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FINITE BEAM ELEMENT WITH PIEZOELECTRIC LAYERS AND FUNCTIONALLY GRADED MATERIAL OF CORE

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Abstract. The paper deals with the beam finite element, which is made from Functionally Graded Material (FGM) core and piezoelectric layers. In order to obtain effective material properties, homogenization of FGM core and homogenization of core and piezoelectric layers are used. In the paper, there is presented the derivation of individual submatrices of local stiffness matrix, where concept of transfer constants is used. Functionality of new FGM finite beam with piezoelectric layers is presented by numerical experiments.

1 INTRODUCTION

New materials have been introduced in the area of mechanisms and mechatronic applications. Contemporary mechanical systems are focusing on minimizing size, active control and low energy consumption. All these attributes can be incorporated into term Micro Electro Mechanical System (MEMS). MEMS applications usually contain multi layers structure and some of these layers can have piezoelectric properties. Piezoelectric structures offer facilities to make motions or they can be used to damp vibrations as an active damper or as an active sensor. For better understanding all these applications, it is inevitable to model such structures.

2 PIEZOELECTRIC CONSTITUTIVE EQUATIONS

Piezoelectric constitutive equations describe the relationship between mechanical and electrical quantities [1]. This relationship is derived in tensor notation, but for practical usage it can be rewritten into matrices notation.

2.1 Tensor notation of piezoelectric constitutive equations

The form of the constitutive equations depends on chosen mechanical and electrical quantities and can be expressed in two basic configurations [2]. The first configuration is expressed by mechanical stress tensor components σ_{kl} and vector components of electric intensity E_k and has a form

$$D_i = \epsilon^{\sigma}_{ik} E_k + d_{ikl}\sigma_{kl} \tag{1}$$

$$\varepsilon_{ij} = d_{ijk} E_k + s^{E}_{ijkl}\sigma_{kl} \tag{2}$$

where ε_{ij} are strain tensor components, D_i are components of electric displacement vector, d_{ikl} are tensor components of piezoelectric constants, ϵ^{σ}_{ik} are components of permittivity tensor on conditions constant mechanical stress and s^{E}_{ijkl} are components of compliance tensor on conditions constant electric intensity.
The constitutive equations can be also expressed by strain tensor components ε_{kl} and vector components of electric intensity E_k and has a form

$$\sigma_{ij} = c^{E}_{ijkl}\varepsilon_{kl} - e_{ijk}E_k \tag{3}$$

$$D_i = e_{ikl}\varepsilon_{kl} + \epsilon^{\varepsilon}_{ik}E_k \tag{4}$$

where new quantities are components of stiffness tensor c^{E}_{ijkl} and components of piezoelectric modulus tensor e_{ijk}.

2.2 Matrix notation of piezoelectric constitutive equations

If we use symmetric properties of individual tensor in constitutive tensor equations, we can rewrite constitutive equations into matrix notation [3]. Then equations (1) and (2) have a form

$$D_i = \epsilon^{\sigma}_{ik}E_k + d_{iq}\sigma_q \tag{5}$$

$$\varepsilon_p = d_{pk}E_k + s^{E}_{pq}\sigma_q \tag{6}$$

and equations (3) and (4) can be rewritten in the form

$$\sigma_p = c^{E}_{pq}\varepsilon_q - e_{pk}E_k \tag{7}$$

$$D_i = e_{iq}\varepsilon_q + \epsilon^{\varepsilon}_{ik}E_k \tag{8}$$

D_i and E_k are vectors with three components, σ_q and ε_q are vectors with six components, matrices s^{E}_{pq} and c^{E}_{pq} have dimension 6×6, matrices d_{iq} and e_{pk} have dimension 3×6 and matrix $\epsilon^{\varepsilon}_{ik}$ has dimension 3×3.

