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A NEW LEVEL SET BASED METHOD FOR TOPOLOGY OPTIMIZATION

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Abstract. Topology optimization has become a very popular field during the last decade, tempting thousands of scholars to dive into this discipline with a diversity of tools, among which level set based method is the celebrated one. But conventional level set based method for topology optimization has the flaw that it is difficult to nucleate holes. In this paper, we incorporate topological sensitivity to help holes creation process. In order to achieve higher order accuracy, trapezoidal method is utilized. Sensitivity filter is also employed to make result a better structure. The effectiveness of the method we propose is manifested in numerical examples at last.

1 INTRODUCTION

Topology optimization has achieved a smashing success over the last decade since a great deal of academic attention has been caught to this relatively new discipline. The field of topology optimization has been stacked with a tremendous amount of interesting and novel works. The objective of topology optimization is to eliminate redundant parts and scout for the optimal layout in a mathematical and scientific way.

Topology optimization provides mechanical engineers a powerful tool to find the optimal layout or material distribution of a structure in conceptual design phase. Sophisticated skills have been applied to topology optimization of discrete structure, on which quite a few studies have pinned their themes [1-3]. However topology optimization of continuum structure is the theme music in this discipline for its extensive and successful applications [4, 5].

Topology optimization of continuum structure can be traced back to the ground-breaking work by Bensøe and Kikuchi [6]. The discipline thrived thereafter and numerous methods emerged and hit the ground running, among which homogenization method [7, 8], SIMP (solid isotropic material with penalization) scheme [9], ESO (evolutionary structural optimization) approach [10-12], Pareto-Optimal tracing method [13, 14] and level set based methods [15-18] are the renowned ones. Intelligent algorithms are also employed tailored to solve topology optimization problems [1, 19, 20].

Besides those specific methods listed above, other facets of topology optimization of continuum structure can also be the subject of research. For instance, coupled with manufacturing constraints, topology optimization has more applicability to engineering problems [19, 20]. Blended with some particular conditions, topology optimization can be customized to obtain optimal structure of compliant mechanisms [17, 23]. A variety of works is geared to numerical instability during solving process of topology optimization [24].

Level set method [25] was proposed by Osher and Sethian [26] and has demonstrated great potentials in solving topology optimization problems [27]. Compared to density-based methods, level set-based methods have the merit of accomplishing topology and shape optimization instantaneously and averting the catch of checkboard problem. In level set based topology optimization method, the optimization process can be viewed as evolution of boundary in time, which is characterized by the Hamilton-Jacobi equation. The evolution of boundary employs information of design sensitivity analysis. For this reason, shape sensitivities are referred to as boundary velocity.

Nonetheless, some flaws reside in conventional level set based methods, impeding the development of level set based method. (1) The computation of level set based method is somewhat cumbersome and takes a considerable amount of time; (2) Re-initialization is often utilized to regularize the level set function, costing a large portion of time of investigators; (3) Time step size is restricted by CFL condition for the sake numerical stability; (4) Difficulties in hole nucleation come forth due to localized boundary changes and results in sub-optimal local minima; (5) Conventional level set based methods could only acquire a first order accuracy, which is insufficient under some circumstance.

Then level set based methods using compactly supported radial basis functions and Galerkin global weak-forms come forward with remedies for some flaws above. A family of compactly supported radial basis functions (CSRBFs) is firstly used to interpolate the level set function, and then augmented to construct the shape function for meshless approximation by satisfying basic requirements. A meshless Galerkin method (MGM) with global weak-forms is established to implement the discretization of the state equations. Armed with the level set based methods using compactly supported radial basis functions and Galerkin global weak-forms, investigators can arbitrarily determine the number of scattered meshless nodes. And modified formula of boundary velocity indicates a natural extension of the normal velocity field to all the nodes inside design domain [16]. As a result, this method is independent of time-step limitation and free of re-initializations. Furthermore, it unifies two different numerical stages involved in processing the discrete level set propagation and the shape derivative analysis, saving a significant portion of time.

