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MOMENTUM INTERPOLATION FOR QUASI ONE-DIMENSIONAL UNSTEADY LOW MACH NUMBER FLOWS WITH ACOUSTICS

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Abstract. A Rhie-Chow based algorithm for quasi 1-D sound propagation in a low Mach number mean flow is described. It is shown that the proposed Rhie-Chow interpolation method preserves the linear wave equation at first order, giving confidence in its ability to properly simulate flows that feature simultaneously acoustic waves and low Mach number convection.

1 Introduction

When a co-located arrangement of the unknowns is adopted, compressible low Mach number flow calculations may encounter difficulties heavily related to the way of interpolation on the cell or element faces (see *e.g.* [4, 5, 6, 7, 8, 9, 12, 18]). In particular, loss of accuracy may arise for unsteady calculations, when convective and acoustic waves propagate together at very different time and space scales, with possible interactions.

Versions of Godunov-type schemes that remain accurate at low Mach number are designed such as to conform to some low Mach number continuous asymptotic properties (see *e.g.* [4, 5, 6, 7, 8, 9]). One of these properties is that the thermodynamic and acoustic pressures should be constant in space at the convective scale [15]. Denoting by $\mathrm{M_r}$ a reference Mach number in the flow, it was shown in [9] for steady flow calculations, that this requests the $1/\mathrm{M_r^2}-$scaling of the pressure gradient term in the face velocity or the face mass flux to be preserved. If suitable boundary conditions are chosen, the checkerboard decoupling problem that may arise at low Mach number is thus avoided. As shown in [4, 6, 8], another asymptotic property provides insights for the design of Godunov-type schemes that remain accurate at low Mach number. This property is the linear acoustic energy conservation in the low Mach number regime, which holds if periodic boundary conditions are adopted. Assessment of the growth rate of the discrete linear acoustic energy (due to spurious acoustic waves) was thus used in [6] to design a modified Godunov-type scheme accurate at any Mach number for the compressible Euler system. This scheme generalizes the Low Mach Godunov-type scheme previously proposed in [4]. The guideline in [4, 6] is the study of the behaviour at low Mach number of the first-order modified equation associated to the Godunov scheme applied to the linear wave equation. In [5], this approach is also used in the case of the quasi one-dimensional linear acoustic equation.

In the present study, we propose to justify in the low Mach number regime the momentum interpolation technique, also often called Rhie-Chow interpolation technique [21], applied to the quasi one-dimensionnal compressible Euler equations. Notice that from the momentum equation, the above mentioned $1/\mathrm{M}_\mathrm{r}^2-$scaling of the pressure gradient term in the face velocity is readily satisfied. As a consequence, the momentum interpolation method does not suffer from checkerboard modes. Notice that this is not the case for the scheme proposed in [7] (see also [3]). The ability of the momentum interpolation method to properly simulate acoustic waves in low Mach number flows is justified by expliciting the first-order modified equation associated to this scheme in the case of the quasi one-dimensional linear acoustic equation. It is shown that, at the discrete level, the linear acoustic energy behaviour is similar to that at the continuous level. It is expected that this will be beneficial for accurate calculation of low Mach number flows in the non-linear case, including acoustics, in nozzles with variable cross-section area for which some numerical results are presented.

2 Flow equations

A 1-D flow of air is considered in a nozzle of variable section. Neglecting viscosity effects, the flow model is given by the Euler equations. To ease discussions on the Mach number scaling in the different terms, these equations are given in dimensionless form. Reference pressure p_r, density ϱ_r and velocity v_r thought of as a convective quantity, are introduced. A reference Mach number is then defined as $\mathrm{M}_\mathrm{r} = v_\mathrm{r}/\sqrt{p_\mathrm{r}/\varrho_\mathrm{r}}$. Reference length l_r and duration t_r, thought of as a convective quantity, are also considered, as well as a reference Strouhal number, $\mathrm{St}_\mathrm{r} = (l_\mathrm{r}/v_\mathrm{r})/t_\mathrm{r}$. If the reference length l_r is chosen as $t_\mathrm{r}\sqrt{p_\mathrm{r}/\varrho_\mathrm{r}}$, which is an acoustic length, the reference Strouhal and Mach numbers are related by $\mathrm{St}_\mathrm{r} = 1/\mathrm{M}_\mathrm{r}$. Here however, the possibility is left open for another choice of reference duration, so that we will work with the reference Strouhal number St_r. Denoting by x the dimensionless coordinate in the flow direction, by S the dimensionless cross-section area and by t the dimensionless time, the obtained non-dimensional form of the Euler equations reads

$$\mathrm{St_r}\partial_t(\varrho S) + \partial_x(\varrho v S) = 0, \tag{1a}$$

$$\mathrm{St_r}\partial_t(\varrho v S) + \partial_x\big((\varrho v^2 + \frac{1}{\mathrm{M_r^2}}p)S\big) = \frac{1}{\mathrm{M_r^2}}p\mathrm{d}_x S, \tag{1b}$$

$$\mathrm{St_r}\partial_t(\varrho E S) + \partial_x(\varrho v H S) = 0, \tag{1c}$$

$$E = e + \frac{1}{2}\mathrm{M_r^2}v^2, \quad \varrho H = \varrho E + p, \quad \varrho e = \frac{p}{\gamma - 1}, \tag{1d}$$

where ϱ, p, v, e, E and H represent dimensionless density, pressure, velocity, internal energy, total energy and total enthalpy per unit mass, respectively. Furthermore, γ denotes the ratio of the specific heats.

