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AEROFOIL INVISCID DRAG MINIMIZATION BY CONSTRAINED GLOBAL OPTIMIZATION

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1 INTRODUCTION

Optimization is the process of improving on a current design. In real world problems, historically optimization has often been performed manually where designers use intuition to produce solutions to problems so that the solution performs better than the initial starting point. However, it has now become commonplace to use automated optimization algorithms to allow a more streamlined and strict approach to this process and with the advent of increased computer power available to engineers, expensive optimization problems are being solved within an entirely automated process. Computational fluid dynamics (CFD) is at the forefront of aerodynamic analysis capabilities, and application of numerical optimization algorithms with such analysis is termed aerodynamic shape optimization (ASO), and has already produced notable results [1, 2, 3]. The authors have also presented work in this area, having developed a modularised, generic optimization tool that is applicable to any aerodynamic problem [4, 5, 6].

Solving optimization problems commonly uses either the gradient-based approach (which uses the local gradient as a guide to the search direction) or global search approach (which uses a set of agents who cooperate to interrogate the space). The choice of whether to use a gradient or global system is often made based on objective function evaluation cost, number and nature of constraints, and degree of multimodality within the problem hence, within the field of ASO the gradient approach is more popular where there is a requirement to minimise the number of objective function evaluations – which represent a CFD solution. Furthermore, the issue of constraints is particularly prominent in ASO and often leads to the use of gradient-based algorithms for ease of constraint handling, though at the expense of global optimum locating ability. The objective of this paper, therefore, is to investigate the effect that using a constrained global search algorithm has on ASO results and whether globally optimal feasible solutions can be obtained for a variety of aerofoil drag minimization cases. The issues of cost and convergence properties of a constrained global search algorithm are considered, as well as design space modality. This has implications on robustness and optimality, which are considered for several transonic aerofoil shape optimization cases.

2 CONSTRAINED NUMERICAL OPTIMIZATION

Mathematically, a single objective constrained optimization problem is:

$$\begin{aligned} \underset{\mathbf{x}\in\Re^n}{\text{minimise}} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & \mathbf{g}(\mathbf{x}) \leq \mathbf{0} \\ & \mathbf{h}(\mathbf{x}) = \mathbf{0} \end{aligned} \tag{1}$$

where $\mathbf{x}$ is the solution vector $[x_1, x_2, \ldots, x_n]^T$ where each element of the vector is a design variable, $f(\mathbf{x})$ is the value of the objective function for the given solution vector, $\mathbf{g}(\mathbf{x})$ represents inequality constraints, and $\mathbf{h}(\mathbf{x})$ represents equality constraints. The solution vector is bounded by an upper, U_k, and lower, L_k, bound such that for each x_k where $k = 1, 2, \ldots, n$, the solution must be $L_k \leq x_k \leq U_k$. In certain classes of optimization algorithms, it is typical to transform equality constraints into inequality constraints within some small tolerance: $|\mathbf{h}(\mathbf{x})| - \boldsymbol{\epsilon} \leq \mathbf{0}$.

Gradient-based optimization algorithms use the local gradient as a basis along which to search. The most basic forms of this are the steepest descent, conjugate gradient, and Newton approaches, though in the presence of constraints methods that involve solving the Karush-Kuhn-Tucker (KKT) conditions are preferred. The most widely adopted approach is sequential-quadratic-programming (SQP), which allows the strict enforcement of constraints within the algorithm, and this is the family of numerical optimization algorithms that has found greatest favour within the ASO community [7, 8, 9], primarily driven by the requirement to minimise the number of objective function evaluations. A significant issue when using these types of algorithm though is the termination in local minima, so in a multimodal search space a gradient-based algorithm will run to convergence although the multiple local minima in the problem represent an issue in finding the overall global best solution.

Global search algorithms avoid the issues associated with gradient-based approaches by avoiding the computation and use of the gradient, instead employing the current position of the optimization process in the search space as a method to build an algorithm. The use of global search algorithms within the ASO community is often limited to two dimensions where the objective function evaluation is cheaper. The most popular types of global search algorithm in ASO are evolutionary based (genetic algorithms and differential evolution) [10, 11, 12]. The particle swarm algorithm is the most widely used agent-based system within the wider optimization field, and so has also been used for aerofoil [13] and wing optimizations [14]. When comparing evolutionary- and swarm-based systems, the swarm methods tend to perform more efficiently and effectively than evolutionary algorithms [15, 16].