Conventional level set based methods can execute topology changes of structures effortlessly, but they run into trouble when there are inadequate holes within the structure since they have difficulties in nucleating new holes, which is a corollary of maximum principle. Therefore conventional level set based methods depend on initial guesses of the number of holes. To tackle this difficulty, topological sensitivities could be incorporated in the update of level set function, which facilitates nucleation of holes [28]. Topological sensitivities have been recognized as a promising tool.

With topological sensitivities, this study aims to improve the accuracy of conventional level set based methods. Instead of using the conventional upwind scheme, which only guarantees a first order accuracy, trapezoidal method [36] is employed to promote boundary changes of the structure, leading to second order accuracy. Topological sensitivities and design sensitivities are coupled to constitute boundary velocity. Then trapezoidal method is employed to numerically solve Hamilton-Jacobi equation.

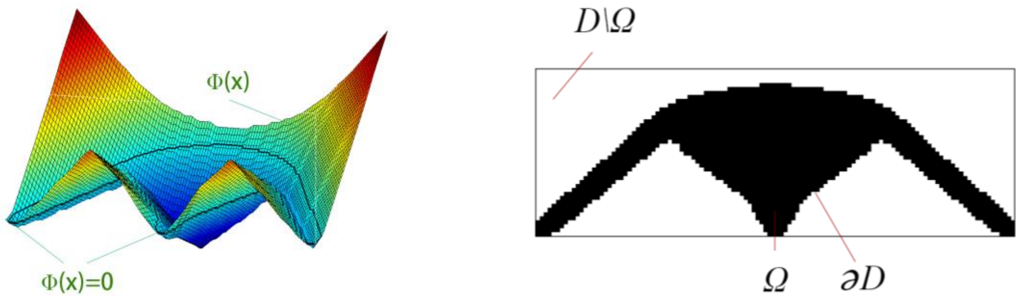
The basic idea of level set function is introduced in section 2, where the relation between the element densities and the level set function is driven home. In section 3, the properties of linear elastic material, which is the stuff where our level set method is implemented, are brought up. We formulate the topology optimization problem with the properties we established in section 4. Then in section 5 we introduce design and topological sensitivities, which are indispensable for the update of level set function. The proposed method of this study is raised in section 6. We also address the augmented Lagrangian multiplier method there. Sensitivity filters are also employed and introduced so as to make the topology optimization problem well-posed in section 6. Finally, we demonstrate the effectiveness of the proposed method by numerical examples in section 7.

2 LEVEL SET METHOD

In level set based method for topology optimization, the free moving boundary of design space is defined as the zero level set. The changes in the free moving boundary are embedded in the evolvement of higher dimensional level set function. As depicted in figure 1(b), *D* is the design domain of the structure where topology optimization operates. Ω is the material domain, which is isolated by boundary $\partial\Omega$. Figure 1(a) shows the 2D interface is the zero level set of 3D level set function, which is a signed distance function. If the structure occupies some domain Ω inside a larger design domain D, the level set function has the following property:

$$\Phi(\boldsymbol{x})\begin{cases} <0 & \text{if } \boldsymbol{x}\in\Omega \\ =0 & \text{if } \boldsymbol{x}\in\partial\Omega \\ >0 & \text{if } \boldsymbol{x}\in(\boldsymbol{D}\setminus\Omega) \end{cases}, \tag{1}$$

where $\partial\Omega$ denotes the free moving boundary.