3 BASIC FEM EQUATIONS FOR PIEZOELECTRIC MATERIAL

To obtain basic FEM equations for piezoelectric material, the principle of virtual work is used. For static system, virtual work of internal and external forces can be expressed in the form

$$\int_V \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dV = \delta\mathbf{u}^{eT}\mathbf{F}^e \tag{9}$$

$\mathbf{F}^e$ is a vector of nodal forces and has a form

$$\mathbf{F}^e = \begin{bmatrix} F_{x1} \dots F_{xn} & F_{y1} \dots F_{yn} & F_{z1} \dots F_{zn} \end{bmatrix}^T \tag{10}$$

Virtual work of electric field (internal work) and electric charge (external work) can be expressed in the form [3]

$$-\int_V \delta\mathbf{E}^T \mathbf{D} dV = \delta\boldsymbol{\phi}^{eT}\mathbf{Q}^e \tag{11}$$

$\mathbf{Q}^e$ is a vector of nodal electric charges and has form

$$\mathbf{Q}^e = \begin{bmatrix} Q_1 \dots Q_n \end{bmatrix}^T \tag{12}$$

Constitutive equations (7) and (8) written in a component form can be rewritten as

$$\boldsymbol{\sigma} = \mathbf{c}^E\boldsymbol{\varepsilon} - \mathbf{e}^T\mathbf{E} \tag{13}$$

$$\mathbf{D} = \mathbf{e}\boldsymbol{\varepsilon} + \boldsymbol{\epsilon}^\varepsilon\mathbf{E} \tag{14}$$

Using shape functions, we can write relationship between components of strain and components of nodal displacements

$$\boldsymbol{\varepsilon} = \mathbf{B}_u\mathbf{u}^e \tag{15}$$

Similarly relationship between components of electric field intensity and components of nodal potential can be written as

$$\mathbf{E} = -\mathbf{B}_\phi\boldsymbol{\phi}^e \tag{16}$$

Using equations (15) and (16) constitutive equations (13) and (14) can be rewritten in a form

$$\boldsymbol{\sigma} = \mathbf{c}^E\mathbf{B}_u\mathbf{u}^e + \mathbf{e}^T\mathbf{B}_\phi\boldsymbol{\phi}^e \tag{17}$$

$$\mathbf{D} = \mathbf{e}\mathbf{B}_u\mathbf{u}^e - \boldsymbol{\epsilon}^\varepsilon\mathbf{B}_\phi\boldsymbol{\phi}^e \tag{18}$$

Virtual strain and virtual electric field intensity can be expressed as

$$\delta\boldsymbol{\varepsilon} = \mathbf{B}_u\delta\mathbf{u}^e \tag{19}$$

$$\delta\mathbf{E} = -\mathbf{B}_\phi\delta\boldsymbol{\phi}^e \tag{20}$$

Equations of virtual work (9) and (11) using equations (17) – (20) can be rewritten in the form

$$\delta\mathbf{u}^{eT}\int_V \left(\mathbf{B}_u^T\mathbf{c}^E\mathbf{B}_u\mathbf{u}^e + \mathbf{B}_u^T\mathbf{e}^T\mathbf{B}_\phi\boldsymbol{\phi}^e\right) dV = \delta\mathbf{u}^{eT}\mathbf{F}^e \tag{21}$$

$$\delta\boldsymbol{\phi}^{eT}\int_V \left(\mathbf{B}_\phi^T\mathbf{e}\mathbf{B}_u\mathbf{u}^e - \mathbf{B}_\phi^T\boldsymbol{\epsilon}^\varepsilon\mathbf{B}_\phi\boldsymbol{\phi}^e\right) dV = \delta\boldsymbol{\phi}^{eT}\mathbf{Q}^e \tag{22}$$