3 Analysis of the momentum interpolation method with the variable cross-section area linear wave equation

3.1 Linear acoustic wave equation and acoustic energy

The linear acoustic wave equation for quasi one-dimensional flows in the low Mach number regime is obtained through a two-scale low Mach number asymptotic analysis.

From Eqs. (1a)-(1b) and (1c)-(1d), one obtains, respectively:

$$\mathrm{St_r}\varrho\partial_t v + \varrho v\partial_x v + \frac{1}{\mathrm{M_r^2}}\partial_x p = 0, \tag{2a}$$

$$\mathrm{St_r}\partial_t(pS) + \partial_x(vpS) = -(\gamma - 1)p\partial_x(vS). \tag{2b}$$

A variable relevant to reveal the behaviour of the flow at the acoustic length scale is introduced as

$$\xi = \mathrm{M_r}x.$$

Then, the pressure is assumed to be expanded as

$$p(x, t, \mathrm{M_r}) = \sum_{n=0}^{N} \mathrm{M_r^n}p^{(n)}(x, \xi, t) + o(\mathrm{M_r^N}), \quad N = 0, 1, 2,$$

and similar expansions are assumed for density ϱ and velocity v.

After substitution of these expansions in Eqs. (2), Eq. (2a) at zeroth-order and Eq. (2b) at first order yield:

$$\mathrm{St_r}\varrho^{(0)}\partial_t v^{(0)} + (\varrho v)^{(0)}\partial_x v^{(0)} + \partial_x p^{(2)} + \partial_\xi p^{(1)} = 0, \tag{3a}$$

$$\begin{aligned}&\mathrm{St_r}S\partial_t p^{(1)} + \partial_x(vpS)^{(1)} + S\gamma p^{(0)}\partial_\xi v^{(0)} = \\ &\qquad -(\gamma - 1)p^{(1)}\partial_x(vS)^{(0)} - (\gamma - 1)p^{(0)}\partial_x(vS)^{(1)} - \gamma(pv)^{(0)}\mathrm{d}_\xi S.\end{aligned} \tag{3b}$$

Keeping terms at the acoustic scale[1], the first-order linear wave equation for quasi 1-D low Mach number flow is obtained as

$$\mathrm{St_r}\partial_t \widetilde{v^{(0)}} + \frac{1}{\widetilde{\varrho^{(0)}}}\partial_\xi p^{(1)} = 0, \tag{4a}$$

$$\mathrm{St_r}\partial_t p^{(1)} + \gamma p^{(0)}\partial_\xi \widetilde{v^{(0)}} = -\gamma p^{(0)}\widetilde{v^{(0)}}\frac{\mathrm{d}_\xi S}{S}. \tag{4b}$$

Introducing the zeroth-order sound speed as

$$c^{(0)} = c^{(0)}(\xi, t) = \sqrt{\frac{\gamma p^{(0)}(t)}{\widetilde{\varrho^{(0)}}(\xi)}},$$

Eqs. (4) yield the following non-linear acoustic wave propagation equation:

$$\mathrm{St_r^2}\partial_{tt}p^{(1)} - \partial_\xi\big((c^{(0)})^2\partial_\xi p^{(1)}\big) = (c^{(0)})^2\partial_\xi p^{(1)}\frac{\mathrm{d}_\xi S}{S}. \tag{5}$$

From Eqs. (4), one also obtains

$$-\partial_\xi(p^{(1)}\widetilde{v^{(0)}}) = \mathrm{St_r}\widetilde{\varrho^{(0)}}\partial_t\left(\frac{(\widetilde{v^{(0)}})^2}{2}\right) + \frac{\mathrm{St_r}}{\widetilde{\varrho^{(0)}}(c^{(0)})^2}\partial_t\left(\frac{(p^{(1)})^2}{2}\right) + p^{(1)}\widetilde{v^{(0)}}\frac{\mathrm{d}_\xi S}{S}. \tag{6}$$

The linear acoustic energy of quasi 1-D low Mach number flows in the nozzle is defined as

$$E_a = \frac{1}{2}\int_0^L \left\{\widetilde{\varrho^{(0)}}\big(\widetilde{v^{(0)}}\big)^2 + \frac{(p^{(1)})^2}{\widetilde{\varrho^{(0)}}(c^{(0)})^2}\right\} S\,\mathrm{d}\xi. \tag{7}$$