(a) level set function (b) design domain

Figure 1. The level set function of a two-dimensional design

In structural optimization, the boundary $\partial\Omega$ is represented as the zero level set

$$\partial\Omega(\boldsymbol{t}) = \{\Phi(\boldsymbol{x}(\boldsymbol{t})) = 0\} \ \ \forall \boldsymbol{x}(\boldsymbol{t}) \in \partial\Omega(\boldsymbol{t}) \tag{2}$$

Differentiating Equation (2), the update scheme of level set function is specified by following Hamilton-Jacobi equation:

$$\frac{\partial\Phi(\boldsymbol{x})}{\partial\boldsymbol{t}} + \boldsymbol{v_n}\left|\nabla\Phi\right| = 0 \ , \tag{3}$$

where $\boldsymbol{v_n} = \boldsymbol{v}\cdot(\nabla\Phi / |\nabla\Phi|)$ is the normal velocity and t is the pseudo-time.

It has manifold positive features to embed a lower dimensional function in a higher dimensional function [25]. And as level set function, signed distance function could simplify the process of update for level set function. Therefore, signed distance function is regularly chosen as initial level set function. The convergence properties of a level set method gradually decay during update process. Therefore, level-set function re-initializations are often used to regularize the level-set function and maintain it as a signed-distance function [29].

To access the design during a topology optimization, the material domain Ω needs to be mapped to the finite element mesh. Most density-based topology optimization methods, as well as most level set based methods use material-fraction intermediate variables or element densities ρ_e, namely, $\varepsilon \le \rho_e \le 1$, where ε is the lower bound to avoid the singularity.

But in this study, we pursue a more distinct and painless scheme to define the density of every element. After the level set function is obtained, we only evaluate the level set value at the center of each element. If the level set value at the center of an element is greater than 0, then this element is void. Contrarily, this element is concrete. This concrete-void distribution can be expressed by the following formula,

$$\rho_e(\phi) = \begin{cases} 1 & \text{if } \Phi(\boldsymbol{x}) \le 0 \\ 0 & \text{if } \Phi(\boldsymbol{x}) > 0 \end{cases} , \tag{4}$$

where $\Phi(\boldsymbol{x})$ and $\rho_e(\phi)$ denote level set value and density of a specific element respectively.

3 DESIGN INTERPOLATION

The element densities ρ_e defined by the value of level set function are used to determine the presence of material in each element. The linear elastic structure is the subject of this study for the sake of simplicity. The finite element discretization of a design domain characterized by the element densities ρ_e lends to a global linear system,

$$Ku = f \ , \tag{5}$$

where K denotes the global linear stiffness matrix, u denotes the displacement vector and f is the external force vector assumed independent of the displacement. The stiffness of each element Ke makes a contribution to the construction of global stiffness matrix K,

$$K(\rho_e) = \sum_e c_e(\rho_e) K_e \ , \tag{6}$$

where $c_e(\rho_e)$ is the factor of Ke and is dominated by the density ρ_e and expressed by

$$\boldsymbol{c}_{\boldsymbol{e}}(\rho_{\boldsymbol{e}}) = \begin{cases} 1 & \text{if } \rho_{\boldsymbol{e}}(\Phi) = 1 \\ 0.01 & \text{if } \rho_{\boldsymbol{e}}(\Phi) = 0 \end{cases} . \tag{7}$$

The introduction of factor $\boldsymbol{c}_{\boldsymbol{e}}(\rho_{\boldsymbol{e}})$ is to escape singularity of global stiffness matrix. Now structural displacement u can be obtained from Equation (5). In this way, the structural behaviour of a design is implicitly parameterized by the level set function Φ.

4 FORMULATION OF TOPOLOGY OPTIMIZATION PROBLEM

Now we are poised to establish the topology optimization problem. As mentioned, the objective of topology optimization is to reduce weight as well as find the optimal distribution of materials. In this study, topology optimization problems entail an optimization objective of compliance $f(u)$, an equilibrium constraint $h(\rho_e, u)$ that bespeaks the linear property of the structure, and a volume constraint $g(\rho_e)$ with maximum volume fraction V_{max}. This topology optimization problem can be written as the following minimization problem,

$$\min \quad f(\rho_e, u) = f^T u \tag{8}$$

$$\text{subject to} \quad h(\rho_e, u) = K(\rho_e)u - f = 0 \tag{9}$$

$$g(\rho_e) = \frac{1}{N_e}\sum_e \rho_e - V_{\max} \le 0. \tag{10}$$

$$\rho_e = 0 \text{ or } \rho_e = 1 \tag{11}$$

where f is the column vector of external force. The element densities $\rho_{\boldsymbol{e}}$ are a function of level set value at the centre of each element and the displacements u are related to the element densities $\rho_{\boldsymbol{e}}$ according to equilibrium constraint (9). The range of element densities only contains two values, 0 or 1.

