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EXACT FORMULAS FOR BENDING OF SANDWICH BEAMS USING THE REFINED ZIGZAG THEORY

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Abstract. The paper presents some exact formulas for the maximum deflection of symmetric beams, subject to different boundary and loading conditions, within the framework of the Refined Zigzag Theory, recently proposed for the analysis of multilayered structures. Considering sandwich-like stacking sequences, these formulas can be simplified further in order to be easily used for practical applications. Based on these results, two studies are conducted: on the contribution of zigzag deformation to the bending of sandwich beams and on the evaluation of shear correction factor based on the Refined Zigzag Theory.

1 INTRODUCTION

Sandwich structures are widely used in several engineering fields due to their high stiffness-to-weight ratio, impact energy absorption and noise reduction. Face-sheets are usually made up of stiff and high-strength materials, metallic or composite. Cores can be cellular (foams), honeycomb or corrugated.

Modeling the mechanical behavior of sandwich structures is quite challenging due to the complex geometry of the core in some cases (honeycomb or corrugated) and to the high transverse anisotropy (faces are usually orders of magnitude stiffer than the core). The use of high-fidelity, three-dimensional finite element models allows successfully addressing both problems but with high computational costs [1]. Beam and plate theories can be used to reduce these costs but an adequate levels of accuracy is not always guaranteed. At first, the core layer has to be substituted with a homogeneous medium with equivalent mechanical properties [2]. Then, particular attention is required for the choice of the beam or plate theory. On one hand, equivalent single layer theories are computationally affordable but provide poor response predictions, in particular in terms of transverse stresses. The First-order Shear Deformation Theory (FSDT) requires the use of shear correction factors to allow calculating accurate global responses (deflection, vibration frequencies, buckling loads) [3]. On the other hand, layerwise theories accurately describe the local and global behavior of sandwich structures but involving a large number of kinematic variables.

An interesting compromise between high accuracy and low computational costs is represented by the Refined Zigzag Theory (RZT) , a recently developed model for the analysis of multilayered composite and sandwich beams and plates [4-6]. The theory is an improvement of FSDT since it takes into account the normal distortion (*zigzag effect*), typical of multilayered structures, through the enrichment of the FSDT in-plane displacement field with *zigzag* additional terms. The through-the-thickness shape of these terms is described by the so-called *zigzag functions* whereas the amplitude of the normal distortion is measured by the *zigzag rotations*, two kinematic variables that are additional with respect to the classical five of FSDT. Several papers have already been published demonstrating the accuracy and computational efficiency of RZT-based analyses. Analytic approaches (of the Navier's type or with the Rayleigh-Ritz method [7-8]) and FEM solutions [9,10] have been used to evaluate the static response, the free vibration modes and the buckling loads of multilayered composite and sandwich structures. Nevertheless, a more practical engineering approach for the solution of the same problems would require simple formulas based on the key material and geometry properties of the structure.

The present work aims at developing, in the framework of RZT, exact-analytic formulas for the static response of beams in planar bending. The focus will be on sandwich-like stacking sequences and on typical boundary and loading conditions. This study will also be the occasion to investigate how the normal distortion (zigzag effect) contributes to the deflection of sandwich beams. Moreover, the derived formulas will be used to obtain an expression for the shear correction factor of the Timoshenko beam theory.

2 THE REFINED ZIGZAG THEORY FOR BEAMS

In this section, the basic assumptions of the RZT for beams are reviewed and the equations necessary for the subsequent investigation are derived. For further details, refer to [4].

Consider a beam of length L, cross-sectional area $A = 2h \times b$ made of N perfectly bonded orthotropic layers; each layer is denoted by the superscript *(k)*. The beam is referred to the Cartesian coordinate system (*x*,*y*,*z*), where $x \in [x_a, x_b]$ is the beam longitudinal axis, and $z \in [-h, h]$ the thickness coordinate. The thickness of the *k*th layer is $2h^{(k)}$. $E_x^{(k)}$ and $G_{xz}^{(k)}$ denote, respectively, the *k*th layer axial and transverse-shear moduli. Only planar deformations in the (*x*,*z*) plane are considered. The displacement field of RZT is

$$u_x^{(k)}(x,z) = u(x) + z\,\theta(x) + \phi^{(k)}(z)\,\psi(x), \qquad u_z^{(k)}(x,z) = w(x) \tag{1}$$