Assume that $c^{(0)}$ is constant in time. Notice that this amounts to suppose that the thermodynamic pressure $p^{(0)} = p^{(0)}(t)$ is constant, which is a common assumption for open flows (see *e.g.* the discussion by Rehm and Baum [20]). Then, from Eqs. (6) and (7), the time evolution of the acoustic energy in quasi 1-D low Mach number flows follows,

$$\mathrm{St_r}\mathrm{d}_t E_a = -\big[(p^{(1)}\widetilde{v^{(0)}}S)(L) - (p^{(1)}\widetilde{v^{(0)}}S)(0)\big]. \tag{8}$$

3.2 The Rhie-Chow-like interpolation method

The x axis along the nozzle of length L is divided into N cells of length Δx. A finite volume formulation in co-located arrangement is applied. For sake of presentation, $\widetilde{v^{(0)}}$ and $p^{(1)}$ are designated by u and q hereafter.

[1] $\widetilde{\cdot}$ denotes the large scale average, *i.e.* the average on $|\boldsymbol{x}| < 1/\mathrm{M_r}$ as $\mathrm{M_r} \to 0$. This average operation allows to separate features at the acoustic length scale from those at the convective length scale through the so-called sublinear growth lemma ; see *e.g.* Klein [11] or Meister [17] for details.

Suppose that Eqs. (4) are discretized into

$$\frac{u_i^{n+1}-u_i^n}{\tau}+\frac{a}{\mathrm{M_r}}(q_{i+1/2}^n-q_{i-1/2}^n)=0, \tag{9a}$$

$$\frac{q_i^{n+1}-q_i^n}{\tau}+\frac{b}{\mathrm{M_r}}(u_{i+1/2}^{n+1}-u_{i-1/2}^{n+1})+\frac{b}{\mathrm{M_r}}u_i^{n+1}\frac{S_{i+1/2}-S_{i-1/2}}{S_i}=0, \tag{9b}$$

where $\tau=\Delta t/\Delta x$, $a=1/(\mathrm{St_r}\widetilde{\varrho^{(0)}})$ and $b=\gamma p^{(0)}/\mathrm{St_r}$. In particular, it is thus assumed that the variations of $\widetilde{\varrho^{(0)}}$ in space are negligible at the large acoustic length scale.

Here, the momentum interpolation method is referred to as the Rhie-Chow-like interpolation method since it is applied to the velocity equation. For Eqs. (9), this method is formulated as follows. The procedure consists firstly in the identification of the non-linear convective term B_i in the momentum (or velocity) equation the face velocity is derived from. Since Eq. (9a) is linear, B_i is zero. Thus,

$$B_i=\frac{u_i^{n+1}-u_i^n}{\tau}+\frac{a}{\mathrm{M_r}}(q_{i+1/2}^n-q_{i-1/2}^n). \tag{10}$$

Eq. (10) must be compared to Eq. (20) (see Section 4.2), which is the discretized momentum equation as formulated for the momentum interpolation method. The second step in the Rhie-Chow-like interpolation method consists in the assumption that an equation similar to (10) may be written on the face $i+1/2$,

$$B_{i+1/2}=\frac{u_{i+1/2}^{n+1}-u_{i+1/2}^n}{\tau}+\frac{a}{\mathrm{M_r}}(q_{i+1}^n-q_i^n).$$

The third step is then to use a central interpolation on the convective terms. Here,

$$B_{i+1/2}=\frac{1}{2}(B_i+B_{i+1}), \tag{11}$$

from which the face velocity expression is derived. In the present case Eq. (11) results simply in $B_{i+1/2}=0$, and thus

$$0=\frac{u_{i+1/2}^{n+1}-u_{i+1/2}^n}{\tau}+\frac{a}{\mathrm{M_r}}(q_{i+1}^n-q_i^n).$$

The definition of $q_{i+1/2}$ must be given. Notice that this choice is independent of the face velocity definition provided by the momentum interpolation. In the present study, the face pressure is defined by central interpolation. The reason of this choice will be made clear in the following section by examining the first-order modified acoustic wave equation. The momentum interpolation scheme for the asymptotic linear acoustic wave equation, completed by a central interpolation of the pressure, is summarized as

$$\frac{u_i^{n+1}-u_i^n}{\tau}+\frac{a}{\mathrm{M_r}}(q_{i+1/2}^n-q_{i-1/2}^n)=0, \tag{12a}$$

$$\frac{q_i^{n+1}-q_i^n}{\tau}+\frac{b}{\mathrm{M_r}}(u_{i+1/2}^{n+1}-u_{i-1/2}^{n+1})+\frac{b}{\mathrm{M_r}}u_i^{n+1}\frac{S_{i+1/2}-S_{i-1/2}}{S_i}=0, \tag{12b}$$

$$\frac{u_{i+1/2}^{n+1}-u_{i+1/2}^n}{\tau}+\frac{a}{\mathrm{M_r}}(q_{i+1}^n-q_i^n)=0, \tag{12c}$$

