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DYNAMIC ANALYSIS OF MODERATELY THICK DOUBLY CURVED SHELLS VIA EFFICIENT 3D ELEMENTS

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Abstract. The dynamic study of doubly curved shells is a subject of undoubted interest. Among them, there are a lot of shell structures with the shape of hyperbolic paraboloid, elliptic paraboloid, or velaroidal shells; however, the dynamic analysis of this kind of structure is rather limited. Most studies refer to shallow and thin shells. In this work, following the theoretical approach to the problem, we perform a study of the vibrational frequencies of transverse oscillations of moderately thick shells by 3D elements. We formulate a close variant of the 20-nodes serendipity element and we apply it to the dynamic study of these kinds of structures. We also discuss the different kinds of mass matrices, lumped and consistent mass matrices, proposing a variant of the classical lumped one.

1 INTRODUCTION

Vibration of shell structures is an interesting topic that has been manifested in many aspects of civil engineering and aeronautics. Shell structures have the ability to withstand significant loads precisely because of their curvature, which causes the membrane stresses tangent to the middle surface to be responsible for providing additional resistance, differentiating them from other structural elements such as plates.

Among the shell structures, the doubly curved ones, such as the hyperbolic paraboloid, the elliptic paraboloid, and the two-sheets paraboloid, are significant structures in the field of civil engineering. However, it is generally not possible to obtain the closed form solutions of the equilibrium equations of shells in a general curvilinear coordinate system including shear deformation; instead, approximate methods are used [1].

In this paper, we address the dynamic study of doubly curved shells such as the elliptic paraboloid and the hyperbolic paraboloid. These kinds of surfaces cannot be formulated along orthogonal coordinate lines and they have not been studied analytically or with approximate methods recently. These surfaces have been used in several constructions in civil engineering [2], and therefore they are of significant practical interest.

The treaties of Leissa [3] and Soedel [4] are the first complete references to the dynamic analysis of plates and shells. These references study analytical solutions in very particular cases of geometry and loading conditions. Except in the annexes of this work, the Finite Element Method (FEM) is not used.

FEM has been and is a very powerful tool for solving differential or integral equations in the context the structural calculation [5]. The work of Yang [6] shows the number of variants and techniques that have been developed since its inception in the 1960s to improve the behaviour of the various elements. The dynamic study of doubly curved shells is of particular interest because of their complex geometry, which cannot be parameterized along the lines of curvature. Studies of these shells using the FEM approach have been collected in various works such as Liew [7,8] and Stavridis [9,10] who use degenerated shell elements with several enhancements to avoid locking phenomenon but usually consider shallow shells and neglect shear deformation.

The reappearance of the 3D finite element in structural analysis is relatively recent and may come, in part, coupled with the large computing power of today's computers.

Advantages of these types of elements compared to the most common degenerated shell elements, originally proposed by Ahmad [11], include the possibility of using 3D constitutive laws and the fact that no kinematic assumption is used in deriving strain-displacement relations, whereas the developments of some or other terms are usually dispensed with, sometimes without much foundation, in most shell theories.

However, most of the publications concerning 3D elements for the study of shells consider elements of eight nodes. There have been many studies that attempt to relieve locking (membrane, shear, and trapezoidal locking) that appears in their formulation. See for example the work of Schwarze [12], which has an interesting description of this type of element.

In this paper we have preferred to use high order finite elements to address the problem. One advantage of higher order elements compared to lower order ones is that the appearance of the different types of locking is much less significant. In fact, the computational efficiency of low-order elements is achieved by countless improvements to avoid zero energy modes by means of stabilization methods, hybrid models, and so on. Specifically we have tested the 20-node finite element (in its dynamic formulation),for which we have only found references to the classic formulation.

Next we will present the equations defining the problem and its numerical approximation, showing some examples of interest.