The selection of gradient-based or global search algorithms is highly dependent on the optimization case analysed, specifically the degree of modality present. The presence of a multimodal search space is dependent on the extent of the surface representation and the fidelity of the flow analysis tool, although the true aerodynamic optimization problem is independent of these two modules. As such, if given any surface representation method, if a multimodal space can be proved using this then it logically follows that a multimodal space must exist for the true aerodynamic optimization problem, and this has been shown for aerofoil optimizations [17, 13, 7] and aircraft topology [18] optimizations, however Chernukhin and Zingg [18] have also shown that for a B-spline based parameterization of the surface, drag minimization of the RAE2822 aerofoil has one global optimum. It is therefore imperative to have a parameterization that can represent all possible shapes.

3 CONSTRAINED GLOBAL SEARCH ALGORITHM

This work investigates the extent to which a global search algorithm can optimize a constrained aerodynamic shape optimization problem, so a suitable constraint handling system (separation-sub-swarm) has been combined with a gravitational search algorithm (GSA) [19] swarm mechanism that allows efficient optimization to be performed. The constraint handling framework developed by the authors is specifically applicable to GSA and takes into account the global transfer of data within the GSA algorithm by creating two separate swarms depending on the feasibility of agents. The separate swarms then either minimise the constraint violation by a particle swarm mechanism, or optimize the true objective function by the GSA mechanism. The infeasible swarm uses particle swarm to allow transfer of data out from the feasible space to the infeasible particles by a set of feasibility rules [20]. The correct transfer of data between search agents is the key to efficiency of agent-based global search algorithm.

The algorithm requires a system of N agents to be initialised within the bounds of the design variables. The velocity of the agents is also initialised to be a random value that is smaller than the bounds of the search space. For each agent, the objective function and the values of the constraints need to be calculated. The objective function associated with the n-th agent is:

$$\overline{f(\mathbf{x}_n)} = \begin{cases} f(\mathbf{x}_n) & \text{if } \mathbf{x_n} \text{ is feasible} \\ \sum_{j=1}^{G} \max\{0, g_j(\mathbf{x}_n)\} + \sum_{j=1}^{H} |h_j(\mathbf{x}_n)| & \text{else} \end{cases} \tag{2}$$

The particle's best ever position, $\mathbf{p}_n$, and the swarm's best ever position, $\mathbf{s}$, need to be updated, which is done by comparing the current position with the best positions by the following rules:

1. if current and best positions are feasible, the one with best fitness wins

2. if either the current or best positions are feasible and the other infeasible, the feasible position wins

3. if current and best positions are infeasible, the one with the minimum constraint violation wins

For the N_f feasible particles only, the mass of the i-th feasible particle is calculated based on a particle's feasible fitness:

$$m_i(t) = \frac{\text{fit}_i(t) - \text{worst}(t)}{\text{best}(t) - \text{worst}(t)} \tag{3}$$

$$M_i(t) = \frac{m_i(t)}{\sum_{j=1}^{N_f} m_j(y)} \tag{4}$$

where the best and worst fitnesses are from the feasible particles only. The force acting on particle i from particle j, where both particles are feasible is:

$$F_{i,j}^d(t) = G(t)\left(\frac{M_i(t) \times M_j(t)}{R_{i,j}(t) + \epsilon}\right)(x_j^d(t) - x_i^d(t)) \tag{5}$$

where $G(t) = G_0 \exp(-\alpha t/T)$. The total force acting on the i-th feasible particle is:

$$F_i^d(t) = \sum_{j=1, j\neq i}^{\min\{N_f, Kbest\}} \text{rand}_j F_{i,j}^d(t) \tag{6}$$

The acceleration of the feasible particles uses the cognitive and social memory parameters of particle swarm combined with the global transfer of data through the force mechanism of gravitational search to facilitate faster convergence:

$$a_i^d(t)_{gsa} = F_i^d(t)/M_i(t) \tag{7}$$

$$a_i^d(t)_{pso} = c_1 r_{1_i}(p_i^d - x_i^d(t)) + c_2 r_{2_i}(s^d - x_i^d(t)) \tag{8}$$

$$a_i^d(t) = (a_i^d(t)_{gsa} + a_i^d(t)_{pso})/2 \tag{9}$$

where $\mathbf{p}_i$ and $\mathbf{s}$ are the particle's and swarm's best positions ever, which are always feasible in this part of the optimization process. The cognitive, c_1, and social, c_2, parameters do add additional parameters to the problem but it is expected that these have the same value as the cognitive and social parameters is the infeasible search (equation 10).