From equations (21) and (22) we obtain finite element equations for static piezoelectric analysis

$$\begin{bmatrix} \mathbf{K}^e_{uu} & \mathbf{K}^e_{u\phi} \\ \mathbf{K}^e_{\phi u} & \mathbf{K}^e_{\phi\phi} \end{bmatrix} \begin{bmatrix} \mathbf{u}^e \\ \boldsymbol{\phi}^e \end{bmatrix} = \begin{bmatrix} \mathbf{F}^e \\ \mathbf{Q}^e \end{bmatrix} \tag{23}$$

where

$$\mathbf{K}^e_{uu} = \int_V \mathbf{B}_u^T \mathbf{c}^E \mathbf{B}_u dV \tag{24}$$

$$\mathbf{K}^e_{u\phi} = \int_V \mathbf{B}_u^T \mathbf{e}^T \mathbf{B}_\phi dV \tag{25}$$

$$\mathbf{K}^e_{\phi u} = \int_V \mathbf{B}_\phi^T \mathbf{e} \mathbf{B}_u dV \tag{26}$$

$$\mathbf{K}^e_{\phi\phi} = -\int_V \mathbf{B}_\phi^T \boldsymbol{\epsilon}^\varepsilon \mathbf{B}_\phi dV \tag{27}$$

4 BEAM FINITE ELEMENT WITH PIEZOELECTRIC LAYERS

2D beam element with piezoelectric layers, where beam core is made from functionally graded material is shown in Fig. 1, where only electrical degree of freedom (electric potentials ϕ_i on 4 electrodes) are depicted. Mechanical degrees of freedom in each node are two displacements (in direction x a y) and rotation (in plane $x-y$) [6]. The height of beam core made from FGM is h, the height of piezoelectric layer is h_p, the depth and the length of the beam element are b and L, respectively.
Material properties of FGM core are function of longitudinal and transversal coordinate x and y, material properties of piezoelectric layers are constants.

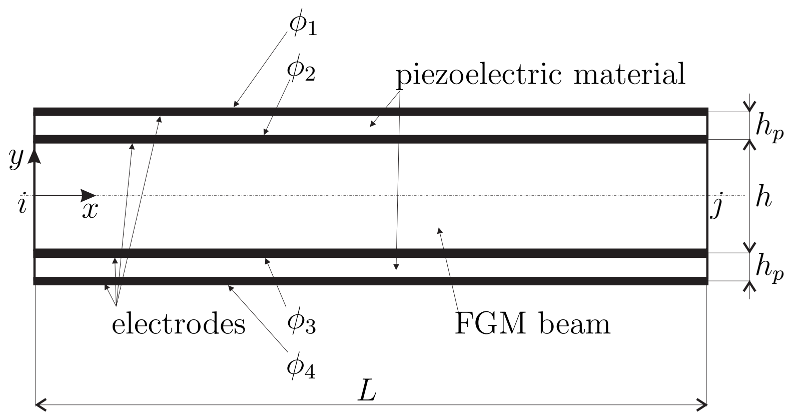

Figure 1: Electric DOF in 2D beam element

In order to derive individual submatrices of the beam element with piezoelectric layers and FGM core, two steps in homogenization process have to be performed. At first, homogenization of material properties of FGM core have to be performed, where direct integration method is used [6]. In the next step, homogenization of piezoelectric layers and homogenized FGM core (from step one) is performed. After this two step homogenization, material properties of the beam vary through the length of the beam as a function of longitudinal coordinate x and are constant in transversal direction.

4.1 Equations for structural analysis

The structural submatrix $\mathbf{K}^e_{uu}$ for the beam element with piezolayers can be expressed in a form

$$\mathbf{K}^e_{uu} = \begin{bmatrix} k'_u & 0 & 0 & -k'_u & 0 & 0 \\ & k'_{v2} & k'_{v3} & 0 & -k'_{v2} & k_{v2} \\ \mathrm{S} & & k'_{v33} & 0 & -k'_{v3} & k_{v3} \\ & \mathrm{Y} & & k'_u & 0 & 0 \\ & & \mathrm{M} & & k'_{v2} & -k_{v2} \\ & & & & & k_{v23} \end{bmatrix} \tag{28}$$

where individual components contain the influence of FGM core stiffness and also the influence of piezolayers stiffness. The calculation of components is identical for classical multilayer or FGM beam without piezoelectric layer and is described in [6].