where $u_x^{(k)}$ and u_z are the displacements in the directions of the *x*- and *z*-axis, respectively, the kinematic variables are the uniform axial displacement, $u(x)$, the deflection, $w(x)$, the average cross-sectional (bending) rotation, $\theta(x)$, and the *zigzag* rotation, $\psi(x)$. This variable represents the magnitude of the zigzag displacement, $\phi^{(k)}\,\psi(x)$, which models the cross-sectional distortion. The zigzag function, $\phi^{(k)} \equiv \phi^{(k)}(z, h^{(k)}, G_{xz}^{(k)}, G)$, has units of length, is a piecewise linear, C^0-continuous function of the thickness coordinate and of the stacking sequence. $\phi^{(k)}$ is defined in terms of its layer-interface values, $\phi_{(i)}$ $(i = 0,1,...,N)$, and is linear with the thickness coordinate, z, within the *k*th layer between the two values $\phi_{(k-1)}$ and $\phi_{(k)}$. The interfacial values of $\phi^{(k)}$ are defined as follows

$$\begin{aligned}&\phi_{(k)}=\phi_{(k-1)}+2h^{(k)}\beta^{(k)},\quad \beta^{(k)}\equiv\phi_{,z}^{(k)}=\left(G/G_{xz}^{(k)}-1\right)\quad\left(k=1,...,N\right)\\&\phi_{(0)}=0,\qquad \phi_{(N+1)}=0\end{aligned}\tag{2}$$

where

$$G\equiv\left(\frac{1}{2h}\int_{-h}^{h}\frac{dz}{G_{xz}^{(k)}}\right)^{-1}=\left(\frac{1}{2h}\sum_{k=1}^{N}\frac{2h^{(k)}}{G_{xz}^{(k)}}\right)^{-1}\tag{3}$$

represents a weighted-average transverse shear modulus of the total laminate. The complete derivation of Eqs. (2) and (3) can be found in [4]. The definition of $\phi^{(k)}$ can be extended in order to be also valid for some particular cases. For homogeneous single-layer beams, $\phi^{(k)}$ would vanish according to Eqs. (2) and (3), thus leading to the Timoshenko beam theory but, by adopting the Homogeneous-Limit Modelling [6], the zigzag function is modified in order to make RZT capable of predicting highly accurate response quantities including the strains and stresses. Moreover, the zigzag function may be naturally and accurately modified for the case of laminates having external layers with low transverse shear moduli [8].
The linear strain-displacement relations yield the strain field of RZT

$$\varepsilon_x^{(k)}(x,z)=u_{,x}+z\,\theta_{,x}+\phi^{(k)}\,\psi_{,x},\qquad \gamma_{xz}^{(k)}(x,z)=w_{,x}+\theta+\beta^{(k)}\,\psi\tag{4}$$

If each layer is orthotropic with the orthotropy axes aligned with the Cartesian axes, the beam exhibits a plane-stress behavior in the (x,z) plane, and the transverse normal stress $\sigma_z^{(k)}$ is smaller than the axial and transverse shear stresses, the kth-layer constitutive relations are

$$\sigma_x^{(k)}=E_x^{(k)}\varepsilon_x^{(k)},\qquad \tau_{xz}^{(k)}=G_{xz}^{(k)}\gamma_{xz}^{(k)}\tag{5}$$

The beam is subject to distributed transverse load, $q^b(x)$ and $q^t(x)$, (units of force/length) applied at the bottom and top beam surfaces. The end cross-sections are subject to the prescribed axial (T_{xa}, T_{xb}) and transverse shear (T_{za}, T_{zb}) tractions. Equilibrium equations of the beam according to RZT can be obtained using the Principle of Virtual Works (PVW) [4]

$$N_{x,x}=0,\qquad V_{x,x}=-q,\qquad M_{x,x}-V_x=0,\qquad M_{\phi,x}-V_\phi=0\tag{6}$$

where $q\equiv q^b+q^t$ and

$$\left[N_x,M_x,M_\phi,V_x,V_\phi\right]\equiv\int_A\left[\sigma_x^{(k)},z\,\sigma_x^{(k)},\phi^{(k)}\sigma_x^{(k)},\tau_{xz}^{(k)},\beta^{(k)}\tau_{xz}^{(k)}\right]dA\tag{7}$$

are the stress resultants. Consistent boundary conditions, also obtained with the PVW, read as

$$\left.\begin{aligned}u(x_\alpha)=\bar u_\alpha\ \text{or}\ N_x(x_\alpha)=\bar N_{x\alpha},\qquad \theta(x_\alpha)=\bar\theta_\alpha\ \text{or}\ M_x(x_\alpha)=\bar M_{x\alpha}\\ w(x_\alpha)=\bar w_\alpha\ \text{or}\ V_x(x_\alpha)=\bar V_{x\alpha},\qquad \psi(x_\alpha)=\bar\psi_\alpha\ \text{or}\ M_\phi(x_\alpha)=\bar M_{\phi\alpha}\end{aligned}\right\}\quad(\alpha=a,b)\tag{8}$$

