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PROPAGATION OF WAVES IN INFINITE BEAMS: PML APPROACH

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Abstract. The problem of transverse vibration of an infinite beam on elastic foundation involves the elastodynamic wave equation over an unbounded domain. This paper deals with such a semi-infinite Bernoulli-Euler beam subjected to a harmonically varying concentrated load at its origin. A common application of this beam is in the investigation of railroad tracks. Since modelling of unbounded domains with radiation damping requires absorption of the waves, the Perfectly Matched Layer (PML) approach, which provides an appropriate solution technique for such problems, is employed here. This method has been widely used for electromagnetic wave problems. It has also been utilized for some dynamic soil-structure-interactions as well as for simulation of earthquake ground motions.

For the general case of infinite domains with inhomogeneous media application of the initial and boundary conditions is difficult to effect in closed form. In addition, irregularities in geometry, material, and viscoelasticity of the foundation will necessitate a numerical procedure. The PML technique along with finite element discretization can offer a suitable means of dealing with such problems. In this manner the response is carried out by dividing the domain into two segments; a bounded portion and an artificial, wave absorbing one. This framework is implemented with a displacement-based Finite Element Method (FEM) in the context of a Galerkin scheme. The first step in the PML process is to transform the governing equations into frequency domain. In this domain the spatial variable is stretched, in complex coordinates, resulting in damping of the outgoing waves. The decaying function used for this purpose is selected in such a manner as to prevent reflecting waves.

Most PML applications so far have dealt with lower order governing differential equations. The case of beams on elastic foundation, considered here, involves a fourth-order equation. The process starts out by first reducing the fourth order equation into four first-order equations. The auxiliary variables thus introduced lead to internal moments and shear forces. The governing equations are then transformed back into the time domain. A weak form of the governing equations is spatially discretized by the use of standard FEM. This leads to the final form of the governing equations of the system. The latter equations are solved by a step-by-step algorithm,

carried out in MATLAB environment. The efficiency and accuracy of the results obtained by the method described here are validated by comparing the results to existing regular numerical solutions for some simple problems. Parametric studies are conducted on the effect of various PML parameters and of the stretching functions in order to establish the best form of these parameters.

1 INTRODUCTION

Simulation of wave propagation over unbounded domains is of importance in a variety of fields of science and engineering, ranging from quantum mechanics [1] and electromagnetic waves [2], to seismology [3] and soil-structure interaction [4, 5]. In such problems, the spatial domain is often very large compared to the characteristic dimension, the wavelength. So, the solution of the wave equation requires imposition of a radiation condition in any unbounded direction: waves should radiate outwards from a source toward an unbounded direction, without any spurious wave motion in the reverse direction.

In numerical simulations when domain discretization methods are used, especially in the presence of material heterogeneity or geometric irregularities, large spatial domains must be reduced to a finite one. This domain reduction is typically accomplished through a geometric truncation of the physically bounded domain, along with an artificial boundary that absorbs the outgoing waves, for the modelling of the unbounded domain. In 1994, the problem of absorbing boundaries for wave equations was transformed in a seminal paper by Berenger [6]. An absorbing boundary layer is a layer of artificial absorbing material that is placed adjacent to the edges of the grid, completely independent of the boundary condition. When a wave enters the absorbing layer, it is attenuated by absorption and decays exponentially. Even if it reflects off the boundary the returning wave after one round trip through the absorbing layer is reduced exponentially. The problem with this approach is that, whenever there is a transition from one material to another, waves generally reflect. The transition from non-absorbing to absorbing material is no exception. So, instead of having reflections from the grid boundary, the reflections are from the absorbing boundary. Berenger has shown that a special absorbing medium can be constructed so that waves do not reflect at the interface. This so called perfectly matched layer, or PML was originally derived for electromagnetism (Maxwell's equations). But the same idea is applicable to other wave equations. PML has been formulated for other linear wave equations, such as the scalar wave equation or Helmholtz' equation [7], linearized Euler equations [8], wave equation for poroelastic media [9], and elastodynamic wave equation [10]. Basu and Chopra [11] developed the PML technique in conjunction with a displacement based finite element technique for one-dimensional rods on elastic foundations. Kang and Kallivokas [12] used a displacement-stress, mixed finite element formulation for semi-discrete forms resulting from PML equations for a 1-D rod. The latter two studies deal with second-order governing differential equations.

The present study, however, applies the PML approach to semi-infinite beams on elastic foundations, involving fourth-order differential equations. The dynamical problem of infinite beams on elastic foundations is an important topic frequently encountered in airport runways as well as highway and railway engineering. Early researchers attempted at closed form solutions of such problems. Timoshenko [13], Kenney [14], Hetenyi [15] and Sun [16] are among the many early researchers who tried to obtain analytic solutions for specific cases. Later researchers have tried to use more general solution techniques, such as finite elements [17]. Due to the infinite domain nature of such problems an attractive solution process is provided by the PML procedure.