The acceleration of the j-th infeasible particles is done by a pure particle swarm approach:

$$a_j^d(t) = c_1 r_{1_j}(p_j^d - x_j^d(t)) + c_2 r_{2_j}(s^d - x_j^d(t)) \tag{10}$$

The updating procedure for all of the particles is:

$$v_n^d(t+1) = \text{rand}_n v_n^d(t) + a_n^d(t) \quad (11) \qquad x_n^d(t+1) = x_n^d(t) + v_n^d(t+1) \quad (12)$$

If a particle exceeds the boundary of the search space then this is not an infeasible particle but is a particle without a solution so is reinitialised in its last position with a zero velocity.

The 3S-GSA algorithm is independent of objective function evaluation, so for the position of any given agent that agent need only know a value for the objective function. In the aerodynamic shape optimization process implemented here the objective function evaluation requires one flow solution to obtain the aerodynamic forces which are then used to evaluate the objective and constraints. The nature of agent-based search systems means that the objective of each particle is independent of other particles, and this makes the algorithm ideal for parallelisation, which is done in the MPI environment. The algorithm is independent of objective function so the agents call a wrapper which perturbs the surface and CFD mesh, followed by a flow solver to obtain the forces which form the objective function and constraints. Once the agents have the objective and constraint values, the master process carries out the position update by the 3S-GSA algorithm.

4 AERODYNAMIC SHAPE OPTIMIZATION

The aerodynamic shape optimization process involves integrating an effective shape parameterization and mesh deformation technique with a numerical simulation method and an effective optimizer. These are generally developed in a modular manner to allow the effective development of code independent techniques which have varying flexibility and cost in terms of aerodynamic flexibility, number of degrees of freedom and complexity of the optimization scheme. Work at the University of Bristol has been produced in this area, where development of a general shape parameterization [4, 5] and mesh deformation method [21, 22] has been linked to a parallelised FSQP optimizer [23], where the objective function is analysed by a finite-volume, inviscid, upwind flow solver. This unified shape parameterization and deformation technique is employed here to allow fully flexible aerodynamic shape optimization.

4.1 Geometry Parameterization

The shape parameterization and mesh deformation module must be flexible enough to allow sufficient design space investigation, robust enough to be applicable to any geometry, and efficient enough to maximise design space coverage with a minimum number of design parameters. To satisfy these requirements, an efficient domain element shape parameterization method has been developed by the authors and presented previously for CFD-based shape optimization [4, 5, 6]. The parameterization technique, surface control and volume mesh deformation all use radial basis functions (RBFs), wherein global interpolation is used to provide direct transfer of domain element movements into unified deformations of the design surface and the CFD mesh, which is deformed in a high-quality fashion. The method requires very few design variables to allow free-form design; the authors have recently used a mathematical approach to derive efficient design variables [24, 25], and those design variables are used here.

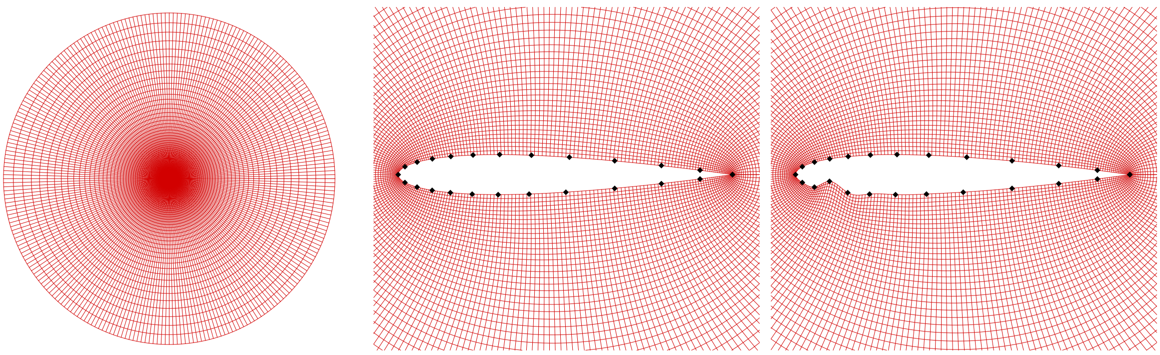
From a deformation of a set of control points (as given in figure 1), a set of interpolation points (which is the aerodynamic mesh) are deformed by:

$$\Delta\mathbf{X}_a = \mathbf{H}_{ac}\Delta\mathbf{X}_c \tag{13}$$

where the interpolation matrix is formed between a multiplication of a matrix of basis functions between the interpolation and control points, $\mathbf{A}$, and the inverse of a matrix of basis functions between the control points, $\mathbf{H}$. Compactly supported basis functions are preferred for mesh deformation, and Wendland's C^2 function [26] is used here. This matrix is never actually constructed, but applied row by row for each evaluation point (see Morris *et al.* [4]).

4.2 Flow-Solver

The flow-solver used is a structured multiblock finite-volume, unsteady, inviscid upwind code[27] using the flux vector splitting of van Leer[28]. Convergence acceleration is achieved through multigrid [29]. A single block O-mesh was generated, using a conformal mapping approach. Figure 1 shows two views of the 257×97 point mesh and the surface based control points which decouple the design variables from the surface and mesh, which extends to 40 chords at farfield. All surface cells have an aspect ratio of one.



(a) Farfield (b) Nearfield (c) Control point deformation

Figure 1: 257×97 O-mesh for NACA0012 aerofoil with surface based control points

4.3 Cases Considered

The primary factor investigated in this work is the performance of a global search algorithm when used for constrained aerodynamic shape optimization. This also requires consideration of: 1)optimizer convergence; 2) optimizer robustness; 3) optimizer efficiency and cost; 4) design space modality. To investigate these factors, aerodynamic shape optimization problems using the optimizer described are tested, using various numbers of design variables for compressible inviscid optimization. All cases were run with strict constraints (1% margins were allowed):

Objective:	Minimize drag (C_D)
	with respect to control point positions
Constraint 1 (lift):	$C_L \geq 0.99\ C_L^{initial}$
Constraint 2 (pitching moment):	$\lvert C_M \rvert \leq 1.01\ \lvert C_M^{initial} \rvert$
Constraint 3 (internal volume):	$V \geq 0.99\ V^{initial}$

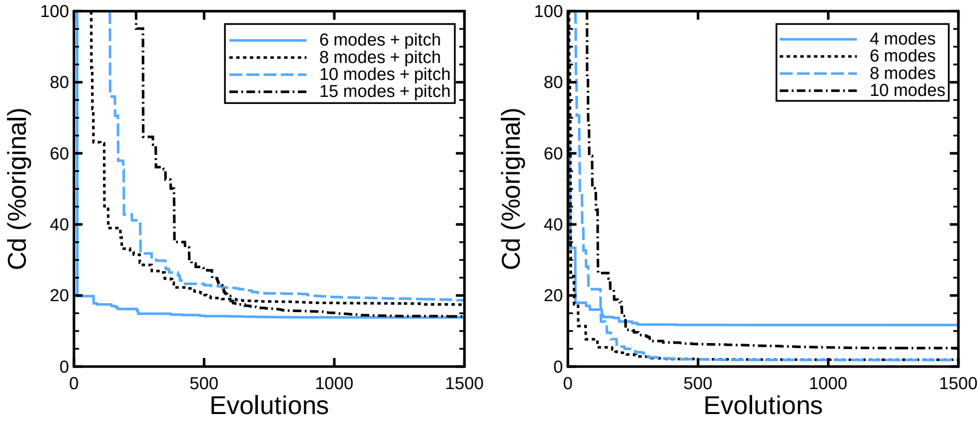
The aerofoil optimization cases considered are given in table 1, where the constraints given refer to the greater-than or less-than constraints above. The pitch degree of freedom moves the control points globally, whereas the modes derived from SVD move the control points subject to those design variables. Cases 1 and 2 are used to investigate the transonic optimization of the same aerofoil and investigate whether shock free inviscid solutions result in different flow conditions. Case 3 [30] is a symmetric zero lift drag minimization case so has no pitch parameters. Finally, case 4 is a standard optimization case and was chosen to allow deviation from a symmetric initial starting aerofoil. This case may also represent a typical transonic aerofoil optimization problem.