where

$$\left[\bar N_{x\alpha},\bar M_{x\alpha},\bar M_{\phi\alpha},\bar V_{x\alpha}\right]\equiv\int_A\left[T_{x\alpha},zT_{x\alpha},\phi^{(k)}T_{x\alpha},T_{z\alpha}\right]dA\quad(\alpha=a,b)\tag{9}$$

are the prescribed-stress resultants at the beam ends. The constitutive equations, expressing the relation between stress resultants and derivatives of the kinematic unknowns, are

$$\begin{Bmatrix} N_x \\ M_x \\ M_\phi \end{Bmatrix} = \begin{bmatrix} A_{11} & B_{12} & B_{13} \\ B_{12} & D_{11} & D_{12} \\ B_{13} & D_{12} & D_{22} \end{bmatrix} \begin{Bmatrix} u_{,x} \\ \theta_{,x} \\ \psi_{,x} \end{Bmatrix}, \qquad \begin{Bmatrix} V_x \\ V_\phi \end{Bmatrix} = \begin{bmatrix} \overline{G}A & \left(G-\overline{G}\right)A \\ \left(G-\overline{G}\right)A & \left(\overline{G}-G\right)A \end{bmatrix} \begin{Bmatrix} w_{,x}+\theta \\ \psi \end{Bmatrix} \tag{10}$$

where the stiffness coefficients are defined as

$$\left(A_{11}, B_{12}, D_{11}\right) \equiv \int_A E_x^{(k)}\left(1, z, z^2\right) dA, \quad \left(B_{13}, D_{12}, D_{22}\right) \equiv \int_A E_x^{(k)} \phi^{(k)}\left(1, z, \phi^{(k)}\right) dA, \quad \overline{G} \equiv \frac{1}{2h}\int_{-h}^{h} G_{xz}^{(k)} dz \tag{11}$$

By substituting Eqs. (10) into Eqs. (6), the equilibrium equations expressed in terms of the kinematic variables can be written

$$\begin{aligned} A_{11}u_{,xx} + B_{12}\theta_{,xx} + B_{13}\psi_{,xx} &= 0 \\ \overline{G}A\left(w_{,xx}+\theta_{,x}\right) + \left(G-\overline{G}\right)A\psi_{,x} &= -q \\ B_{12}u_{,xx} + D_{11}\theta_{,xx} + D_{12}\psi_{,xx} - \overline{G}A\left(w_{,x}+\theta\right) - \left(G-\overline{G}\right)A\psi &= 0 \\ B_{13}u_{,xx} + D_{12}\theta_{,xx} + D_{22}\psi_{,xx} - \left(G-\overline{G}\right)A\left(w_{,x}+\theta\right) - \left(\overline{G}-G\right)A\psi &= 0 \end{aligned} \tag{12}$$

3 EXACT FORMULAS FOR BEAMS DEFLECTION

Solution of Eqs. (12), subject to boundary conditions (8), depends on the applied loads.

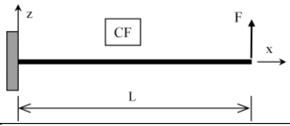
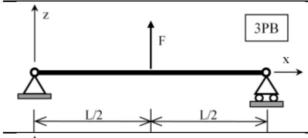
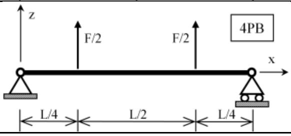
Let us consider first a beam subject to concentrated forces and moments only ($q(x)=0$). A general solution of Eqs. (12) can be found in the following form

$$\begin{aligned} u(x) &= \left(-C_8 + C_3C_7 - \frac{C_2C_7}{R^2D_{11}^*}\right)\left(a_1\cosh(Rx) + a_2\sinh(Rx)\right) - \frac{C_2C_7a_3}{2D_{11}^*}x^2 + a_6x + a_7 \\ w(x) &= \left[\frac{C_3}{R} - \frac{C_2}{R^3D_{11}^*} + \frac{1}{R}\left(\frac{C_2C_5}{D_{11}^*} - C_4\right) + R\left(C_6 - C_3C_5\right)\right]\left(a_1\sinh(Rx) + a_2\cosh(Rx)\right) + \\ &\quad - \frac{C_2a_3}{6D_{11}^*}x^3 - \frac{a_4}{2}x^2 + \left[\left(\frac{C_2C_5}{D_{11}^*} - C_4\right)a_3 - a_5\right]x + a_8 \\ \theta(x) &= \left(-C_3 + \frac{C_2}{R^2D_{11}^*}\right)\left(a_1\cosh(Rx) + a_2\sinh(Rx)\right) + \frac{C_2a_3}{2D_{11}^*}x^2 + a_4x + a_5 \\ \psi(x) &= a_1\cosh(Rx) + a_2\sinh(Rx) + a_3 \end{aligned} \tag{13}$$