4.2 Equations for electric analysis

Electric field intensity in piezoelectric layer is constant and for top layer can by expressed as [4, 5]

$$E_1 = -\frac{\partial \phi}{\partial y} = \frac{\phi_2 - \phi_1}{h_p} \tag{29}$$

and for bottom layer as

$$E_2 = -\frac{\partial \phi}{\partial y} = \frac{\phi_4 - \phi_3}{h_p} \tag{30}$$

Both components of electric field intensity can be written in a form

$$\mathbf{E} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = -\begin{bmatrix} 1/h_p & -1/h_p & 0 & 0 \\ 0 & 0 & 1/h_p & -1/h_p \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = -\mathbf{B}_\phi \boldsymbol{\phi}^e \tag{31}$$

For 1D problems the matrix of material properties for electric field – permittivity is reduced to only one material property ϵ^ε, but because the beam element contains two identical layers, we can write

$$\boldsymbol{\epsilon}^{\boldsymbol{\varepsilon}} = \begin{bmatrix} \epsilon^\varepsilon & 0 \\ 0 & \epsilon^\varepsilon \end{bmatrix} \tag{32}$$

Then the equation (27) has a form

$$\mathbf{K}^e_{\phi\phi} = -\int_V \mathbf{B}^T_\phi \boldsymbol{\epsilon}^\varepsilon \mathbf{B}_\phi dV = -\int_L \mathbf{B}^T_\phi \boldsymbol{\epsilon}^\varepsilon \mathbf{B}_\phi A_p dx \tag{33}$$

where A_p is cross-section of one piezoelectric layer, i.e. $A_p = bh_p$.
After some mathematical manipulations the equation (33) can be expressed as

$$\mathbf{K}_{\phi\phi}^e = \begin{bmatrix} -\frac{A_p L \epsilon^\varepsilon}{h_p^2} & \frac{A_p L \epsilon^\varepsilon}{h_p^2} & 0 & 0 \\ \frac{A_p L \epsilon^\varepsilon}{h_p^2} & -\frac{A_p L \epsilon^\varepsilon}{h_p^2} & 0 & 0 \\ 0 & 0 & -\frac{A_p L \epsilon^\varepsilon}{h_p^2} & \frac{A_p L \epsilon^\varepsilon}{h_p^2} \\ 0 & 0 & \frac{A_p L \epsilon^\varepsilon}{h_p^2} & -\frac{A_p L \epsilon^\varepsilon}{h_p^2} \end{bmatrix} \tag{34}$$

4.3 Coupling of structural and electrical analysis

Piezoelectric material properties express coupling between mechanical and electrical field - matrices **e** or **d**. The relationship between these two matrices can be expressed by elasticity matrix. In 1D problem in $x - y$ plane (in index notation $x_1 - x_2$) we have only one material property – e_{21} or d_{21}, where index 2 represents direction of piezoelectric layer polarization and also the direction of electric field intensity vector and index 1 defines direction of mechanical deformation. The relationship between these two quantities is reduced to expression $e_{21} = d_{21}E_p$ [7, 8], where E_p is Young modulus of elasticity of piezoelectric material. In reality, relationship between matrices **e** and **d** is more complicated and the quantity e_{21} computed from matrix **d** and elastic matrix for 3D system and the quantity e_{21} computed from d_{21} and E_p have different values. Therefore if we have quantities e_{21} and d_{21} computed from matrix relationship, it is better to used them then simplified relationship.
Piezoelectric material properties of both piezoelectric layers are defined as

$$\mathbf{e} = \begin{bmatrix} e_{21} & 0 & -ye_{21} \\ e_{21} & 0 & -ye_{21} \end{bmatrix} \tag{35}$$