2 FORMULATION

The governing equation of motion of an infinite beam on elastic foundation, Fig. 1(a), can be derived by invoking the Bernoulli-Euler beam theory for the free body diagram, Fig. 1(b). The magnitude of the continuously distributed support reaction is proportional to the deflection of the beam. That is, the elastic foundation yields a resistive force $q(x,t) = -k_s v$.

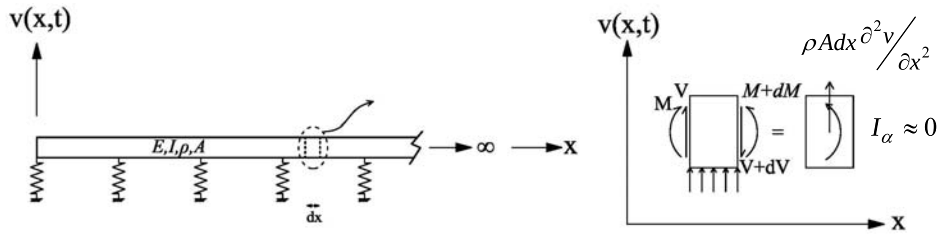

Fig. 1 (a) Semi–infinite beam on elastic foundation under transverse motion
(b) Free body diagram of the beam element

Using equilibrium of forces and moments for the element in Fig. 1 (b) and invoking the moment curvature relation we get the equation of motion as:

$$EI\frac{\partial^4 v}{\partial x^4} + k_s v + \rho A\frac{\partial^2 v}{\partial t^2} = p \tag{1}$$

In this equation E is Young's modulus, I moment of inertia of the beam, A its cross-sectional area, ρ its density and k_s stiffness of the foundation.

3 PML PROCESS

The PML approach has been used for longitudinal waves. Its application to the beam problem is a simple extension of the latter. Considering the time-harmonic displacement function of the form $v(x,t) = \hat{v}(x)\exp(i\omega t)$, Eq. (1) can be transformed to frequency domain resulting in:

$$EI\frac{d^4\hat{v}}{dx^4} + k_s\hat{v} - \omega^2\rho A\hat{v} = \hat{p} \tag{2}$$

Here $\hat{v}$ is Fourier transform of the subtended function v which is in terms of x, and x is the coordinate of the beam within a semi-infinite domain, $0 \le x < \infty$. The Fourier transform is taken with respect to t, leading to $\hat{v}$ which is a function of x only. Eq. (2) can be modified by stretching the spatial variable to complex coordinates. To this end we divide the domain into two segments; a finite domain of interest L_{BD}, and an artificial truncated domain or PML of finite length L_{PML}, Fig. 2.

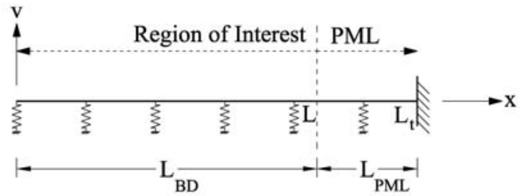

Fig. 2 Truncated semi-infinite domain with a PML segment

The solution of the semi-infinite beam has both propagating and non-propagating terms. To decay this solution within the PML and prevent wave reflections, Eq. (2) is recast by stretching the real coordinate x to $\tilde{x}$ with the aid of mapping:

$$\tilde{x} := \int_0^x \lambda(s)ds \tag{3}$$

In Eq. (3) λ is a nowhere-zero, continuous complex-valued coordinate stretching function;

$$\lambda(x) = f_1(x) + \frac{1}{i\omega} f_2(x) \qquad x \in [0 \quad L_t] \tag{4}$$

$f_1(x)=1+f^e(x)$ and $f_2(x)=\sqrt{\omega c/r}\, f^p(x)$. $c=\sqrt{EI/\rho A}$ is the wave pseudo-velocity and $r=\sqrt{EI/k_s}$ is the characteristic length of the beam. The so called evanescent and propagating attenuation functions $f^e(x)$ and $f^p(x)$ are functions that must be chosen so as they will attenuate the solution of the propagating and non-propagating terms simultaneously and eliminate the wave reflections. Here they are chosen as polynomials of the form:

$$f^e(x) = f^p(x) = \begin{cases} 0 & : \quad x \in [0 \quad L_{BD}] \\ f_0\left(\dfrac{x-L_{BD}}{L_{PML}}\right)^m & : \quad x \in [L_{BD} \quad L_t] \end{cases} \tag{5}$$

