a) $K_\theta < 1$ Reabsorption b) $K_\theta > 1$ Growth

first step for future works in the implementation of material models using FEM with the capability to grow or reabsorb as a response of different stress levels in order to predict zones of bone formation associated to growth and bone mass loose zones associated to reabsorption.

5 CONCLUSIONS

- It was presented an study and a revision of the theory for thermoelastic materials with growth and its treatment in the context of continuum mechanics.

- An example of finite adaptive heart growth due to abnormal loads was analyzed, and results obtained were discussed. Also, the results obtained matched with previous results presented by other authors showing agreement.

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