5 SENSITIVITY ANALYSIS

In this section, we will introduce design sensitivities [30] and topological sensitivities [28], on which the update of level set function are based. In the case of traction-free boundary conditions on the moving boundary, design sensitivity of the compliance objective $f(\rho_e, u)$ can be expressed as follows [31]:

$$\frac{\partial f(\rho_e, u)}{\partial \rho_e} = -c_e(\rho_e) u_e^T k_e u_e, \tag{12}$$

where $\boldsymbol{c}_{\boldsymbol{e}}(\rho_{\boldsymbol{e}})$ is the factor and determined by Equation (7) to avoid singularity. This formula of design sensitivity differs from the one used in SIMP because the densities have been polarized in this study while densities in SIMP are in need of penalization to avoid intermediate results. The design sensitivity of the volume constraint $g(\rho_e)$ simply equals unity:

$$\frac{\partial g(\rho_e)}{\partial \rho_e} = 1. \tag{13}$$

The topological sensitivities furnish for any point of the domain the sensitivity of the problem in creating a small hole in that point [32]. Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain, whose boundary ∂D is fairly smooth, i.e. a unit vector n exist almost everywhere, except possibly in a finite set of null measure. Let $\Omega_e \subset \mathbb{R}^2$ be a new domain so that $\Omega_\varepsilon = \Omega - \bar{B}_\varepsilon$, whose boundary is denoted by $\partial D_\varepsilon = \partial D \cup \partial B_\varepsilon$, where $\bar{B}_\varepsilon = B_\varepsilon \cup \partial B_\varepsilon$ is a ball of radius ε centered on the point $\hat{x} \in \Omega$. Therefore, a new domain Ω_e with a small hole $\bar{B}_\varepsilon$ has been created, as shown in figure. It is assumed that a given shape functional $\psi(\Omega_\varepsilon)$, associated to the topologically perturbed domain, admits the following topological asymptotic expansion

$$\psi(\Omega_\varepsilon) = \psi(\Omega) + f(\varepsilon)\delta_T^*(\hat{x}) + o(f(\varepsilon)), \tag{14}$$

where $\psi(\Omega)$ is the shape functional associated to the unperturbed domain, and $f(\varepsilon)$ is a positive function such that $f(\varepsilon) \to 0$ when $\varepsilon \to 0$. The function $\hat{x} \mapsto \delta_T^*(\hat{x})$ is called the topological sensitivity of ψ at $\hat{x}$. Hence topological sensitivity can be regarded as a first order correction of $\psi(\Omega)$ to approximate $\psi(\Omega_\varepsilon)$. After the rearrangement of Equation (14), we get

$$\frac{\psi(\Omega_\varepsilon) - \psi(\Omega)}{f(\varepsilon)} = \delta_T^*(\hat{x}) + \frac{o(f(\varepsilon))}{f(\varepsilon)}. \tag{15}$$

The limit passage $\varepsilon \to 0$ in the above expression results in

$$\delta_T^*(\hat{x}) = \frac{\psi(\Omega_\varepsilon) - \psi(\Omega)}{f(\varepsilon)}. \tag{16}$$