The expression $\mathbf{e}^T\mathbf{E}$ defines mechanical stress caused by piezoelectric effect. In the beam elements, internal quantities are not mechanical stress but internal forces and moments, then the first and the third column of matrix (35) multiplied by corresponding components of $\mathbf{B}_u$ and $\mathbf{B}_\phi$ as well as corresponding components of displacement $\mathbf{u}$ and potential $\boldsymbol{\phi}$ represents axial forces and bending moments, respectively.
Description of piezoelectric behavior by e_{21} is more suitable for sensor equation – matrix $\mathbf{K}_{\phi u}^e$, description by d_{21} is more suitable for actuator equation – matrix $\mathbf{K}_{u\phi}^e$, i.e.

$$\mathbf{e} = \begin{bmatrix} d_{21}E_p & 0 & -yd_{21}E_p \\ d_{21}E_p & 0 & -yd_{21}E_p \end{bmatrix} \tag{36}$$

Using equations (35) and (36) we can write (25) and (26) in form

$$\mathbf{K}_{u\phi}^{e} = \begin{bmatrix} -\frac{A_p d_{21} E_p}{h_p} & \frac{A_p d_{21} E_p}{h_p} & -\frac{A_p d_{21} E_p}{h_p} & \frac{A_p d_{21} E_p}{h_p} \\ 0 & 0 & 0 & 0 \\ \frac{A_y d_{21} E_p}{h_p} & -\frac{A_y d_{21} E_p}{h_p} & \frac{A_y d_{21} E_p}{h_p} & -\frac{A_y d_{21} E_p}{h_p} \\ \frac{A_p d_{21} E_p}{h_p} & -\frac{A_p d_{21} E_p}{h_p} & \frac{A_p d_{21} E_p}{h_p} & -\frac{A_p d_{21} E_p}{h_p} \\ 0 & 0 & 0 & 0 \\ -\frac{A_y d_{21} E_p}{h_p} & \frac{A_y d_{21} E_p}{h_p} & -\frac{A_y d_{21} E_p}{h_p} & \frac{A_y d_{21} E_p}{h_p} \end{bmatrix} \tag{37}$$

and

$$\mathbf{K}_{\phi u}^{e} = \begin{bmatrix} -\frac{A_p e_{21}}{h_p} & 0 & \frac{A_y e_{21}}{h_p} & \frac{A_p e_{21}}{h_p} & 0 & -\frac{A_y e_{21}}{h_p} \\ \frac{A_p e_{21}}{h_p} & 0 & -\frac{A_y e_{21}}{h_p} & -\frac{A_p e_{21}}{h_p} & 0 & \frac{A_y e_{21}}{h_p} \\ -\frac{A_p e_{21}}{h_p} & 0 & \frac{A_y e_{21}}{h_p} & \frac{A_p e_{21}}{h_p} & 0 & -\frac{A_y e_{21}}{h_p} \\ \frac{A_p e_{21}}{h_p} & 0 & -\frac{A_y e_{21}}{h_p} & -\frac{A_p e_{21}}{h_p} & 0 & \frac{A_y e_{21}}{h_p} \end{bmatrix} \tag{38}$$

where parameter A_y represents first moment of cross-section of piezoelectric layer

$$A_y = \frac{1}{2} A_p (h + h_p) \tag{39}$$

4.4 FEM equations for the beam element with piezoelectric layers

FEM equations for beam element with piezoelectric layer and FGM core for static analysis have classical form

$$\begin{bmatrix} \mathbf{K}_{uu}^{e} & \mathbf{K}_{u\phi}^{e} \\ \mathbf{K}_{\phi u}^{e} & \mathbf{K}_{\phi\phi}^{e} \end{bmatrix} \begin{bmatrix} \mathbf{u}^{e} \\ \boldsymbol{\phi}^{e} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^{e} \\ \mathbf{Q}^{e} \end{bmatrix} \tag{40}$$

where individual submatrices are defined by (28), (34), (37) and (38), vector of nodal unknowns is defined as

$$\begin{bmatrix} \mathbf{u}^{e} \\ \boldsymbol{\phi}^{e} \end{bmatrix} = \begin{bmatrix} u_i & v_i & \varphi_i & u_j & v_j & \varphi_j & \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{bmatrix}^T \tag{41}$$