$$q_{i+1/2}^n=\frac{1}{2}(q_i^n+q_{i+1}^n). \tag{12d}$$

When the cross-section area S is constant, scheme (12b)-(12c) is a classical scheme used to solve the linear wave equation on a staggered cartesian mesh (see Sections 3.3, 3.4.3 and 6 in [7] in the 2D cartesian case). This staggered scheme - *i.e.* centered pressure and staggered velocities - is sometimes called MAC scheme and was firstly proposed by [10] to solve the incompressible Navier-Stokes system. In particular, it is proven in [7] that the MAC scheme applied to the linear wave equation is accurate at low Mach number on a 2D cartesian mesh. It is also underlined that this scheme does not suffer from any checkerboard modes due to the even/odd decoupling. These results together with Eqs. (12) justify the good behaviour of the Rhie-Chow scheme on a 1D/2D/3D cartesian mesh at low Mach number, at least when the cross-section area S is constant in the 1-D variable cross-section case.

3.3 First-order acoustic wave equation by the Rhie-Chow-like interpolation method

From Eqs. (12b)-(12d), one readily obtains:

$$\frac{q_i^{n+1}-2q_i^n+q_i^{n-1}}{\tau^2}-\frac{ab}{\mathrm{M_r^2}}(q_{i+1}^n-2q_i^n+q_{i-1}^n)=\frac{ab}{\mathrm{M_r^2}}\left(\frac{S_{i+1/2}-S_{i-1/2}}{S_i}\right)\left(\frac{q_{i+1}^n-q_{i-1}^n}{2}\right). \tag{13}$$

Notice that the first-order modified equation associated with Eq. (13) is

$$\partial_{tt}q-\frac{ab}{\mathrm{M_r^2}}\partial_{xx}q=\frac{ab}{\mathrm{M_r^2}}\frac{\mathrm{d}_xS}{S}\partial_xq, \tag{14}$$

or else,

$$\partial_{tt}q-ab\partial_{\xi\xi}q=ab\frac{\mathrm{d}_\xi S}{S}\partial_\xi q, \tag{15}$$

which is identical to Eq. (5), under the assumption of a density $\widetilde{\varrho^{(0)}}$ constant at the acoustic length scale. As a direct consequence of this result, the acoustic energy equation (8) is retrieved at the discrete level, with a second-order error when periodic boundary conditions are adopted. This confirms the suitability of the Rhie-Chow-like interpolation method for the calculation of acoustic wave propagation in the 1-D variable cross-section case at low Mach number.

4 Analysis of the momentum interpolation method with the Euler equations

4.1 Pressure correction algorithmic framework

The algorithm used for solving the Euler equations (1) was detailed in our earlier work [18]. Here, it is briefly recalled in order to make the paper self-contained.

To simplify the presentation, the velocity is assumed to be positive. Each time-step $n \to n{+}1$ is decomposed into iterations denoted by the superscript k. At the first iteration of the time-step n, one has $k = n$. The superscripts $\star\star$, $\star$ and $\prime$ denote 'pre-predicted', predicted and corrected quantities of each iteration k. ψ denotes the slope limiter. Practically, no more than five iterations are allowed, and the so-called Bounded Central slope limiter was chosen in the unsteady low Mach number flow calculations. The ratio $\Delta t/\Delta x$ is denoted by τ.

4.1.1 'Pre-prediction' step: Construction of the common transporting velocity

• $\varrho_i^{\star\star}$ from

$$\frac{\mathrm{St_r}}{2\tau}(3\varrho_i^{\star\star} - 4\varrho_i^n + \varrho_i^{n-1}) + \frac{S_{i+1}}{S_i}\big[\varrho_i^{\star\star} + \frac{1}{2}\psi_i(\varrho^k)(\varrho_i^k - \varrho_{i-1}^k)\big]v_{i+1/2}^k \\ - \frac{S_{i-1/2}}{S_i}\big[\varrho_{i-1}^{\star\star} + \frac{1}{2}\psi_{i-1}(\varrho^k)(\varrho_{i-1}^k - \varrho_{i-2}^k)\big]v_{i-1/2}^k = 0$$

• $(\varrho v)_i^{\star\star} = \varrho_i^{\star\star} v_i^k \quad , \quad (\varrho E)_i^{\star\star} = \dfrac{p_i^k}{\gamma - 1} + \dfrac{1}{2}\varrho_i^{\star\star}(v_i^k)^2$

• Transporting face velocity:

$$v_{i+1/2}^{\mathrm{T}} = (\varrho v)_{i+1/2}^{\star\star}/\varrho_{i+1/2}^{\star\star} \quad \text{where} \quad \varrho_{i+1/2}^{\star\star} = \frac{1}{2}(\varrho_{\mathrm{L}}^{\star\star} + \varrho_{\mathrm{R}}^{\star\star})$$

How $(\varrho v)_{i+1/2}^{\star\star}$ is defined is the matter of Sec. 4.2.