where $C_i\ \left(i=1,...,8\right)$, D_{11}^* and R are functions of the stiffness coefficients defined in Eqs. (11) whereas the $a_i\ (i=1,...,8)$ unknown constants are determined from the boundary conditions, Eqs. (8) (refer to [4] for further details). Solution (13) is valid not only when forces and moments are applied to the beam ends but also when there are concentrated loads at n internal points within the beam span. In the latter case, for each of the $(n+1)$ beam segments between concentrated loads, the exact solution is in the form of Eqs. (13), thus $8(n+1)$ unknown constants a_i have to be determined: 8 conditions will be obtained at the beam ends (Eqs. (8)), the remaining $8n$ conditions will enforce continuity of the kinematic variables (u, w, θ, ψ) and continuity (or jumps) of the resultant forces and moments (N_x, V_x, M_x, M_ϕ) at each of the n internal points.

We consider three example problems that are important for practical applications (see Table 1). In particular, the three-point-bending and four-point-bending loading schemes are widely adopted for the experimental characterization of sandwich beams [11].

Table 1: Loading and boundary conditions of the considered example problems with concentrated loads.

Geometry, loading and boundary conditions	Boundary conditions at x=0	Boundary conditions at internal points	Boundary conditions at x=L
	$u=w=\theta=\psi=0$	/	$N_x=M_x=M_\phi=0$ $V_x=F$
	$w=N_x=M_x=M_\phi=0$	$u, w, \theta, \psi, N_x, M_x, M_\phi$ continuous, V_x with a jump (F)	$w=N_x=M_x=M_\phi=0$
	$w=N_x=M_x=M_\phi=0$	$u, w, \theta, \psi, N_x, M_x, M_\phi$ continuous, V_x with a jump ($F/2$)	$w=N_x=M_x=M_\phi=0$

Application of the solving procedure described above leads, for the case of symmetric stacking sequence ($B_{12}=B_{13}=0$), to the following expression of the maximum deflection

$$w_{\max} = \alpha\frac{FL^3}{D_{11}} + \beta\frac{FL}{\overline{G}A} + \frac{F2h}{GA}\chi\zeta\left(RL\right) \tag{14}$$

where α and β are numeric coefficients, χ is a coefficient depending on the stacking sequence and ζ is a function of the stacking sequence and of the length of the beam, L, through the product RL. For symmetric laminations, χ and R are defined as follows

$$\chi \equiv \left[\left(\frac{\overline{G}-G}{G}\right)+\left(\frac{D_{12}}{D_{11}}\right)\right]^2 \Big/ \left(\frac{\overline{G}-G}{G}\right)R2h, \qquad R \equiv \sqrt{AG\left(G-\overline{G}\right)D_{11}\Big/\overline{G}\left(D_{12}^2-D_{11}D_{22}\right)} \tag{15}$$

α, β and ζ (Eq. (14)) are different for the considered problems, see Table 2.

Table 2: Coefficients α and β and function ζ for problems CF, 3PB and 4PB (Table 1). Simplified expressions for ζ are reported for the RL values that guarantee errors below 1% with respect to the exact definition.

	α	β	ζ	Simplified expression for ζ	
CF	1/3	1	$RL-\tanh\left(RL\right)$	$RL-1$ (RL>3)	RL (RL>100)
3PB	1/48	1/4	$RL/4-1/2\ \tanh\left(RL/2\right)$	$RL/4-1/2$ (RL>3)	$RL/4$ (RL>200)
4PB	11/768	1/8	$RL/8-e^{RL/4}\left(e^{RL/2}-1\right)\big/2\left(e^{RL}+1\right)$	$RL/8$ (RL>14)	

When $q(x)$ is the only external load, there is no general solution for the bending problem; depending on the boundary conditions and on the function $q(x)$, an exact analytic expression for the kinematic variables could even not exist. The classical case of a simply supported beam with sinusoidal transverse load $q(x) = q_0 \sin(\pi x / L)$ is here considered. Boundary conditions on both the beam ends prescribe that $w = N_x = M_x = M_\phi = 0$. An exact solution for Eqs. (12) can be found in the trigonometric form

$$w(x) = w_0 \sin(\pi x / L), \qquad (u(x), \;\; \theta(x), \;\; \psi(x)) = (u_0, \;\; \theta_0, \;\; \psi_0)\cos(\pi x / L) \tag{16}$$

Substituting Eqs. (16) into Eqs. (12), a set of algebraic equations is obtained that provides the amplitudes w_0, u_0, θ_0, and ψ_0 [4]. After some further manipulations, the maximum deflection of the beam (for symmetric stacking sequence) is found to be

$$w_{\max} = w_0 = \frac{q_0 L^4}{\pi^4 D_{11}} + \frac{q_0 L^2}{\pi^2 \overline{G}A} + \frac{q_0 L^2}{\pi^2 GA} \chi R 2h \frac{(RL)^2}{(RL)^2 + \pi^2} \tag{17}$$

Some observations can be made on the exact formulas for maximum deflection of symmetric beams obtained using the Refined Zigzag Theory, Eqs. (14) and (17).