Table 1: Test cases considered

Case	Ref.	Aerofoil	Freestream parameters	Design variables	Constraints
1	[4]	NACA0012	$M_\infty = 0.65$ $\alpha = 5°$	Pitch + 6,8,10,15 SVD modes	C_L, C_M, Volume
2	-	NACA0012	$M_\infty = 0.70$ $\alpha = 3°$	Pitch + 6,8,10,15 SVD modes	C_L, C_M, Volume
3	[30]	NACA0012	$M_\infty = 0.85$ $\alpha = 0°$	4,6,8,10 symmetric SVD modes	Volume
4	[31]	RAE2822	$M_\infty = 0.73$ $\alpha = 2.7°$	Pitch + 6,8,10,15 SVD modes	C_L, C_M, Volume

5 RESULTS

3S-GSA has been run for all cases with 96 particles, 1500 timesteps, G_0 =30, α =10, $c_1 = c_2$ =2.0, which have all been set based on performance of analytical test cases. The results of the drag minimization problems and the implications of those results are discussed below, though the performance of the optimizer needs consideration. The convergence results of cases 2 and 3 are given in figure 2 which show that the optimization algorithm has allowed a balance between exploration and exploitation in all cases. Towards the start of the optimization process, the particles look to explore the design space which sends many of them infeasible, though these particles are soon brought back to effectively search the feasible design space. This is represented in the convergence plots by no results in the first few iterations (such as in case 2), though when the particles are optimizing the feasible space, they still explore it thoroughly and ultimately lead to a fully explored design space with a fully exploited optimal solution.

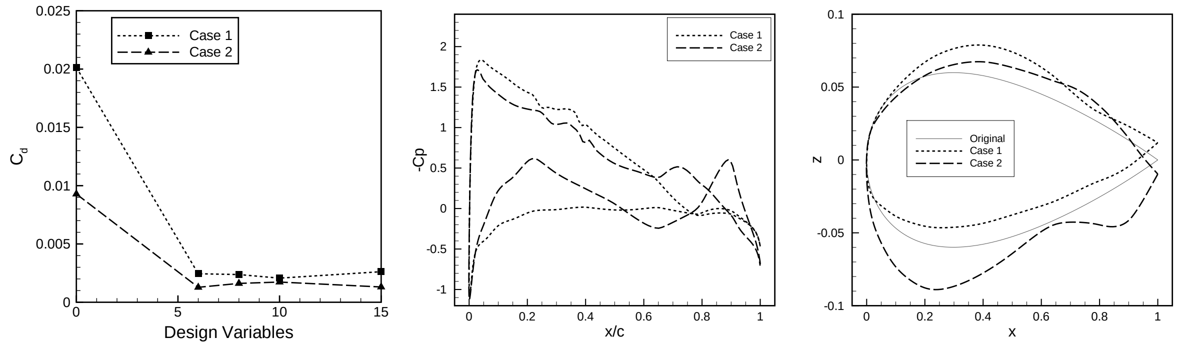


(a) Case 2 (b) Case 3

Figure 2: Optimizer convergence

5.1 Effect of Different Flow Conditions - Cases 1 and 2

Cases 1 and 2 (figure 3) represent the same shape optimization problem, though different flow conditions which results in different flow properties. The flow here is inviscid so the majority of drag at transonic Mach numbers of two-dimensional aerofoils is due to wave drag. The reduction of the wave drag therefore produces the most sharp reduction in overall drag over the aerofoil so the minimum drag solution is the shock free solution. Assuming that a shock free solution is possible at the given Mach number, the globally optimal solution should always be a shock free solution to eliminate as much as possible the wave drag. This can be seen in both the cases considered here even though they are at different flow conditions. The upper surface shock has successfully been eliminated from the solution, though the exact shapes differ between the two cases which maybe due to spurious drag that is as a result of numerical noise at the two flow conditions, or the optimizer not exactly exploiting the global minima. Nonetheless, the minimum drag solution, with no shock, has still been found in both cases.