Equation (2) can be rewritten in terms of the stretched coordinate $\tilde{x}$, i.e., by replacing x with $\tilde{x}$. Thus the governing equation of the perfectly matched medium (PMM) becomes:

$$EI\frac{d^4\hat{v}}{d\tilde{x}^4} + k_s\hat{v} - \omega^2\rho A\hat{v} = \hat{p} \tag{6}$$

Since it is more convenient to solve the problem in terms of the real variable x, the coordinate is transformed back from the complex coordinate $\tilde{x}$. Assuming continuity of $\lambda(x)$ and invoking the chain rule of differentiation results in:

$$\frac{d}{d\tilde{x}} = \frac{1}{\lambda}\frac{d}{dx} \tag{7}$$

To proceed with the solution of the governing equation, the fourth-order differential equation is first cast into four first-order equations by introducing the following auxiliary variables

$$\hat{\phi}=\frac{1}{\lambda}\frac{d\hat{v}}{dx},\quad \hat{\psi}=\frac{1}{\lambda}\frac{d\hat{\phi}}{dx},\quad \hat{q}=\frac{1}{\lambda}\frac{d\hat{\psi}}{dx} \tag{8}$$

Here rotation ϕ, curvature $\psi = M/EI$ and $q = V/EI$ are in terms of x. The differential operator of Eq. (6) can be rewritten in terms of x using Eq.(7):

$$\frac{d^4}{d\tilde{x}^4}=\frac{1}{\lambda}\frac{d}{dx}\left[\frac{1}{\lambda}\frac{d}{dx}\left[\frac{1}{\lambda}\frac{d}{dx}\left(\frac{1}{\lambda}\frac{d}{dx}\right)\right]\right] \tag{9}$$

Similarly Eq. (6) can be recast in the frequency domain by substituting the auxiliary variables into Eq. (9):

$$EI\frac{1}{\lambda}\frac{d\hat{q}}{dx}+k_s\hat{v}-\omega^2\rho A\hat{v}=\hat{p} \tag{10}$$

Multiplying both sides of Eq. (10) by $\lambda(x)$, as defined in Eq.(4), the following equation will result:

$$EI\frac{d\hat{q}}{dx}-\omega^2\rho Af_1\hat{v}+i\omega\rho Af_2\hat{v}+k_s f_1\hat{v}+\frac{1}{i\omega}k_s f_2\hat{v}=0 \tag{11}$$

PMM equations (8) and (11) are in frequency domain and in terms of x and the auxiliary variables.

4 THE GOVERNING EQUATION IN TIME DOMAIN

By inverting Eq. (8) back in time domain we have:

$$\begin{aligned}
\frac{\partial^2 v}{\partial x\partial t}&=f_1\frac{\partial\phi}{\partial t}+f_2\phi\\
\frac{\partial^2 \phi}{\partial x\partial t}&=f_1\frac{\partial\psi}{\partial t}+f_2\psi\\
\frac{\partial\psi}{\partial x}&=f_1 q+f_2\overline{q}\rightarrow q=\frac{1}{f_1}\frac{\partial\psi}{\partial x}-\frac{f_2}{f_1}\overline{q}
\end{aligned} \tag{12}$$

In this equation

$$\overline{q}=\int_0^t q(\tau)d\tau \tag{13}$$

Similarly inverse Fourier Transform [28] of Eq. (11) yields the time domain form of that equation as:

$$EI\frac{\partial q}{\partial x}+\rho Af_1\frac{\partial^2 v}{\partial t^2}+\rho Af_2\frac{\partial v}{\partial t}+k_s f_1 v+k_s f_2\overline{v}=0 \tag{14}$$

In this equation

$$\bar{v} = \int_0^t v(\tau) d\tau \tag{15}$$

Thus, a set of four time domain equations (12.a,b,c) and (14) with PML absorbing boundaries in the x direction can be solved by appropriate discretization schemes. Equation (14) is implemented using the standard displacement-based finite element method and the pertinent matrices for the elements are generated by the Galerkin methods. A weak form of Eq. (14) is derived by multiplying it by an arbitrary weighting function w residing in an appropriate admissible space. Upon integration by parts over the entire computational domain Ω gives:

$$\begin{aligned}&\int_\Omega w\rho A f_1 \ddot{v} dx + \int_\Omega w\rho A f_2 \dot{v} dx + \int_\Omega w k_s f_1 v dx + \\ &\int_\Omega w k_s f_2 \bar{v} dx + \left[wEIq \right]_{\partial\Omega} = \int_\Omega w_{,x} EIq dx\end{aligned} \tag{16}$$