and vector of nodal loads is defined as

$$\begin{bmatrix} \mathbf{F}^{e} \\ \mathbf{Q}^{e} \end{bmatrix} = \begin{bmatrix} F_{xi} & F_{yi} & M_i & F_{xj} & F_{yj} & M_j & Q_1 & Q_2 & Q_3 & Q_4 \end{bmatrix}^T \tag{42}$$

where Q_1, Q_2, Q_3 and Q_4 are electric charge on electrodes 1, 2, 3 and 4, respectively.

5 NUMERICAL EXAMPLES

Efficiency of the beam element with piezoelectric layers and FGM core will be presented in two simple examples. In both examples Young modulus $E(x, y)$ of FGM core is characterized by following function of coordinate x and y

$$\begin{aligned} E(x,y) =& \frac{58000000000x^3y^2}{3} - \frac{580000x^3}{3} - 2900000000x^2y^2+ \\ &+ 29000x^2 + 5800000y^2 + 255 \qquad \text{[GPa]} \end{aligned} \tag{43}$$

After the first homogenization step of FGM core using direct integration method, which is described in [6], we can write for homogenized Young modulus for axial loading $E_L^{NH}(x)$ and for bending $E_L^{MH}(x)$ of FGM core

$$E_L^{NH}(x) = -\frac{290000x^3}{9} + \frac{14500x^2}{3} + \frac{910}{3} \qquad \text{[GPa]} \tag{44}$$

$$E_L^{MH}(x) = \frac{290000x^3}{3} - 14500x^2 + 342 \qquad \text{[GPa]} \tag{45}$$

PZT5A was chosen as piezoelectric material with following orthotropic material properties (polarization of piezoelectric material is in z direction):

- mechanical properties:
  - Young modulus: $E_1 = 61$GPa, $E_2 = 61$GPa, $E_3 = 53.2$GPa
  - Poisson's ratio: $\mu_{12} = 0.35$, $\mu_{13} = 0.38$, $\mu_{23} = 0.38$
  - shear modulus : $G_{12} = 22.6$GPa, $G_{13} = 21.1$GPa, $G_{23} = 21.1$GPa
- piezoelectric properties: $d_{31} = -171 \times 10^{-12}$C/N, $d_{33} = 374 \times 10^{-12}$C/N, $d_{15} = 584 \times 10^{-12}$C/N, $d_{24} = 584 \times 10^{-12}$C/N
- relative permittivity under conditions of constant mechanical stress: $\epsilon_{11}^{\sigma} = 1728.8$, $\epsilon_{22}^{\sigma} = 1728.8$, $\epsilon_{33}^{\sigma} = 1694.9$

We can transform all these material parameters into suitable forms (e.g. polarization of beam is in direction 2) and for the beam element we get: $E_p = 61$GPa, $e_{21} = -6.102$C/m^2, $d_{21} = -171 \times 10^{-12}$C/N, $\epsilon^{\varepsilon} = 8.111 \times 10^{-9}$F/m.

For homogenization of whole beam, i.e. FGM core and piezoelectric layers with material properties mention above, we have to use second homogenization by multilayer method and we can write for homogenized Young modulus for axial loading and for bending of whole beam

$$E_L^{NH}(x) = -\frac{725000x^3}{27} + \frac{36250x^2}{9} + \frac{4733}{18} \qquad \text{[GPa]} \tag{46}$$

$$E_L^{MH}(x) = \frac{4531250x^3}{81} - \frac{453125x^2}{54} + \frac{48301}{216} \qquad \text{[GPa]} \tag{47}$$

Geometry parameters of the beam are: $L = 0.1$m, $h = 0.01$m, $b = h$, $h_p = 0.001$m – Fig. 1.