- In Eqs. (14) and (17), the first deflection term is the one due to bending according to Bernoulli-Euler beam theory, the second term is the additional contribution due to transverse shear deformability according to Timoshenko beam theory ($\overline{G}A$ is the transverse shear stiffness of the beam), thus the third term represents the further deflection caused by the so-called zigzag effect modeled by RZT.
- For both bending and shear contributions to deflection, the dependency on stacking sequence and the dependency on the beam length can be easily distinguished whereas the zigzag additional term has a more complex structure. In particular, this term does not depend simply on the length, L, but on a non-dimensional beam length, RL, where R is a stacking-sequence related coefficient (Eqs. (15)).
- For very short beams subjected to concentrated loads (Eq. (14)), the bending deflection contribution ($\propto L^3$) in negligible with respect to the shear one ($\propto L$). The zigzag contribution is also negligible, since any ζ function is $\propto L^3$ for small L (Table 2). Similarly, for the beam subject to distributed sinusoidal load (Eq. (17)), the bending and the zigzag contributions are $\propto L^4$ in the limit for short beams and are thus negligible with respect to the shear deflection that is $\propto L^2$.
- When considering long beams, it is the bending deflection term to prevail. This contribution is $\propto L^3$ for the case of concentrated loads (Eq. (14)), $\propto L^4$ for distributed loads (Eq. (17)). Both transverse shear and zigzag terms can be neglected being $\propto L$ or $\propto L^2$ for concentrated and distributed loads, respectively.
- The use of Eqs. (14) and (17) involves some stiffness coefficients of the RZT (in particular G, D_{12}, D_{22}) that can be somehow difficult to calculate whereas both D_{11} and $\overline{G}A$ are, respectively, the usual bending and transverse shear stiffness coefficients.

Taking into account these observations, the problem of sandwich beams will be considered in the next Section in order to derive easier-to-use formulas for the maximum deflection and to investigate the importance of the zigzag effect at different beam-slenderness regimes.

3 SIMPLIFIED FORMULAS FOR SANDWICH BEAMS DEFLECTION

A symmetric sandwich beam is considered. Each face has thickness h_F and is made of an isotropic material (E_F is the Young modulus and G_F the shear modulus), the core has thickness h_C, Young modulus E_C and shear modulus G_C. The following ratios are introduced

$$r_E \equiv E_C/E_F\,, \qquad r_G \equiv G_C/G_F\,, \qquad r_h \equiv h_F/h_C\,, \qquad \lambda \equiv L/2h \tag{18}$$

For common sandwich structures, the following assumptions can be made

$$r_E \le 1/100, \qquad r_G \le 1/100 \tag{19}$$

whereas no limitations are considered for r_h, thus including in the analysis sandwich beams with thick faces. Using Eqs. (18) and (19), the following simplified expressions are obtained in order to compute the zigzag contribution to maximum deflection in Eqs. (14) and (17)

$$\begin{gathered} R2h \simeq R_0\left(1+2r_h\right), \quad RL \simeq R_0\left(1+2r_h\right)\lambda, \quad \chi \simeq \frac{2r_h}{R_0\left(1+2r_h\right)\left(2r_h+r_G\right)}\left(\frac{12r_h^3+18r_h^2+6r_h+r_E}{8r_h^3+12r_h^2+6r_h+r_E}\right)^2 \\ G \simeq G_F r_G\left(1+2r_h\right), \quad R_0 \simeq \sqrt{6\frac{G_F}{E_F}}\frac{1}{r_h}\sqrt{r_G\frac{\left(8r_h^3+12r_h^2+6r_h+r_E\right)}{\left(3r_h+2r_E\right)\left(2r_h+r_G\right)}} \end{gathered} \tag{20}$$

The use of Eqs. (14) and (17) together with Eqs. (20) provides the maximum deflection for the considered sandwich beams expressed in terms of known or easy-to-calculate quantities.