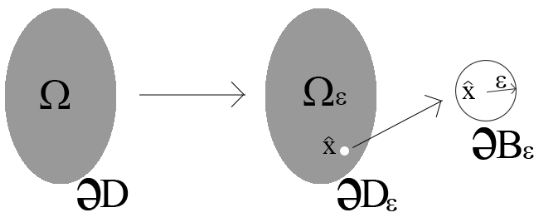


Figure 2. Original topological sensitivity concept

Though geometrical interpretation of topological sensitivity helps construct an intuition, to put topological sensitivity to good use, we have to have a nice numerical formula. According to [28], the topological sensitivity of the compliance objective function $f(\rho_e, u)$ in two dimensions with traction-free boundary conditions on the nucleated hole and the unit ball as the model hole is

$$\delta_T c = \frac{\pi(\lambda + 2\mu)}{2\mu(\lambda + \mu)}(4\mu u_e^T k_e u_e + (\lambda - \mu) u_e^T \left(k_{Tr}\right)_e u_e), \tag{17}$$

where $u_e^T \left(k_{Tr}\right)_e u_e$ is the finite element approximation to the product $tr(\sigma)tr(\varepsilon)$ where σ and ε are the stress tensor and strain tensor respectively. In (17), λ and μ are the Lamé constants for the solid material. The topological sensitivity of the volume constraint $g(\rho_e)$ with a model hole of unit ball is

$$\delta_T g(\rho_e) = -\pi. \tag{18}$$

Armed with all the backgrounds needed, we are ready to set forth our method to solve topology optimization problem in the next section. Two numerical examples will also be exhibited afterwards.

6 THE PROPOSED METHOD

Generally, the negative of design sensitivity is passed into update scheme of level set function as the normal velocity v_n for the boundary of the design. To evade a local minimum with insufficient holes, topological sensitivity is blended in update scheme of level set function. We employ the improved trapezoidal method to solve Hamilton-Jacobi equation of update scheme of level set function so as to obtain higher order accuracy, while conventional level set based method for topology optimization usually utilizes the forward Euler.

Forward Euler is of first-order accuracy and is given for the time discretization by

$$\frac{\Phi_{n+1}-\Phi_n}{\Delta t}=-v_n\left|\nabla\Phi^n\right|, \tag{19}$$

where Δt is the step size in time, v_n is the normal velocity. CFL condition is a well-known hassle which bothers investigators very often and unfortunately to which forward Euler is subject. In order to alleviate this situation, backward Euler can be

Based on standard level set equation (3), a new numerical update scheme is derived as follow [36],

$$\Phi_{n+1}=\Phi_n-\frac{\Delta t}{2}(v_n\left|\nabla\Phi^{n+1}+\nabla\Phi^n\right|)-\omega g, \tag{20}$$

where ω is a positive parameter for balancing the influence of topological sensitivity. However, the proposed method is free from CFL condition because it is an implicit scheme to update level set function.

v_n and g are the velocity terms based on design and topological sensitivities. In order to satisfy volume constraint, they are chosen using the design sensitivity and topological sensitivities of the Lagrangian

$$L=f(\rho_e,u)+\lambda^k\left(V^k-V_{\max}\right)+\frac{1}{2\Lambda^k}\left(V^k-V_{\max}\right)^2, \tag{21}$$

where λ^k and Λ^k are parameters which change with each iteration k of the optimization problem. Compared to other methods for constrained optimization problem solving, augmented Lagrangian multiplier method has the edges that it largely preserves smoothness and reduces the possibility of ill conditioning [33]. With the augmented Lagrangian multiplier method [34], the updating scheme for Lagrangian multipliers λ and Λ is expressed by

$$\lambda^{k+1}=\lambda^k+\frac{1}{\Lambda^k}(V^k-V_{\max}),\quad \Lambda^{k+1}=\alpha\Lambda^k, \tag{21}$$

where Vk is the volume of the structure at the k's iteration and $\alpha\in(0,1)$ is a fixed parameter.