2 THEORETICAL APPROACH

It is desirable that analytical solutions may be found to any problem that may arise. For this purpose, in a structural problem, we must solve the equilibrium equations. In our case, tensor equations that define the problem are [13]:

$$\tilde{n}^{\alpha\beta}\,|_{\alpha} - \left(b_{\gamma}^{\beta} m^{\gamma\alpha}\right)|_{\alpha} - b_{\alpha}^{\beta} q^{\alpha} + p^{\beta} = \rho h \ddot{v}^{\beta} \quad ,\rho = 1,2. \tag{1}$$

$$b_{\alpha\beta}\tilde{n}^{\alpha\beta} - b_{\alpha\beta} b_{\gamma}^{\beta} m^{\gamma\alpha} + q^{\alpha}|_{\alpha} + p^{3} = \rho h \ddot{v}_{3}$$

$$m^{\alpha\rho}|_{\alpha} - q^{\rho} + c^{\rho} = \rho \frac{h^{3}}{12} \ddot{w}^{\rho} \quad ,\rho = 1,2.$$

where $\tilde{n}^{\alpha\rho}$are the resultant membrane effective stresses,$b_{\alpha\beta}$andb_{γ}^{ρ} are the components of the curvature tensor in covariant and contra-covariant form referred to the medium surface, $q^{\alpha}$are the resultant generalized shear stresses, $p^{\beta}$are the components of load force vectors,$m^{\alpha\rho}$are the components of the moment tensor, and v and w are the deflections and the rotations respectively. The vertical bar represents the covariant derivative with respect to the tangent vectors of the middle surface of the shell, which are not necessarily orthogonal.

Let us remember that the relationship between effective stresses (pseudo stresses) and resultant generalized stresses, which are not necessarily symmetrical , is obtained through the curvature tensor. See for example [14].

Expressing these equations in function of the real resultant stresses and strains and neglecting the products $b_{\gamma}^{\rho} m^{\gamma\alpha}$ and $b_{\alpha\beta} b_{\gamma}^{\beta}$, without are not the most general equations, we obtain:

$$N_{,\alpha}^{\alpha\beta} + \Gamma_{\alpha\rho}^{\beta} N^{\alpha\rho} - b_{\alpha}^{\beta} q^{\alpha} + p^{\beta} = \rho h \ddot{v}^{\beta} ,$$
$$N^{\alpha\beta} b_{\alpha\beta} + q_{,\alpha}^{\alpha} + p = \rho h \ddot{v}_3,$$
$$M_{,\alpha}^{\alpha\beta} + \Gamma_{\alpha\beta}^{\beta} M^{\alpha\beta} - q^{\beta} = \rho \frac{h^3}{12} \ddot{w}^{\beta},$$
$$\varepsilon_{\alpha\beta}\left(N^{\alpha\beta} - b_{\rho}^{\alpha} M^{\rho\beta}\right) = 0. \tag{2}$$

These are the developed equilibrium equations of a shell element. The last of these equations, which we have not developed, is the condition of symmetry of the so-called effective stresses. We see clearly that generalized stresses are not symmetrical.

If we wanted to solve a real physical problem, we would have to express these equations in terms of their physical components. For a second order tensor, the following is satisfied:

$$\mathcal{N}^{\alpha\beta} = \sqrt{\frac{a_{\beta\beta}}{a^{\alpha\alpha}}} N^{\alpha\beta}, \tag{3}$$

where $\mathcal{N}^{\alpha\beta}$are the physical components of the tensor $N^{\alpha\beta}$. So we would have to compute the derivatives of $\sqrt{\frac{a^{\alpha\alpha}}{a_{\beta\beta}}}$ and the quantities $\sqrt{\frac{a^{\alpha\alpha}}{a_{\beta\beta}}}$ in order to obtain the differential equations that define the physical problem.

Let us note that we have a system of five equations but eight unknowns; we must emphasize that the study of the equilibrium of a shell element involving bending stresses is an indeterminate problem not solvable by equations of statics.

We must turn to the constitutive relations to raise these equations in terms of pure displacements and establish the corresponding compatibility equations.

The final system is a system of partial derivatives with eight equations and eight unknowns with a high degree of non-linearity and is therefore not resolvable by elemental techniques. In a forthcoming work we will study the feasibility of the method of finite differences.