Another integration-by-parts and invoking (12.c) yields:

$$\begin{aligned}&\int_\Omega w\rho A f_1 \ddot{v} dx + \int_\Omega w\rho A f_2 \dot{v} dx + \int_\Omega w k_s f_1 v dx + \int_\Omega w k_s f_2 \bar{v} dx + \\ &\int_\Omega [w_{,x} EI \frac{1}{f_1}]_{,x} \psi dx + \int_\Omega w_{,x} EI \frac{f_2}{f_1} \bar{q} dx = [w_{,x} EI \frac{1}{f_1} \psi]_{\partial\Omega} - [wEIq]_{\partial\Omega}\end{aligned} \tag{17}$$

5 FINITE ELEMENT IMPLEMENTATION

The weak form is spatially discretized by interpolation of v and w element-wise in terms of nodal quantities using appropriate nodal shape functions. Because for fourth-order differential equations the essential boundary conditions are the values of its first order derivatives, for a beam under flexure transverse displacement, v and rotation θ are the essential boundary conditions. Thus, these two parameters are chosen as nodal quantities leading to a simple two-node beam element with Hermitian shape functions. This leads to the system of equations

$$\left[m \right] \ddot{\underset{\sim}{d}} + \left[c \right] \dot{\underset{\sim}{d}} + \left[k \right] \underset{\sim}{d} + \left[k' \right] \bar{\underset{\sim}{d}} + \underset{\sim}{f}_{\text{int}} = \underset{\sim}{f}_{ext} \tag{18}$$

With

$$\bar{d}(t) = \int_0^t d(\tau) d\tau \tag{19}$$

Where $\underset{\sim}{d}$ is a vector of nodal quantities. Pertinent matrices, internal and external vectors, in the context of FEM, are assembled from the corresponding element-level matrices and vectors:

$$
\begin{aligned}
&[m]^e = \int_{\Omega^e} \underset{\sim}{N}^T \rho A f_1 \underset{\sim}{N} d\Omega && [c]^e = \int_{\Omega^e} \underset{\sim}{N}^T \rho A f_2 \underset{\sim}{N} d\Omega \\
&[k]^e = \int_{\Omega^e} \underset{\sim}{N}^T k_s f_1 \underset{\sim}{N} d\Omega && [k']^e = \int_{\Omega^e} \underset{\sim}{N}^T k_s f_2 \underset{\sim}{N} d\Omega \\
&\underset{\sim}{f}^e_{ext} = \left[\underset{\sim}{N}^T_{,x} M \right]_{\partial\Omega} - \left[\underset{\sim}{N}^T V \right]_{\partial\Omega} && \underset{\sim}{f}^e_{\text{int}} = \underset{\sim}{f}^e_{\text{int},\psi} + \underset{\sim}{f}^e_{\text{int},\bar{q}} \\
&\underset{\sim}{f}^e_{\text{int},\psi} = \int_{\Omega^e} \left[\underset{\sim}{N}^T_{,x} EI \frac{1}{f_1} \right]_{,x} \psi d\Omega && \underset{\sim}{f}^e_{\text{int},\bar{q}} = \int_{\Omega^e} \underset{\sim}{N}^T_{,x} EI \frac{f_2}{f_1} \bar{q} d\Omega
\end{aligned}
\tag{20}
$$

Here $\underset{\sim}{N}$ is a row vector of element-level nodal shape functions, M vector of external moments, V that of shear forces. $\underset{\sim}{f}_{\text{int}}$ is the vector of internal moments $\underset{\sim}{f}^e_{\text{int},\psi}$ plus that of internal shear forces $\underset{\sim}{f}^e_{\text{int},\bar{q}}$. Eq. (19) can be approximated as

$$
\bar{\underset{\sim}{d}}_{n+1} = \bar{\underset{\sim}{d}}_n + \underset{\sim}{d}_{n+1} \Delta t \tag{21}
$$

Therefore, the terms involving $\bar{\underset{\sim}{d}}$ in Eq. (18) may be linearized as

$$
\Delta\left([k'] \bar{\underset{\sim}{d}}_{n+1} \right) = \left([k'].\Delta t \right) \Delta \underset{\sim}{d}_n \tag{22}
$$

Expressing the functions f_1 and f_2 in terms of the natural coordinate ζ rather than x, and the shape functions will further simplify the process of obtaining elemental matrices. To this end the chain rule of differentiation can be invoked replacing x by ζ :

$$
\frac{d^n}{dx^n} \underset{\sim}{N}(x) = \left(\frac{2}{L} \right)^n \frac{d}{d\zeta^n} \underset{\sim}{N}(\zeta) \tag{23}
$$