Based on the Lagrangian we built in Equation (21) and sensitivity results established in Equation (12) and (13), the velocity v_n within element at iteration k of the algorithm is

$$v_n=-\frac{\partial L}{\partial\rho_e}=c_e(\rho_e)u_e^Tk_eu_e-\lambda^k-\frac{1}{\Lambda^k}\left(V^k-V_{\max}\right). \tag{22}$$

The velocity term of g involving topological sensitivity is to produce new holes. It becomes meaningless to create a new hole within void regions. Therefore, we define g as follows

$$g = \begin{cases} \delta_T L & \text{if } \phi < 0 \\ 0 & \text{if } \phi \geq 0 \end{cases}, \tag{23}$$

where $\delta_T L$ is the topological sensitivity of Lagrangian, which is defined by

$$\delta_T L = \delta_T c - \pi(\lambda^k + \frac{1}{\Lambda^k}(V^k - V_{\max})) \tag{24}$$

Sensitivity filter is also a helpful tool to make the result a better structure and is described as follows [35]:

$$\bar{s}_e = \frac{1}{x_e \sum_{f=1}^{N} H_f} \sum_{f=1}^{N} H_f x_f s_f, \tag{25}$$

where s_f is the sensitivity of neighbouring element and H_f is the weight factor and is written as,

$$H_f = r_{\min} - dist(e, f), \left\{ f \in N \middle| dist(e, f) \leq r_{\min} \right\}, e = 1, ..., N. \tag{26}$$

where the operator dist(e, f) is defined as the distance between centre of element e and centre of element f. The convolution operator H_f is zero outside the filter area.

7 NUMERICAL EXAMPLE

In this section, two numerical examples are exhibited to illustrate effectiveness of the method we proposed. Level set function is initially embedded as a signed distance function. It is apparent that in both numerical examples no hole is initially reserved. Following finite element analysis and sensitivity analysis, the proposed method is used to solve Hamilton-Jacobi equation.

In the first case, consider an elastic material with a Young's modulus of E=1 and a Poisson's ratio of $\nu = 0.3$. The design domain is a cantilever with a load of 1 applied at centre of the right side and depicted in figure 3. The left side of the domain is fixed. The width is 3 and the height is 1.

Initially, we choose the boundary of the design domain as the contour of zero level set for the level set function. The objective function is to minimize the structural compliance while the material usage is 50% in volume fraction. The radius of sensitivity filter is 1.5 and if the relative change of objective value between two successive iterations is less than 0.005 then the iteration terminates. Then design sensitivity and topological sensitivity become elements of update scheme (9) while the weight ratio of design sensitivity and topological sensitivity is 1:2.

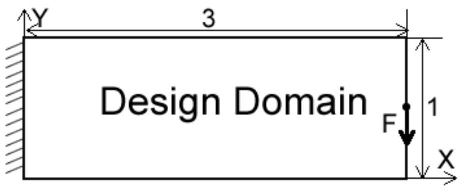
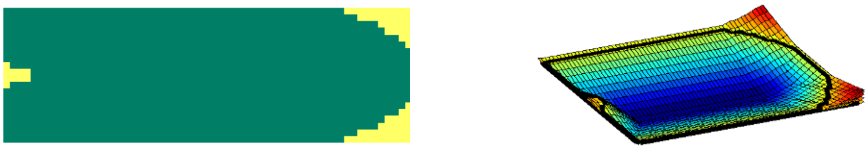


Figure 3. Design domain of the structure in case 1

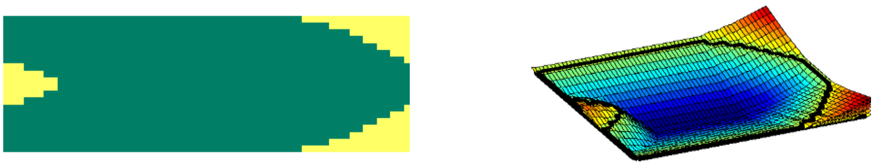
The whole design domain is discretized with a mesh of 60×20. Initial design and corresponding level set surface are given in Figure 4(a). The results given in Figure 4(b)–(d) are intermediate designs, and the optimal configuration and related level set surface are shown in Figure 4(e). The step size is subject to CFL condition. With the help of sensitivity filter, the optimal structure becomes manufacturable. Finally, it takes 53 iterations and achieves a compliance of 180.8087. The history of compliance and volume ratio is depicted in figure 5.