5.1 Example 1 – model of actuator

Fig. 2 shows piezoelectric actuator, where the core made from FGM material and material of piezoelectric layers were defined above. Left side of beam is clamped and right end is free. Action motion is performed by different voltage between electrodes. Because the goal is to determine the deformation caused by voltage on electrodes, we can

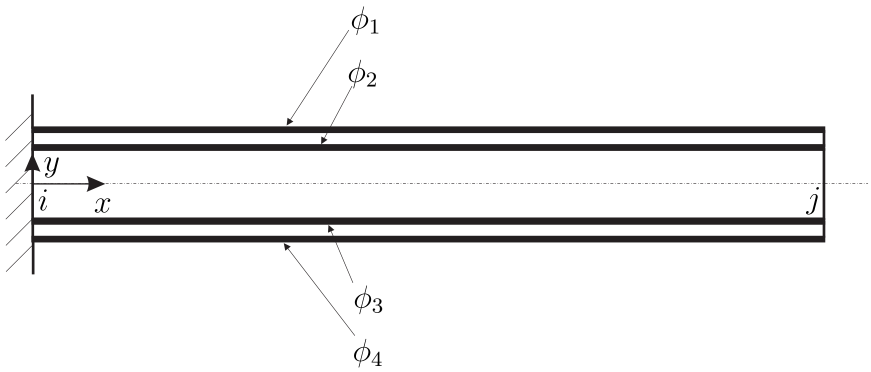

Figure 2: Example 1 – model of actuator

use only one part of equation (40)

$$\mathbf{K}_{uu}\mathbf{u} + \mathbf{K}_{u\phi}\boldsymbol{\phi} = \mathbf{F} \tag{48}$$

Equation (48) describes the whole system, which can be modeled by only one element.

Longitudinal actuator

Inner electrodes are grounded, i.e. $\phi_2 = \phi_3 = 0$V. In order to obtain longitudinal (axial) deformation, top and bottom electrodes have to have electric potential with different sign - we have chosen for top electrode electric potential $\phi_1 = -100$V and for bottom electrode electric potential $\phi_4 = 100$V. For these electric and mechanical conditions and after formation of matrices $\mathbf{K}_{uu}$ and $\mathbf{K}_{u\phi}$ we get:

$$u_j = -6.449 \times 10^{-8}\text{m} \tag{49}$$

In order to verify obtained results, the same example was modeled by FEM program ANSYS, where piezoelectric plane element PLANE223 was used. Modeling of FGM core by plane elements requires relatively fine mesh – total number of elements of FGM core was 5200 and the whole system contains 6400 elements. The end point of beam actuator has deformation in ANSYS model $u_j = -6.05 \times 10^{-8}$m. The difference between our beam model and ANSYS model was caused by fact, that in 2D or 3D elements not only d_{21} constant but the whole piezoelectric matrix is included into the computation.

Bending actuator

Inner electrodes are grounded, i.e. $\phi_2 = \phi_3 = 0$V. Now, outer electrodes have to have the same electric potential, we have chosen $\phi_1 = \phi_4 = 100$V. For these electric and mechanical conditions we get for displacement and rotation at node j:

$$v_j = -1.853 \times 10^{-6}\text{m} \tag{50}$$

$$\varphi_j = -3.809 \times 10^{-5}\text{rad} \tag{51}$$

The same problem was modeled also by program ANSYS and for end point we get $v_j = -1.85 \times 10^{-6}$m. As we can see from obtained results, our beam element results are comparable with ANSYS plane element results.