6 SOLUTION PROCESS

Step-by-step numerical integration schemes, such as the Newmark β can be used along with Newton-Raphson iteration at each time step to enforce the equilibrium and solve the semi-discretized system of Eq. (18). In this way having the solution of Eq. (18) at time t_{n+1} the solution at t_n is obtained by the Newton-Raphson iteration process. This requires a) calculation of ψ_{n+1} and $\bar{q}_{n+1}$, to find $\left\{ \underset{\sim}{f}^e_{\text{int},\psi} \right\}_{n+1}$ and $\left\{ \underset{\sim}{f}^e_{\text{int},\bar{q}} \right\}_{n+1}$, and b) a consistent linearization [18] of $\left\{ \underset{\sim}{f}^e_{\text{int}} \right\}_{n+1}$ at $\underset{\sim}{d}^e_{n+1}$. This is because ψ and $\bar{q}$ are, Eq. (12), are in relation to displacement and internal moments and shear forces. Therefore, Eq. (12.a) is discretized using a backward Euler scheme on ϕ to obtain:

$$
\phi_{n+1} = \frac{1}{Dr(x)} \left[\underset{\sim}{N}_{,x} . \dot{\underset{\sim}{d}}_{n+1} + Nr(x).\phi_n \right] \tag{24}
$$

Here $Dr(x) = \frac{f_1}{\Delta t} + f_2$, $Nr(x) = \frac{f_1}{\Delta t}$ and Δt is the time-step. With the initial condition $\phi_1 = 0$, adding the terms of Eq. (24) together at each time step, the sequence of ϕ_{n+1} can be written in the form of a series:

$$\phi_{n+1} = \sum_{i=1}^{n+1} \frac{Nr^{(i-1)}}{Dr^{(i)}} \underset{\sim}{N}_{,x} . \dot{\underset{\sim}{d}}_{n+2-i} \tag{25}$$

Eq. (12.b) is then discretized in the same fashion on ψ to obtain its series form:

$$\psi_{n+1} = \sum_{i=1}^{n+1} (i) \frac{Nr^{(i-1)}}{Dr^{(i+1)}} \underset{\sim}{N}_{,xx} . \ddot{\underset{\sim}{d}}_{n+2-i} + (\frac{i}{2}) \frac{\partial}{\partial x} \left[\frac{Nr^{(i-1)}}{Dr^{(i+1)}} \right] \underset{\sim}{N}_{,x} . \ddot{\underset{\sim}{d}}_{n+2-i} \tag{26}$$

The time-discrete form of element vector of internal moments is:

$$\left\{ \underset{\sim}{f}^{e}_{\text{int},\psi} \right\}_{n+1} = \int_{\Omega^e} \left[\underset{\sim}{N}^{T}_{,x} EI \frac{1}{f_1} \right]_{,x} \psi_{n+1} d\Omega \tag{27}$$

Linearization of Eq. (27) leads to:

$$\left\{ \Delta \underset{\sim}{f}^{e}_{\text{int},\psi} \right\}_{n} = \left[\int_{\Omega^e} \left(\underset{\sim}{N}^{T}_{,xx} D_{\psi 1} \underset{\sim}{N}_{,xx} - \underset{\sim}{N}^{T}_{,x} D_{\psi 2} \underset{\sim}{N}_{,xx} + \underset{\sim}{N}^{T}_{,xx} D_{\psi 3} \underset{\sim}{N}_{,x} - \underset{\sim}{N}^{T}_{,x} D_{\psi 4} \underset{\sim}{N}_{,x} \right) d\Omega \right] \Delta \ddot{\underset{\sim}{d}}_{n} \tag{28}$$

Where Δ is a differential operator, and

$$D_{\psi 1} = EI \frac{1}{Dr^2} \frac{1}{f_1} \qquad D_{\psi 2} = EI \frac{1}{Dr^2} \frac{d}{dx} \left(\frac{1}{f_1} \right)$$
$$D_{\psi 3} = EI \frac{1}{2} \frac{d}{dx} \left(\frac{1}{Dr^2} \right) \frac{1}{f_1} \qquad D_{\psi 4} = EI \frac{1}{2} \frac{d}{dx} \left(\frac{1}{Dr^2} \right) \frac{d}{dx} \left(\frac{1}{f_1} \right) \tag{29}$$

This linearization leads to a tangent stiffness matrix:

$$\left[m \right]^{e}_{\psi} = \int_{\Omega^e} \left(\underset{\sim}{N}^{T}_{,xx} D_{\psi 1} \underset{\sim}{N}_{,xx} - \underset{\sim}{N}^{T}_{,x} D_{\psi 2} \underset{\sim}{N}_{,xx} + \underset{\sim}{N}^{T}_{,xx} D_{\psi 3} \underset{\sim}{N}_{,x} - \underset{\sim}{N}^{T}_{,x} D_{\psi 4} \underset{\sim}{N}_{,x} \right) d\Omega \tag{30}$$

The latter equation is incorporated in the effective tangent stiffness used in the time-stepping algorithm.