Using these simplified relations, it is also possible to investigate the importance of the zigzag contribution to the maximum deflection at different slenderness regimes. Let us consider, for example, sandwich beams subjected to concentrated loads. Eq. (14) can be simplified as follows in order to highlight the effect of the slenderness ratio, λ

$$w_{\max} = \alpha\frac{F}{bD_0}\lambda^3 + \beta\frac{F}{b\overline{G}}\lambda + \frac{F}{bG}\chi\zeta(\lambda) \tag{21}$$

where $D_0 \equiv D_{11}\Big/\left(b\left(2h\right)^3\right)$. The effect of the material and geometry ratios (r_E, r_G, r_h and λ) on the maximum deflection will be now investigated. Further to Eqs. (19), the following conditions are considered in order to examine realistic cases

$$1/10000 \le r_E = r_G \le 1/100, \qquad 1/100 \le r_h \le 1, \qquad 8 \le \lambda \le 100 \tag{22}$$

In particular, it is supposed that r_E=r_G and problems with λ<8 are not considered since for very short sandwich beams transverse normal deformability (not modeled by RZT) is not negligible [8]. Figures 1-3 illustrate some results for the 3PB problem at different values of λ. BE, SH, and ZZ are the percent contribution due to bending, shear, and zigzag deformation to the maximum deflection, respectively (see Eq. (21)). W^* is the non dimensional maximum deflection, defined as $w_{\max}\Big/\left(FL^3/E_F b\left(2h\right)^3\right)$. The contribution due to shear deformation is below 1% for λ>26 whereas, for the considered range of λ, the main effects are the bending and the zigzag one. The latter is particular important for sandwich beams with high values of

r_h and small values of r_E=r_G (thick faces much stiffer than the core) and does not vanish even if considering really slender beams (λ=100) for which ZZ can reach 60%.

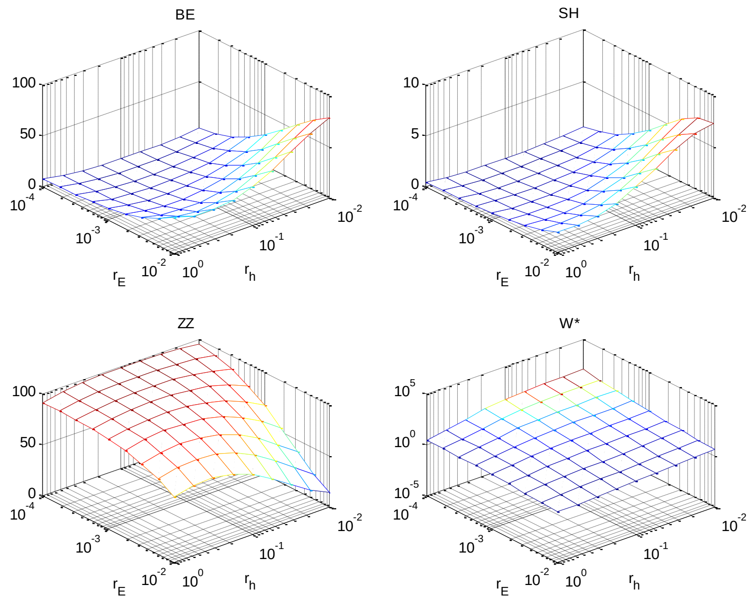


Figure1: Problem 3PB, sandwich-like stacking sequence. Bending, shear, and zigzag contribution to the maximum deflection and non-dimensional maximum deflection for λ=8.

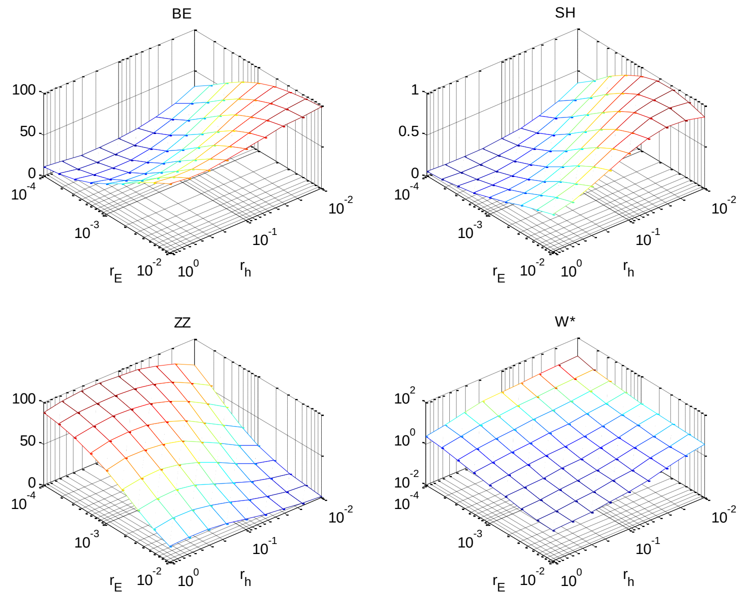


Figure2: Problem 3PB, sandwich-like stacking sequence. Bending, shear, and zigzag contribution to the maximum deflection and non-dimensional maximum deflection for λ=26.