4.1.2 Prediction step

• $p_i^\star = p_i^k$
• $\varrho_i^\star$ from

$$\frac{\mathrm{St_r}}{2\tau}(3\varrho_i^{\star} - 4\varrho_i^n + \varrho_i^{n-1}) + \frac{S_{i+1/2}}{S_i}\big[\varrho_i^{\star} + \frac{1}{2}\psi_i(\varrho^k)(\varrho_i^k - \varrho_{i-1}^k)\big]v_{i+1/2}^{\mathrm{T}} \\ - \frac{S_{i-1/2}}{S_i}\big[\varrho_{i-1}^{\star} + \frac{1}{2}\psi_{i-1}(\varrho^k)(\varrho_{i-1}^k - \varrho_{i-2}^k)\big]v_{i-1/2}^{\mathrm{T}} = 0 \quad (16)$$

• $(\varrho v)_i^\star$ from

$$\begin{aligned}\frac{\mathrm{St_r}}{2\tau}&[3(\varrho v)_i^\star - 4(\varrho v)_i^n + (\varrho v)_i^{n-1}]\\ &+\frac{S_{i+1/2}}{S_i}\big\{(\varrho v)_i^\star + \frac{1}{2}\psi_i((\varrho v)^k)[(\varrho v)_i^k - (\varrho v)_{i-1}^k]\big\}v_{i+1/2}^{\mathrm{T}}\\ &-\frac{S_{i-1/2}}{S_i}\big\{(\varrho v)_{i-1}^\star + \frac{1}{2}\psi_{i-1}((\varrho v)^k)[(\varrho v)_{i-1}^k - (\varrho v)_{i-2}^k]\big\}v_{i-1/2}^{\mathrm{T}}\\ &+\frac{1}{\mathrm{M_r^2}S_i}(S_{i+1/2}p_{i+1/2}^k - S_{i-1/2}p_{i-1/2}^k) = \frac{1}{\mathrm{M_r^2}S_i}p_i^k(S_{i+1/2}-S_{i-1/2})\end{aligned}$$

• $(\varrho E)_i^\star = \dfrac{p_i^k}{\gamma-1} + \dfrac{1}{2}\dfrac{[(\varrho v)_i^\star]^2}{\varrho_i^\star}$ and $(\varrho H)_i^\star = (\varrho E)_i^\star + p_i^k$

4.1.3 Correction step

• p_i' from

$$\frac{\mathrm{St_r}}{2\tau}[3(\varrho E)_i^{k+1} - 4(\varrho E)_i^n + (\varrho E)_i^{n-1}] + \frac{S_{i+1/2}}{S_i}(\varrho v H)_{i+1/2}^{k+1} - \frac{S_{i-1/2}}{S_i}(\varrho v H)_{i-1/2}^{k+1} = 0,$$

where

$$(\varrho v H)_{i+1/2}^{k+1} = (\varrho H)_{i+1/2}^\star v_{i+1/2}^{\mathrm{T}} + H_{i+1/2}^\star(\varrho v)_{i+1/2}' + (\varrho H)_{i+1/2}' v_{i+1/2}^{\mathrm{T}},$$

$(\varrho H)_{i+1/2}^\star$, $H_{i+1/2}^\star$: upwinded in second-order accurate form,

$$(\varrho H)_{i+1/2}' = \frac{\gamma}{\gamma-1}p_{i+1/2}',$$

$$p_{i+1/2}' = f_p^+(M_i^\star)p_i' + f_p^-(M_{i+1}^\star)p_{i+1}', \tag{17a}$$

$$\text{where}\quad f_p^\pm(m) = \begin{cases}\frac{1}{2}(1\pm\mathrm{sign}(m)) & , \ |m|\geq 1\\ \frac{1}{4}(m\pm1)^2(2\mp m)\pm\frac{3}{16}m(m^2-1)^2 & , \ |m|<1\end{cases} \tag{17b}$$

$$(\varrho v)_{i+1/2}' = -\frac{2\tau}{3\mathrm{M_r^2}S_{i+1/2}}[S_{i+1}p_{i+1}' - S_i p_i' - p_{i+1/2}'(S_{i+1}-S_i)]. \tag{17c}$$

Notice that with Eqs. (17a)-(17b), the pressure correction at the face is that given by the AUSM$^+$ scheme (see [14]). Eq. (17c) is the SIMPLE approximation for quasi 1-D flows.