Derivation of $\overline{q}_{n+1}$ and internal shear force $\left\{ \underset{\sim}{f}^{e}_{\text{int},\overline{q}} \right\}_{n+1}$ follow the same pattern. By invoking Eq. (12.c) a recursive expression results in discrete form:

$$\overline{q}_{n+1} = \frac{1}{Dr} \frac{d}{dx} [\psi_{n+1}] + \frac{Nr}{Dr} \overline{q}_{n} \tag{31}$$

$\frac{d}{dx}[\psi_{n+1}]$ is calculated from Eq. (26) as:

$$\frac{d}{dx}[\psi_{n+1}] = \sum_{i=1}^{n+1} (i) \frac{Nr^{(i-1)}}{Dr^{(i+1)}} \underset{\sim}{N}_{,xxx} . \ddot{\underset{\sim}{d}}_{n+2-i} + \left(\frac{3i}{2} \right) \frac{d}{dx} \left[\frac{Nr^{(i-1)}}{Dr^{(i+1)}} \right] \underset{\sim}{N}_{,xx} . \ddot{\underset{\sim}{d}}_{n+2-i} + \left(\frac{i}{2} \right) \frac{d^2}{dx^2} \left[\frac{Nr^{(i-1)}}{Dr^{(i+1)}} \right] \underset{\sim}{N}_{,x} . \ddot{\underset{\sim}{d}}_{n+2-i} \tag{32}$$

In the computer program developed, $\bar{q}_{n+1}$ is computed at each time-step but is dependent on the solution of all past time steps. The discretized form of the vector of internal shear forces for an element is:

$$\left\{ \underset{\sim}{f}{}^{e}_{\mathrm{int},\bar{q}} \right\}_{n+1} = \int_{\Omega^e} \underset{\sim}{N}{}^{T}_{,x} EI \frac{f_2}{f_1} \bar{q}_{n+1} d\Omega \tag{33}$$

Linearization of Eq. (33) gives

$$\left\{ \Delta \underset{\sim}{f}{}^{e}_{\mathrm{int},\bar{q}} \right\}_{n} = \left[\int_{\Omega^e} \left(\underset{\sim}{N}{}^{T}_{,x} D_{\bar{q}1} \underset{\sim}{N}_{,xxx} + \underset{\sim}{N}{}^{T}_{,x} D_{\bar{q}2} \underset{\sim}{N}_{,xx} + \underset{\sim}{N}{}^{T}_{,x} D_{\bar{q}3} \underset{\sim}{N}_{,x} \right) d\Omega \right] \Delta \ddot{\underset{\sim}{d}}_n \tag{34}$$

Where Δ is a differential operator, and

$$D_{\bar{q}1} = EI \frac{1}{Dr^3} \frac{f_2}{f_1}; D_{\bar{q}2} = EI \frac{1}{Dr} \frac{3}{2} \frac{d}{dx} \left(\frac{1}{Dr^2} \right) \frac{f_2}{f_1}; D_{\bar{q}3} = EI \frac{1}{Dr} \frac{1}{2} \frac{d^2}{dx^2} \left(\frac{1}{Dr^2} \right) \frac{f_2}{f_1} \tag{35}$$

Linearization gives the tangent matrix:

$$\left[m \right]^{e}_{\bar{q}} = \int_{\Omega^e} \left(\underset{\sim}{N}{}^{T}_{,x} D_{\bar{q}1} \underset{\sim}{N}_{,xxx} + \underset{\sim}{N}{}^{T}_{,x} D_{\bar{q}2} \underset{\sim}{N}_{,xx} + \underset{\sim}{N}{}^{T}_{,x} D_{\bar{q}3} \underset{\sim}{N}_{,x} \right) d\Omega \tag{36}$$

The effective internal forces for each element in the Newmark β scheme can be written as:

$$\left\{ \tilde{f}^{e} \right\}_{n+1} = \left[m \right]^{e} \ddot{\underset{\sim}{d}}_{n+1} + \left[c \right]^{e} \dot{\underset{\sim}{d}}_{n+1} + \left[k \right]^{e} \underset{\sim}{d}_{n+1} + \left[k' \right]^{e} \bar{\underset{\sim}{d}}_{n+1} + \left\{ \underset{\sim}{f}{}^{e}_{\mathrm{int},\psi} \right\}_{n+1} + \left\{ \underset{\sim}{f}{}^{e}_{\mathrm{int},\bar{q}} \right\}_{n+1} \tag{37}$$