(a) Dimensionality (b) Pressure distributions (c) Surface shapes

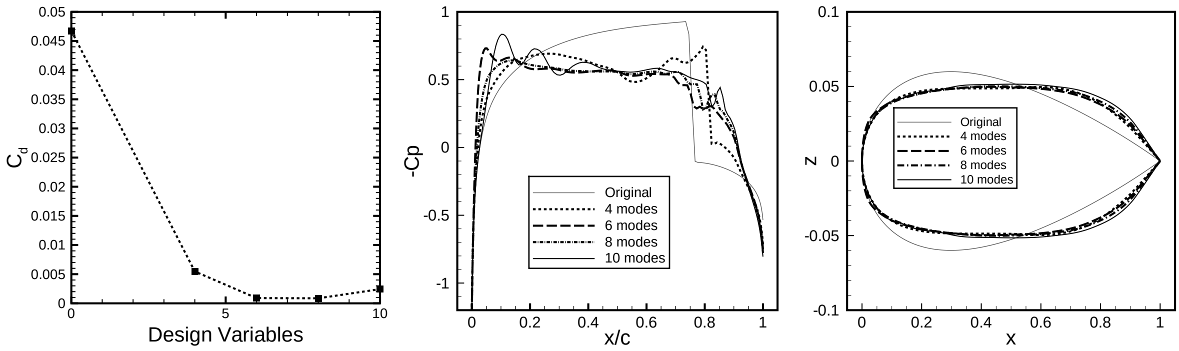
Figure 3: Surface shapes and pressure distributions for 10 modes, and dimensionality effect, for cases 1 and 2

5.2 Effect of Dimensionality

The addition of further design variables expands the size of the design space and therefore provides the optimization algorithm with a more complicated problem. In general, the algorithm has been successful at finding an almost globally optimal solution, however in all the cases considered a feasible solution has always been found which is encouraging. Case 3 (figure 4) represents a difficult optimization where the formation, strength and position of a shock is highly sensitive to the exact surface so the optimizer must have a good exploitation ability to accurately locate an optimal solution that is shock free. In fact this problem is considered to be the first Mach number at which a shock free solution is not possible [30], though six and eight mode cases have resulted in a shock free solution. The authors have also presented results on this case using a finer mesh [32] and demonstrated further drag reductions (99.7% reduction was possible), indicating that the solution is extremely sensitive to mesh density. Finally, the fourth case (figure 5) shows further high quality results from the optimizer, though it appears that a shock free solution is only possible using high numbers of design variables.

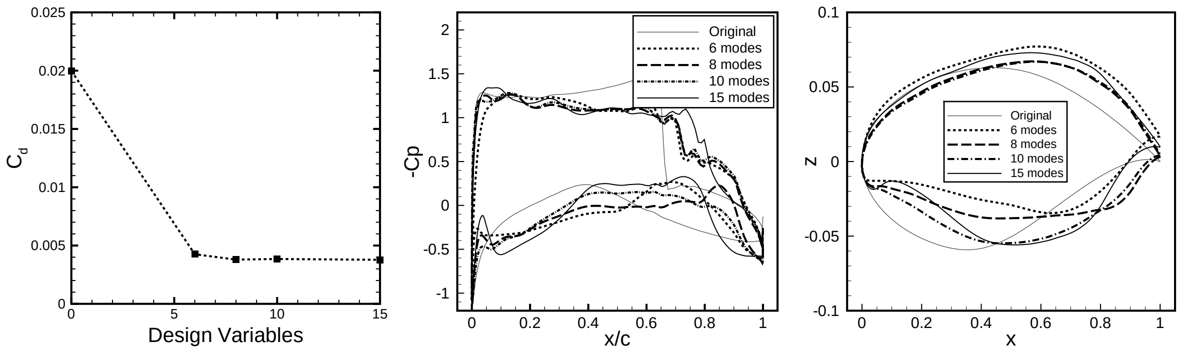
6 CONCLUDING REMARKS

This paper has considered the effect of using a constrained global optimizer on a set of aerodynamic shape optimization transonic test cases. The results indicate high performance of the optimizer, successfully locating a feasible solution for all cases tested, and successfully locating a globally optimal solution the majority of the time. High drag reduction results were observed when considering transonic drag minimization of various aerofoils at various conditions, though the cost of using the global optimizer is high. This high number of solutions was made possible due to the parallelisation of the optimizer, when each agent in the search system is assigned to a processor to evaluate its objective.



(a) Dimensionality (b) Pressure distributions (c) Surface shapes

Figure 4: Dimensionality effect, surface shapes and pressure distributions for case 3



(a) Dimensionality (b) Pressure distributions (c) Surface shapes

Figure 5: Dimensionality effect, surface shapes and pressure distributions for case 4

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