• $(\varrho v)_i'$ from

$$\begin{aligned}\frac{3}{2\tau}(\varrho v)_i' = &-\frac{S_{i+1/2}}{S_i}\big\{(\varrho v)_i' + \frac{1}{2}\psi_i\big((\varrho v)^k\big)[(\varrho v)_i' - (\varrho v)_{i-1}']\big\}v_{i+1/2}^{\mathrm{T}}\\ &+\frac{S_{i-1/2}}{S_i}\big\{(\varrho v)_{i-1}' + \frac{1}{2}\psi_{i-1}\big((\varrho v)^k\big)[(\varrho v)_{i-1}' - (\varrho v)_{i-2}']\big\}v_{i-1/2}^{\mathrm{T}}\\ &-\frac{1}{\mathrm{M_r^2}S_i}[S_{i+1/2}p_{i+1/2}' - S_{i-1/2}p_{i-1/2}' - p_i'(S_{i+1/2}-S_{i-1/2})]\end{aligned}$$

4.1.4 Updates

• Cell quantities:

$$p_i^{k+1} = p_i^k + p_i' \quad , \quad \varrho_i^{k+1} = \varrho_i^\star \left(1 + \frac{p_i'}{p_i^k}\right) \quad , \quad (\varrho v)_i^{k+1} = (\varrho v)_i^\star + (\varrho v)_i'$$

$$(\varrho E)_i^{k+1} = (\varrho E)_i^\star + \frac{p_i^{k+1}}{\gamma - 1} \quad , \quad (\varrho H)_i^{k+1} = (\varrho E)_i^{k+1} + p_i^{k+1}$$

• Cell-face quantities:
Face pressure and face velocity updated by the AUSM$^+$ scheme (see Eqs. (17b) for the definition of $f_p^\pm$):

$$p_{i+1/2}^{k+1} = f_p^+(M_{\mathrm{L}}^{k+1})p_{\mathrm{L}}^{k+1} + f_p^-(M_{\mathrm{R}}^{k+1})p_{\mathrm{R}}^{k+1}$$

and

$$v_{i+1/2}^{k+1} = c_{i+1/2}^{k+1} M_{i+1/2}^{k+1},$$

where the face sound speed $c_{1/2}$ is defined by:

$$c_{1/2} = \min\{\tilde{c}_{\mathrm{L}}, \tilde{c}_{\mathrm{R}}\},$$

with

$$\tilde{c}_{\mathrm{L}} = (c^\star)^2 / \max\{c^\star, v_{\mathrm{L}}\} \;\; , \;\; \tilde{c}_{\mathrm{R}} = (c^\star)^2 / \max\{c^\star, -v_{\mathrm{R}}\} \;\; , \;\; (c^\star)^2 = \frac{2(\gamma - 1)}{\gamma + 1} H.$$

The face Mach number $M_{i+1/2}$ is given by

$$M_{i+1/2} = f_M^+(M_{\mathrm{L}}) + f_M^-(M_{\mathrm{R}}),$$

where

$$f_M^\pm(m) = \begin{cases} \frac{1}{2}(m \pm |m|) \,, & |m| \geq 1 \\ \pm\frac{1}{4}(m \pm 1)^2 \pm \frac{1}{8}(m^2 - 1)^2 \,, & |m| < 1 \end{cases}$$

4.2 The Rhie-Chow-like interpolation method for quasi 1-D flows

The discretized momentum equation (1b) at the pre-prediction step reads:

$$\begin{aligned} \frac{\mathrm{St_r}}{2\tau}[3(\varrho v)_i^{\star\star} - 4(\varrho v)_i^n + (\varrho v)_i^{n-1}] + \frac{S_{i+1/2}}{S_i}\{(\varrho v)_i^{\star\star} + \frac{1}{2}\psi_i((\varrho v)^k)[(\varrho v)_i^k - (\varrho v)_{i-1}^k]\} v_{i+1/2}^k \\ - \frac{S_{i-1/2}}{S_i}\{(\varrho v)_{i-1}^{\star\star} + \frac{1}{2}\psi_{i-1}((\varrho v)^k)[(\varrho v)_{i-1}^k - (\varrho v)_{i-2}^k]\} v_{i-1/2}^k \\ + \frac{1}{\mathrm{M_r^2} S_i}(S_{i+1/2} p_{i+1/2}^k - S_{i-1/2} p_{i-1/2}^k) = \frac{1}{\mathrm{M_r^2} S_i} p_i^k (S_{i+1/2} - S_{i-1/2}). \end{aligned} \tag{18}$$

Setting

$$A_i = \frac{S_{i+1/2}}{S_i} v^k_{i+1/2}$$

and

$$B_i = -\frac{S_{i+1/2}}{S_i}\{\frac{1}{2}\psi_i((\varrho v)^k)[(\varrho v)^k_i - (\varrho v)^k_{i-1}]\}v^k_{i+1/2} + \frac{S_{i-1/2}}{S_i}\{(\varrho v)^{\star\star}_{i-1} + \frac{1}{2}\psi_{i-1}((\varrho v)^k)[(\varrho v)^k_{i-1} - (\varrho v)^k_{i-2}]\}v^k_{i-1/2}, \quad (19)$$

the momentum equation (18) may be re-written for convenience as

$$B_i = A_i(\varrho v)^{\star\star}_i - \frac{2\mathrm{St_r}}{\tau}(\varrho v)^n_i + \frac{\mathrm{St_r}}{2\tau}(\varrho v)^{n-1}_i + \frac{3\mathrm{St_r}}{2\tau}(\varrho v)^{\star\star}_i + \frac{1}{\mathrm{M_r^2}S_i}(S_{i+1/2}p^k_{i+1/2} - S_{i-1/2}p^k_{i-1/2}) - \frac{1}{\mathrm{M_r^2}S_i}p^k_i(S_{i+1/2} - S_{i-1/2}). \quad (20)$$