And the tangent stiffness matrix becomes:

$$\left[\tilde{k} \right]^{e} = \left(\left[k \right]^{e} + \left[k' \right]^{e} \Delta t \right) + \frac{\gamma}{\beta \Delta t} \left[c \right]^{e} + \frac{1}{\beta \Delta t^2} \left(\left[m \right]^{e} + \left[m \right]^{e}_{\psi} + \left[m \right]^{e}_{\bar{q}} \right) \tag{38}$$

It should be noted that the tangent stiffness $\left[\tilde{k} \right]^{e}$ is independent of the solution process, and can thus be computed only once. However, the internal force vector $\left\{ \tilde{f} \right\}_{n+1}$ has to be recomputed at each time step because it is dependent on the solution at the previous steps.

7 RESULTS

Application of the PML approach to a semi-infinite beam on elastic foundation is illustrated for both time-harmonic and transient problems. First, a one-dimensional PML-truncated semi-infinite domain is subjected to a harmonic concentrated load $V(0,t) = p_0 \sin(t)$ of amplitude p_0 applied at the origin. A comparison with regular FE solution of a beam with simulated infinite length (or reasonably long span $L = 200$ m) on elastic foundation with a fixed boundary condition at $x = L$ was undertaken to validate the PML model. The system properties are given in Table 1. The mesh density in MATLAB program for PML was chosen to be similar to that of the bounded domain. For this situation, Fig. 3 (a) and (b), the displacement and rotation along the PML and FE region of the long beam (but of finite length) are practically the same.

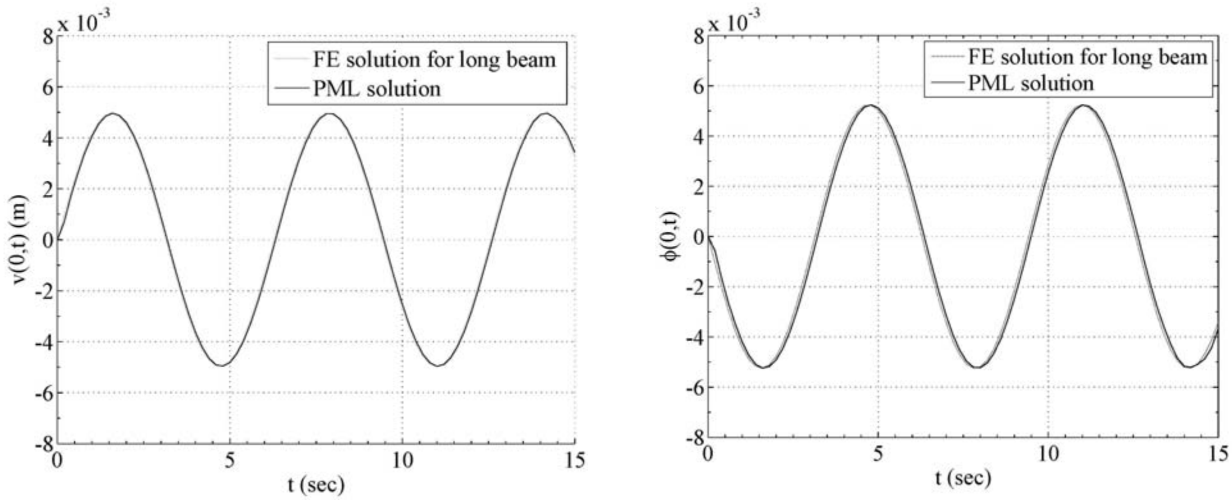

Fig. 3 (a) Displacement at the origin, **(b)** Rotation at the origin

Table 1: Properties of the unbounded dynamical system

Parameter	ρ (kg/m^3)	E (N/m^2)	I (m^4)	k_s (N/m/m)	P_0 (N)
Value	*8000*	200×10^{9}	2000×10^{-8}	20×10^{6}	50×10^{3}