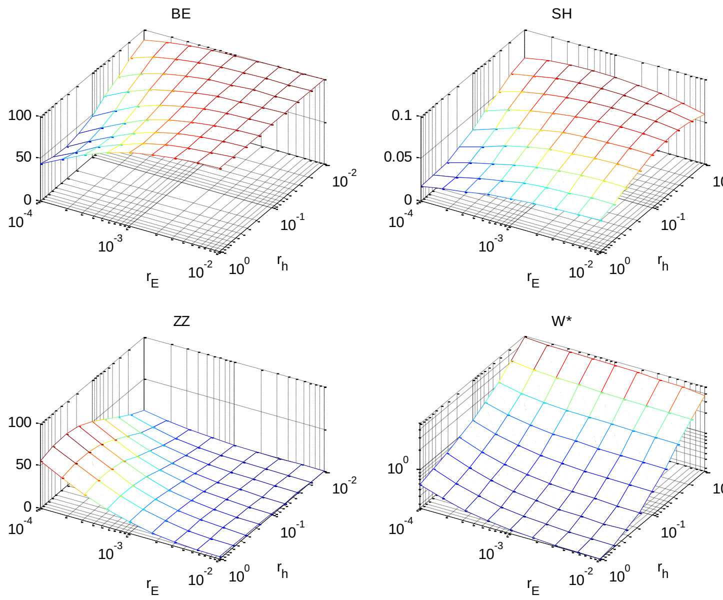


Figure 3: Problem 3PB, sandwich-like stacking sequence. Bending, shear, and zigzag contribution to the maximum deflection and non-dimensional maximum deflection for λ=100.

Similar results are found for the 4PB and CF problems: the minimum value of λ for which SH<1% is 22 and 13, respectively, whereas ZZ at λ=100 reaches 55% and 30%, respectively.

Another important application of Eqs. (14), (17) and (20) is the evaluation of the shear correction factor for sandwich beams, k^2, by using the Refined Zigzag Theory. The definition of k^2 is here based on the equivalence between the maximum deflection computed using the RZT and the maximum deflection computed by Timoshenko beam theory with a corrected transverse shear stiffness. Considering again the case of concentrated loads (Eq. (21)),

$$\alpha\frac{F}{bD_0}\lambda^3+\beta\frac{F}{b\overline{\overline{G}}}\lambda+\frac{F}{bG}\chi\zeta(\lambda)=\alpha\frac{F}{bD_0}\lambda^3+\beta\frac{F}{k^2b\overline{\overline{G}}}\lambda \tag{23}$$

that yields

$$k^2=\left(1+\frac{\overline{\overline{G}}\chi}{\beta G}\frac{\zeta(\lambda)}{\lambda}\right)^{-1} \tag{24}$$

Eq. (24) reveals that the shear correction factor based on RZT depends not only on the stacking sequence (through coefficients $\overline{\overline{G}}$, G and χ and function ζ) but also on the boundary and loading conditions (through coefficient β, see Table 2) and on the slenderness ratio λ. For example, Figure 4 shows the shear correction factor for the three-point bending problem and for different values of λ (Eqs. (22) are still valid). For high values of r_h and low values of r_E, k^2 can change of about an order of magnitude in the considered range of λ whereas, when r_h is low and r_E is high, k^2 seems to be independent of the slenderness ratio.

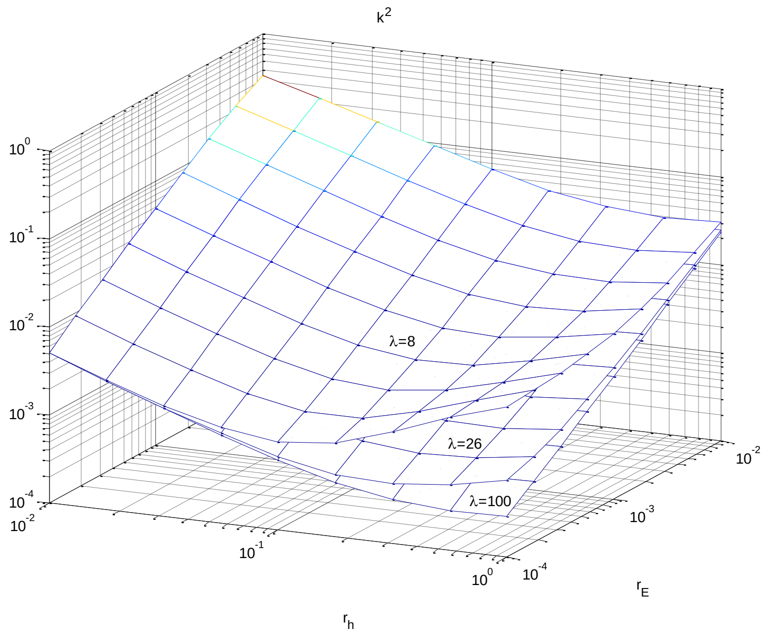


Figure 4: Problem 3PB, sandwich-like stacking sequence. Shear correction factor for different values of λ.