The first step for momentum interpolation is to assume that a similar equation holds on the face $i + 1/2$,

$$B_{i+1/2} = A_{i+1/2}(\varrho v)^{\star\star}_{i+1/2} - \frac{2\mathrm{St_r}}{\tau}(\varrho v)^n_{i+1/2} + \frac{\mathrm{St_r}}{2\tau}(\varrho v)^{n-1}_{i+1/2} + \frac{3\mathrm{St_r}}{2\tau}(\varrho v)^{\star\star}_{i+1/2} + \frac{1}{\mathrm{M_r^2}S_{i+1/2}}(S_{i+1}p^k_{i+1} - S_i p^k_i) - \frac{1}{\mathrm{M_r^2}S_{i+1/2}}p^k_{i+1/2}(S_{i+1} - S_i), \quad (21)$$

where $A_{i+1/2}$ and $B_{i+1/2}$ have yet to be defined. Several alternatives for definition of $A_{i+1/2}$ and $B_{i+1/2}$ were studied in our earlier work [18], where unsteady low Mach number flows including acoustic calculation were considered. We observed that the definition of $A_{i+1/2}$ and $B_{i+1/2}$ affects directly the time consistency of the scheme (see also *e.g.* [12, 19]). Two ways of interpolation that ensure that the steady state (if any) does not depend on the time-step were identified: the Rhie-Chow-like interpolation method and the method of interpolation suggested by Lien and Leschziner [13]. In [18] we also recognized that the method of interpolation suggested by Choi [2] is identical to that by Shen *et al.* [22], and that this method gives time-step dependent steady state. Notice that the method proposed by Li and Gu [12] is based on the reformulation of the method by Choi [2] in the time-marching framework. In the present semi-implicit framework, the method by Li and Gu [12] can therefore be identified to that by Choi [2]. In the following, the Rhie-Chow-like interpolation method is presented in the context of quasi one-dimensional flows.

To define the face velocity by momentum interpolation, $A_{i+1/2}$ and $B_{i+1/2}$ in Eq. (21) have to be defined. As recognized in [19] for incompressible flows, and in [18] for low Mach number compressible flows, the method presented for steady problems in the pioneering work of Rhie and Chow [21], is also suitable for solving unsteady problems. This method consists in the linear interpolation of the convective terms of the discretized momentum equation (see Eqs. (20)-(21)), as follows:

$$\frac{B_{i+1/2}}{A_{i+1/2}} = \frac{1}{2}\left(\frac{B_i}{A_i} + \frac{B_{i+1}}{A_{i+1}}\right). \tag{22}$$

Setting

$$a_{i+1/2} = 1 + \frac{1}{A_{i+1/2}} \frac{3\mathrm{St_r}}{2\tau},$$

the mass flux obtained from Eq. (22) is

$$(\varrho v)^{\star\star}_{i+1/2} = \frac{1}{2a_{i+1/2}} \left(\frac{B_i}{A_i} + \frac{B_{i+1}}{A_{i+1}}\right) - \frac{1}{a_{i+1/2}A_{i+1/2}S_{i+1/2}\mathrm{M_r^2}}(S_{i+1}p^k_{i+1} - S_i p^k_i)$$
$$+ \frac{p^k_{i+1/2}}{a_{i+1/2}A_{i+1/2}S_{i+1/2}\mathrm{M_r^2}}(S_{i+1} - S_i) + \frac{\mathrm{St_r}}{2\tau a_{i+1/2}A_{i+1/2}} \left[4(\varrho v)^n_{i+1/2} - (\varrho v)^{n-1}_{i+1/2}\right]. \tag{23}$$

To define $A_{i+1/2}$ in Eq. (23), it is usual to set (see *e.g.* [16])

$$\frac{1}{A_{i+1/2}} = \frac{1}{2}\left(\frac{1}{A_i} + \frac{1}{A_{i+1}}\right). \tag{24}$$

5 Numerical experiments

A five-meter long converging-diverging nozzle is considered. The cross-section area (dimensions in meter) is given by

$$S(x) = \begin{cases} 0.1, & 0 \le x \le 10/28 \\ 0.1\left\{0.4 + 0.6\left[2\left(\frac{x-\frac{55}{28}}{\frac{45}{28}}\right)^2 - \left(\frac{x-\frac{55}{28}}{\frac{45}{28}}\right)^4\right]\right\}, & 10/28 \le x \le 100/28 \\ 0.1, & 100/28 \le x \le 5 \end{cases}$$