To illustrate the proposed method an effective one, Pareto-optimal tracing method [13] is applied to the same previous numerical example. For Pareto optimal tracing method, it takes 224 iterations to achieve the optimal structure with a compliance of 195.2919, which means the proposed method is more efficient to obtain the optimal design than Pareto-optimal tracing method.

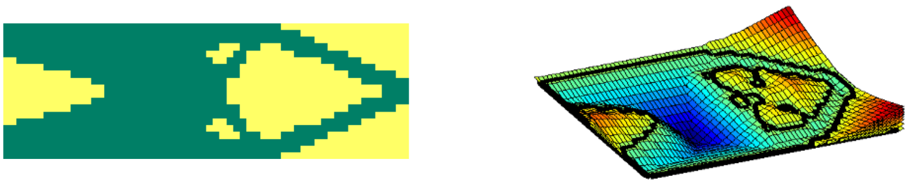
(a) Initial design



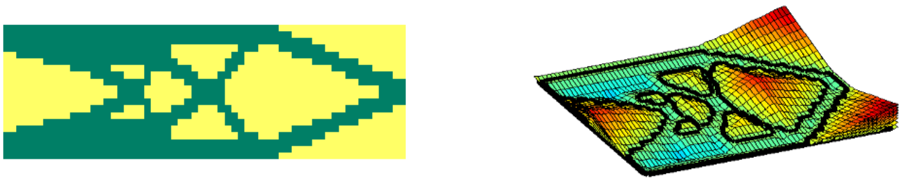
(b) Intermediate design



(c) Intermediate design



(d) Intermediate design



(e) Optimal design

Figure 4. Zero level set contours and level set surface (60×20)

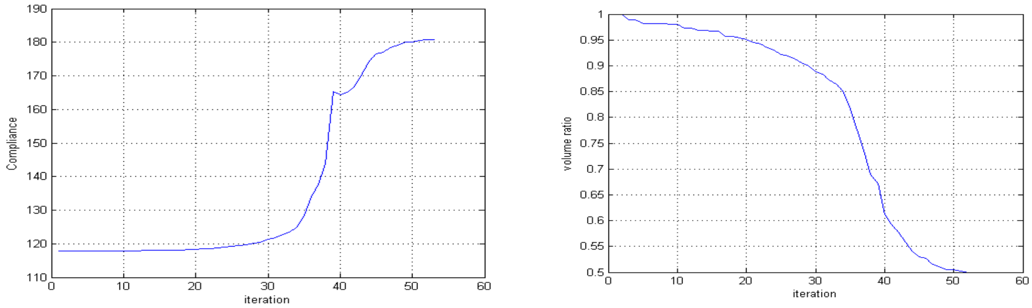


Figure 5. History of compliance and volume ratio in the first case

In the second case, which is depicted in figure 6, the topology optimization of MBB beam is introduced. The load is applied vertically in the upper left corner and the supports sit at the lower left and right corners. Properties of materials resemble the ones in the first case.

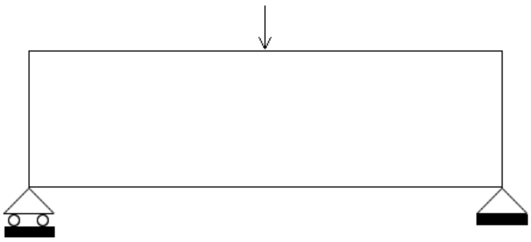
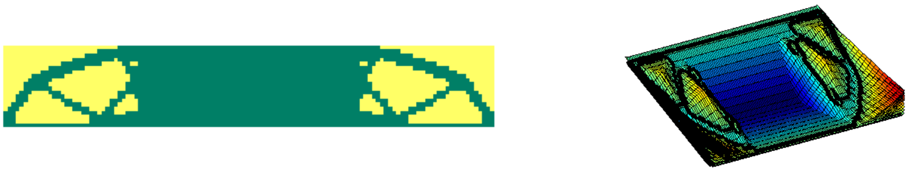


Figure 6. Design domain of structure in case 2.