5.2 Example 2 – model of sensor

Fig. 3 shows model of sensor, which is loaded by external forces F_x and F_y. Material parameters are identical as in Example 1. The goal is to determine electric voltage, that is induced on electrodes caused by mechanical deformation of piezoelectric material. The

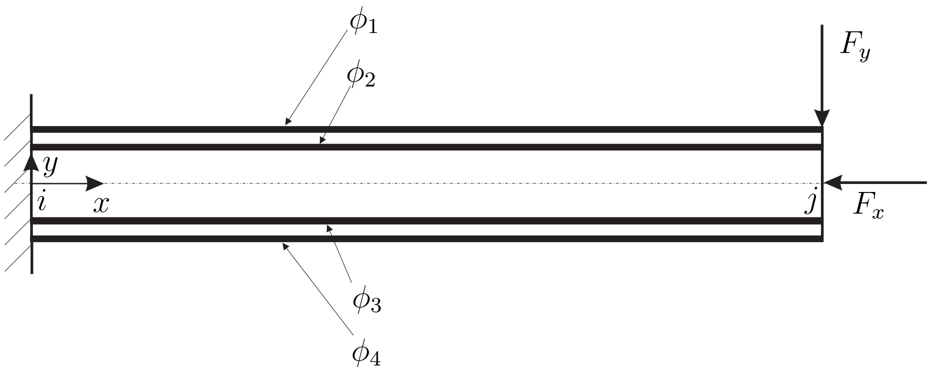

Figure 3: Example 2 – model of sensor

whole procedure can be divided into two steps. In the first step the deformation of system is computed from equation

$$\mathbf{K}_{uu}\mathbf{u} + \mathbf{K}_{u\phi}\boldsymbol{\phi}_a = \mathbf{F} \tag{52}$$

where $\boldsymbol{\phi}_a$ represents electric potential, where system works as actuator. Because both layers work like sensor and induced voltage is very small, we can equation (52) rewrite into the equation

$$\mathbf{K}_{uu}\mathbf{u} = \mathbf{F} \tag{53}$$

and after solution we get nodal displacements $\mathbf{u}$ of system.
To obtain electric potential, we use following equation

$$\mathbf{K}_{\phi u}\mathbf{u} + \mathbf{K}_{\phi\phi}\boldsymbol{\phi}_s = \mathbf{0} \tag{54}$$

where $\boldsymbol{\phi}_s$ is electric potential of sensor, e.q. $\boldsymbol{\phi}_s = \boldsymbol{\phi}$.

Axial force

Inner electrodes are grounded, i.e. $\phi_2 = \phi_3 = 0\text{V}$. System is loaded by axial compression force 100N. From equation (53) we can calculate unknown displacement

$$u_j = -3.09 \times 10^{-9}\text{m} \tag{55}$$

then the nodal vector of displacements has the form

$$\mathbf{u} = \begin{bmatrix} 0 & 0 & 0 & -3.09 \times 10^{-9} & 0 & 0 \end{bmatrix} \tag{56}$$

By equation (54) we get unknown electric potential (and also voltage) ϕ_1 and ϕ_4

$$\phi_1 = 0.0232\text{V} \tag{57}$$
$$\phi_4 = -0.0232\text{V} \tag{58}$$

The same problem was solved by code ANSYS where plane element PLANE223 was used with the following results: $\phi_1 = 0.0227\text{V}$ a $\phi_4 = -0.0227\text{V}$ (mesh density was the same as in Example 1).

Transversal force

Inner electrodes are grounded, i.e. $\phi_2 = \phi_3 = 0\text{V}$. Direction of transversal force is $-y$ and its magnitude is 100N.
Computational process was the same as in first case and the nodal vector of displacements has form

$$\mathbf{u} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1.062 \times 10^{-6} & -1.615 \times 10^{-5} \end{bmatrix} \tag{59}$$

In the second step, unknown electric potential are computed

$$\phi_1 = -0.668\text{V} \tag{60}$$
$$\phi_4 = -0.668\text{V} \tag{61}$$

Results by code ANSYS with the same mesh and the same element are: $\phi_1 = -0.688\text{V}$ a $\phi_4 = -0.688\text{V}$.

6 CONCLUSIONS

The paper presents new beam finite element with piezoelectric layers, where core of the beam can be made of FGM materials. Such combinations of materials is very attractive for machatronic applications, because material composition of FGM core can be optimized for design stress state and deformation can be controlled by voltages on electrodes. The beam finite element can be used for analysis of such systems very effectively and accurately.

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