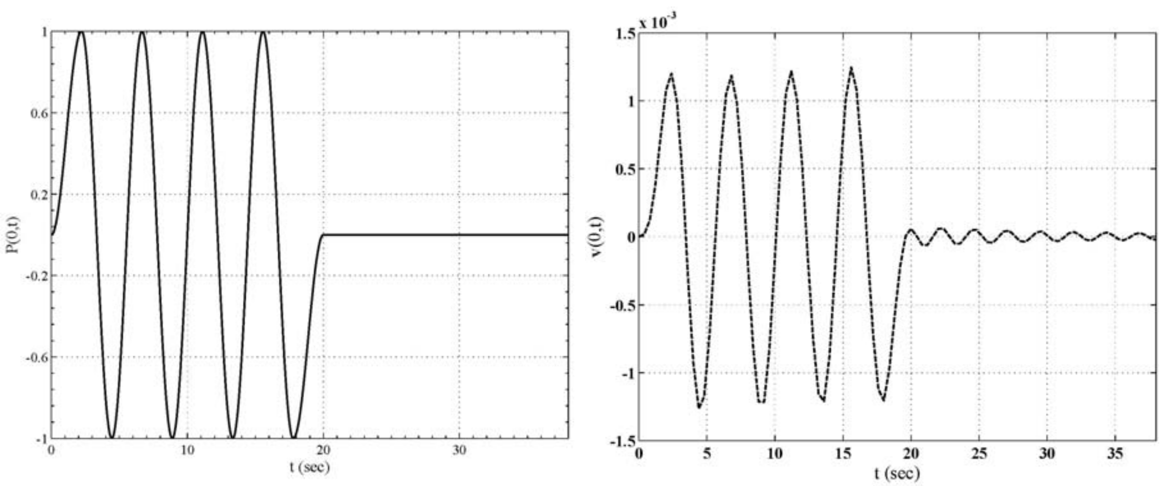

Fig. 4 (a) Plot of applied shear load at the origin, **(b)** PML solution for displacement at the origin

Since the main purpose of the PML is to simulate a radiation boundary in time domain, another specified loading time history was also tried. This was a shear force applied at the origin of the beam in the form of a cosine half-cycle that died at $t = 20\ \ (\text{sec})$, Fig 4 (a). The system was assumed to be initially at rest. The vertical displacement of the beam at $x = 0$ was computed under the induced shear force, with the PML model of Fig. 4 (b).

8 CONCLUSIONS

The concept of PML has been employed for semi-infinite Bernoulli-Euler beams on elastic foundation in the context of elastodynamics by utilizing a solution process similar to that of a 1-D rod problem. The higher order governing differential equation of perfectly matched medium (PMM) for the beam problem is derived by a modification of the equations for an elastic medium. The modification was motivated by a continuous, complex-valued, coordinate stretching. The time domain equations were obtained by the aid of ADEs (Auxiliary Differential Equations) and selecting stretching functions with simple dependence on the factor $i\omega$. This facilitated the transformation of the time-harmonic equations into the time domain. The PML equations were implemented numerically by a displacement-based FEM. The equilibrium equations were discretized in time by the Newmark method and were solved at each time step using a Newton-Raphson iteration scheme. Because the tangent stiffness matrix employed in this scheme was independent of the solution process it had to be computed only once at the start of the analysis. The time-discrete form of ADEs gives the internal shear and moment leading to a mass tangent matrix that could be incorporated into the effective tangent stiffness of the algorithm.

The solutions admitted by the PMM are of the same form as those admitted by elastic media, but with stretched coordinates replacing the real coordinates. The solutions involve both propagating and non-propagating terms. By choosing appropriate decaying functions in Eq. (5), the solution in the PMM takes the form of a corresponding elastic medium, but with an imposed spatial attenuation. The stretching function of the PMM in terms of attenuation functions has been matched at the interface so as not to allow any reflected waves to be generated when waves travel from the bounded domain into the PML zone. To achieve this end the attenuation function should be chosen so as to increase from zero at the bounded-domain-PML interface to a maximum value at the end of the layer. Although the rapid increase of the decaying function or the higher maximum value at the end causes smaller reflections of the waves back towards the bounded domain in the analytical solution of PMM, the discretization process causes some reflections at the interface of elements. Therefore slowly varying functions produce smaller reflections. This limitation can be overcome in such a manner to increase the mesh density of PMM with higher order decaying functions as well as to increase the depth of PMM with lower order ones. The efficiency and accuracy of the results is validated through a rudimentary trial-and-error procedure and by comparing the results to numerical solutions of a long beam. This showed that accurate results have been obtained from the PML model with small bounded domains at low computational costs. The method is applicable to different initial and boundary conditions.

If the space outside of the region of interest is assumed to be homogeneous, the radiating solutions in the infinite space take the wave form which can be decayed by a stretching function. Although it is not possible to use PML analysis for exterior heterogeneities, this does not impose any limitation to the solution process as any arbitrary heterogeneity within the computational region of interest can be handled. The time-domain equations obtained from PML process show that it is not limited to the particular form of harmonic loading and is applicable to impulse loads as well as arbitrary time histories.

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