The design domain is discretized with a mesh of 120×20. Process of topology optimization for MBB beam is illustrated in figure 7. Compliance and volume ratio over iterations are demonstrated in figure 8. SIMP is a great foil for the proposed method so as to manifest its effectiveness. We applied the topology optimization code using SIMP of Sigmund [35] to structure of the second case. In this test, penalty factor is set to 3 and filter radius is equal to 2. The optimized structure using SIMP method and proposed method is demonstrated in figure 9 and details of results are shown in table 1. With less iterations and compliance, the optimized structure using proposed method is more manufacturable so that proposed method outdoes the SIMP approach.

(a) Initial design

(b) Intermediate design



(c) Intermediate design

(d) Intermediate design

(e) Optimal design

Figure 7. Zero level set contours and level set surface (120×20)

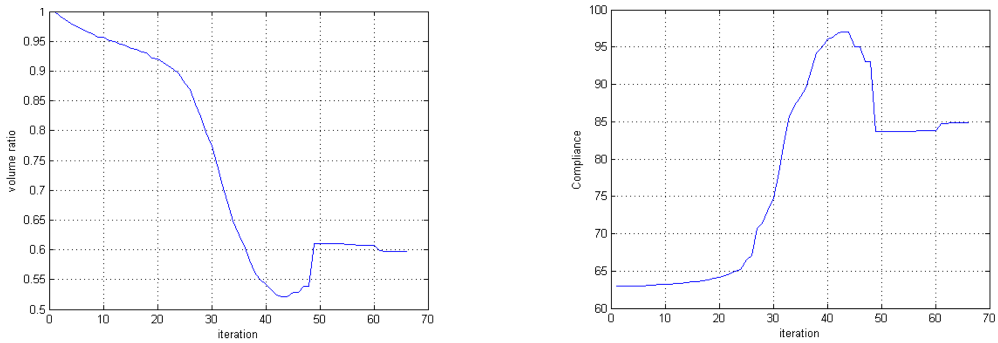
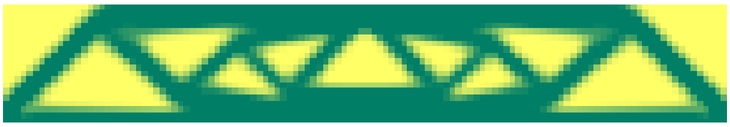
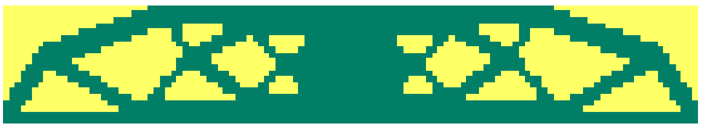


Figure 8. Compliance and volume Ratio over iterations in the second case



(a) The optimized structure in second case using SIMP method



(b) The optimized structure in second case using proposed method

Figure 9. Contrast of optimized structures using SIMP and proposed method

Table 1. Contrast of general results using SIMP and proposed method

	Number of iteration	Compliance
SIMP approach	96	87.2017
Proposed method	66	84.8166

8 CONCLUSION

A level set based method for topology optimization is introduced in this paper. We use trapezoidal method to solve Hamilton-Jacobi equation and incorporate topological sensitivity to nucleate holes. Topological and design sensitivities filter is also applied to make optimal result smooth. At last two numerical examples are presented to prove this method an effective one, compared to Pareto-optimal tracing method and SIMP approach

9 ACKNOWLEDGEMENT

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