A downstream propagating Gaussian acoustic pulse in a five-meter long pipe is generated through a superimposition onto a mean flow, with constant density $\varrho_0 = 1.2046\ \mathrm{kg/m^3}$, velocity $v_0 = 0.030886\ \mathrm{m/s}$ and pressure $p_0 = 101\ 300\ \mathrm{Pa}$, of a perturbation of pressure δp, density $\delta\varrho = \delta p/c_0^2$, and velocity $\delta v = \delta p/(\varrho_0 c_0)$, where $c_0 = \sqrt{\gamma p_0/\varrho_0}$. At $t = 0$,

$$\delta p = 200\ \exp\left[-\frac{(x-0.2)^2}{2\sigma^2}\right]\ \mathrm{Pa},\ \text{where}\quad \sigma = 2\times 10^{-2}\ \mathrm{m}.$$

In this test case, the Mach number of the background flow is 10^{-4}. The time-step is chosen so that the acoustic CFL number is about 5, which is allowed by the semi-implicit algorithm used, see Section 4.1. The grid is uniform with $2\ 500$ cells.

The pulse propagation through the nozzle is presented in Fig. 1. A soon as the pulse is entering the varying nozzle section, its intensity is evolving as a decreasing function of the cross-section area. So, it reaches its highest level at the nozzle throat. Since there exists no dissipation mechanism, this behaviour is simply the signature of the expected constancy of the acoustic power through the different sections of the nozzle.

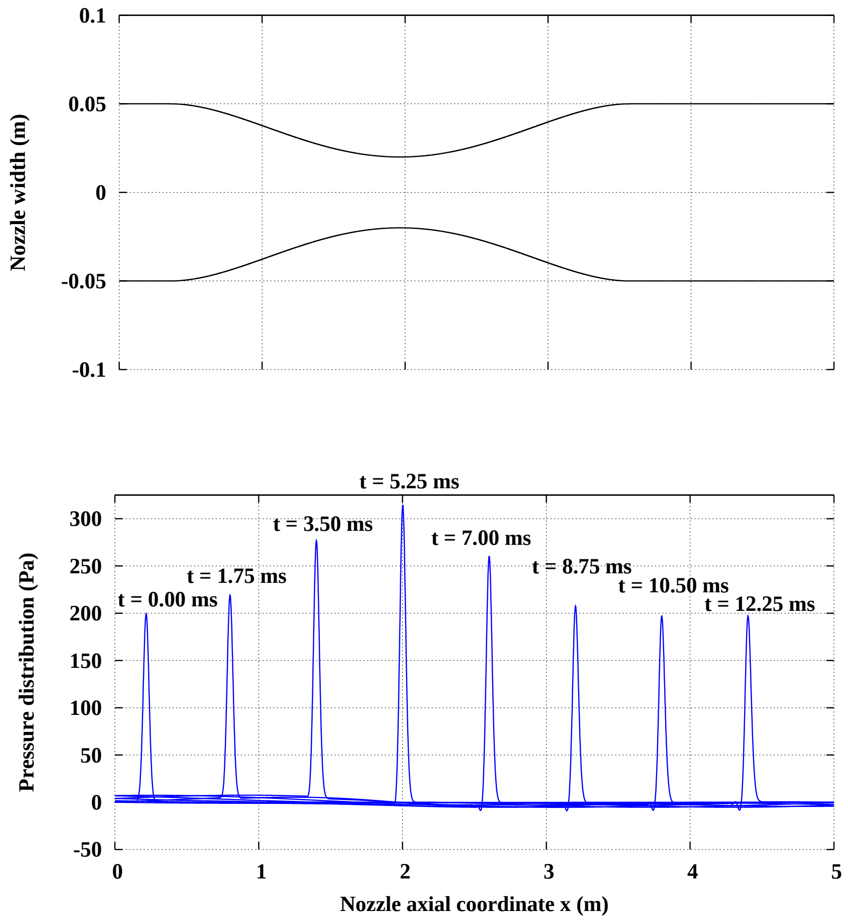

Figure 1: Downstream propagation of an acoustic pulse in a five-meter long nozzle, *cf.* Sec. 5. Pressure distribution at $t = 0.00$ ms, $t = 1.75$ ms, $t = 3.50$ ms, $t = 5.25$ ms, $t = 7.00$ ms, $t = 8.75$ ms, $t = 10.50$ ms and $t = 12.25$ ms.

6 Conlusion

This preliminary study is encouraging. Indeed, it demonstrates that the proposed Rhie-Chow based algorithm for quasi 1-D sound propagation in a nozzle with a low Mach number mean flow is able to produce physically sound results. Since in the present configuration the Mach number is very low, it is expected that the Doppler effect reported by Campos and Lau [1] is absent in the simulated flow configuration. Future work will concentrate on assessing the capability of the proposed algorithm to accurately capture the Doppler shift that must be observed as soon as the Mach number of the backflow is sufficiently increased.

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