This result contradicts those definitions of the shear correction factor that only depend on the stacking sequence [3].

An interesting case is the one for which *RL* is high. *RL* is a non-dimensional beam length that appears in the definition of ζ (Table 2). Eqs. (20) show that *RL* is proportional to the beam slenderness, λ, through the coefficient $R_0\left(1+2r_h\right)$ that only depends on the stacking sequence. This coefficient is shown in Figure 5 as a function of r_h and r_E: when r_h is small (and especially when r_E is high), $R_0\left(1+2r_h\right)$ exhibits values that are three orders of magnitude higher than those in the rest of the considered (r_h,r_E) domain. This means that, for a fixed λ, *RL* is higher for low values of r_h and high values of r_E.

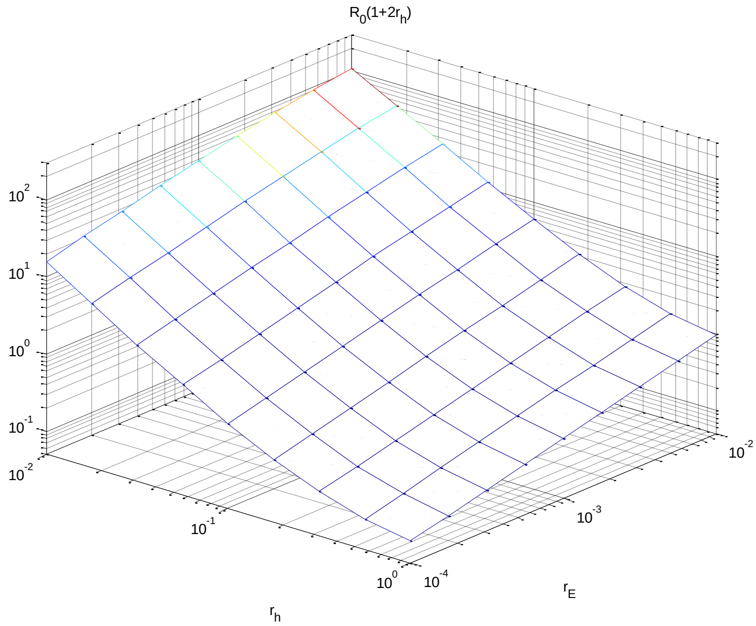


Figure 5: Sandwich-like stacking sequence. Coefficient $R_0\left(1+2r_h\right)$.

When *RL* is high, the function ζ can be substituted by simplified expressions, in particular those reported in Table 2 that are proportional to *RL*. As a consequence, the expression of k^2, Eq. (24), can be greatly simplified

$$k^2 = \left(1+2\frac{r_h}{r_G}\left(\frac{12r_h^3+18r_h^2+6r_h+r_E}{\left(8r_h^3+12r_h^2+6r_h+r_E\right)\left(1+2r_h\right)}\right)^2\right)^{-1} \tag{25}$$

This definition depends on the stacking sequence only since the dependency on boundary conditions, loads and beam slenderness ratio cancels out using the simplified expressions of ζ. In other words, when r_h is low and r_E is high, *RL* is high and the shear correction factor depends on the stacking sequence only, as shown by Eq. (25) and by Figure 4. A similar discussion is valid for the case of distributed loads: when *RL* is high, the same expression for the shear correction factor is found, Eq. (25).

6 CONCLUSIONS

In this paper, the Refined Zigzag Theory (RZT) has been used to investigate the static response of sandwich beams to transverse loads. The focus is on the derivation of simplified and easy-to-use formulas for the maximum deflection of symmetrically laminated sandwich beams under different boundary and loading conditions.

The basic definitions and relations of RZT have been reviewed in order to set the framework for the present study. Exact solutions of the equilibrium equations have been derived for simply supported and cantilevered beams with symmetric stacking sequence and subjected to concentrated and distributed transverse loads. In particular, explicit formulas for the maximum deflection have been obtained. When considering sandwich-like stacking sequences, these formulas can be simplified in order to be used easily for practical applications.

Based on the simplified formulas, the effect of zigzag deformation on the maximum deflection of sandwich beams has been studied for different geometry and material properties (face-to-core thickness and stiffness ratios and beam slenderness). The numerical results have shown that, also for beams with high length-to-thickness ratios, the zigzag contribution to the beam deflection can be relevant, especially for sandwich beams with thick faces and a high face-to-core stiffness ratio. Moreover, an expression for the shear correction factor of the Timoshenko beam theory has been derived for sandwich beams, showing that the correction does not depend only on the stacking sequence but also on boundary/loading conditions and on the beam slenderness.

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