3 PROBLEM FORMULATION USING 3D FINITE ELEMENTS. STIFFNESS MATRIX

For the reasons set out above, the 20-nodes serendipity element has been chosen for the dynamic study of doubly curved shells. To define the shape functions, the isoparametric formulation is used in a coordinate system with the origin at the center of the element (see Figure 1).

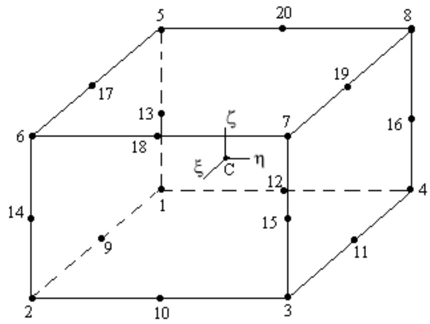

Figure 1: Twenty-nodes serendipity finite element

In a compact form these equations can be written in the well-known form:
For corner nodes, j=1,…,8,

$$N_j = \frac{1}{8}(1+\xi_j\xi)(1+\eta_j\eta)(1+\zeta_j\zeta)(\xi_j\xi+\eta_j\eta+\zeta_j\zeta-2). \tag{4}$$

For corner nodes, j=1,…,8,

$$N_j = \frac{1}{4}(1-\xi^2)(1+\eta_j\eta)(1+\zeta_j\zeta). \tag{5}$$

For mid-sides nodes, j=10,12,14,16,

$$N_j = \frac{1}{4}(1-\eta^2)(1+\xi_j\xi)(1+\zeta_j\zeta). \tag{6}$$

For mid-sides nodes, j=9,11,13,15,

$$N_j = \frac{1}{4}(1-\zeta^2)(1+\xi_j\xi)(1+\eta_j\eta), \tag{7}$$

for mid-sides nodes j=17,18,19,20.

Using FEM, we have at least two coordinate systems: a Cartesian coordinate system $(x\ \ ,y\ \ ,z\ \ )$ and other isoparametric system (ξ,η,ζ).

To solve the dynamic problem we have to find the stiffness matrix and the mass matrix of the element.

In matrix form we have to solve

$$|\mathbf{K}-\omega^2\mathbf{M}| = 0, \tag{8}$$

where K is the stiffness matrix and M the mass matrix.

The stiffness matrix $[k_e]$ of the element in local coordinates can be expressed as:

$$[\boldsymbol{k_e}] = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1}[\boldsymbol{B}]^T[\boldsymbol{D}][\boldsymbol{B}]det[\boldsymbol{J}]d\xi d\eta d\,\zeta\ , \tag{9}$$

where the matrix B is the relation between strains and displacements. $det[J]$ is the determinant of the Jacobian matrix and D is the constitutive matrix.

The relationship between strains and displacement $\{\boldsymbol{\varepsilon_0}\}$ inthe3D elasticity is:

$$\{\boldsymbol{\varepsilon_0}\} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{Bmatrix}. \tag{10}$$

If these strain-displacements relations are used, we would define the classical formulation of the 20-nodes serendipity element. In this formulation, the curvature of the shell is expressed through the coordinates of the nodes of the elements (Jacobian matrix), but without defining a curvilinear local reference frame at the middle surface of the shell.

In this work, we prefer to work with a curvilinear local reference frame at the middle surface of the shell for a better definition of the middle surface and to study the performance of the new element in dynamic shell analysis (see Figure 4).

A first step to keep in mind is to recall the expression of the strains in a curvilinear system.

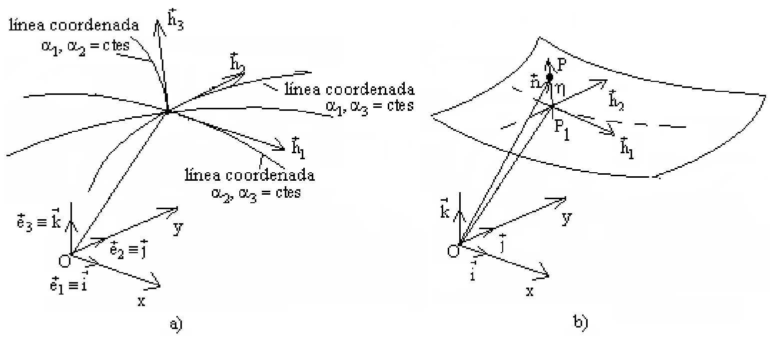

Figure 2: Curvilinear local reference frame at the middle surface of the shell

The expressions relating strains and displacements in curvilinear coordinates are [15]:

$$\varepsilon_{\alpha_i} = \frac{\partial}{\partial\alpha_i}\left(\frac{v_i}{\sqrt{g_{ii}}}\right) + \frac{v_n}{\sqrt{g_{nn}g_{ii}}}\frac{\partial\sqrt{g_{ii}}}{\partial\alpha_n} \tag{11}$$

$$\gamma_{ij} = \sqrt{\frac{g_{ii}}{g_{jj}}}\frac{\partial}{\partial\alpha_j}\left(\frac{v_i}{\sqrt{g_{ii}}}\right) + \sqrt{\frac{g_{jj}}{g_{ii}}}\frac{\partial}{\partial\alpha_i}\left(\frac{v_j}{\sqrt{g_{jj}}}\right),$$

where $\boldsymbol{\varepsilon}_{\boldsymbol{\alpha}_i}$are the normal strains, $\boldsymbol{\gamma}_{ij}$are the engineering shear strains, $\boldsymbol{g}_{ij}$are the components of the metric tensor referred to the shell space, $\boldsymbol{\alpha}_i$are the intrinsic coordinates of the surface, and $\boldsymbol{v}_i$are the displacements. These expressions are written in terms of the physical components so we can directly interpolate the displacements.

When proposing FEM in curvilinear coordinates, we want to write these equations, as far as possible, in matrix form for displacements and strains. Here we will make the approximation consisting in assimilating the metric tensor of the shell space $\boldsymbol{g}_{ij}$to the reference shell surface $\boldsymbol{a}_{ij}$. Therefore, we write:

$$\{\varepsilon^o\} = \begin{Bmatrix} 0 & \frac{\partial\sqrt{a_{11}}}{\partial\alpha_2} & \frac{1}{\sqrt{a_{11}a_{33}}}\frac{\partial\sqrt{a_{11}}}{\partial\alpha_3} \\ \frac{\partial\sqrt{a_{22}}}{\partial\alpha_1} & 0 & \frac{1}{\sqrt{a_{33}a_{22}}}\frac{\partial\sqrt{a_{22}}}{\partial\alpha_3} \\ \frac{\partial\sqrt{a_{33}}}{\partial\alpha_1} & \frac{\partial\sqrt{a_{33}}}{\partial\alpha_2} & 0 \\ -\frac{\partial\sqrt{a_{11}}}{\partial\alpha_2} & -\frac{\partial\sqrt{a_{22}}}{\partial\alpha_1} & 0 \\ 0 & -\frac{1}{\sqrt{a_{33}a_{22}}}\frac{\partial\sqrt{a_{22}}}{\partial\alpha_3} & 0 \\ -\frac{1}{\sqrt{a_{11}a_{33}}}\frac{\partial\sqrt{a_{11}}}{\partial\alpha_3} & 0 & 0 \end{Bmatrix} \begin{Bmatrix} u^1 \\ u^2 \\ u^3 \end{Bmatrix} \tag{12}$$

$$\begin{Bmatrix} \frac{1}{\sqrt{a_{11}}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{a_{22}}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{a_{33}}} \\ 0 & \frac{1}{\sqrt{a_{22}}} & 0 & \frac{1}{\sqrt{a_{11}}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{a_{33}}} & 0 & \frac{1}{\sqrt{a_{22}}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{a_{33}}} & 0 & 0 & 0 & \frac{1}{\sqrt{a_{11}}} & 0 & 0 \end{Bmatrix} \begin{Bmatrix} u^1_{,\alpha 1} \\ u^1_{,\alpha 2} \\ u^1_{,\alpha 3} \\ u^2_{,\alpha 1} \\ u^2_{,\alpha 2} \\ u^2_{,\alpha 3} \\ u^3_{,\alpha 1} \\ u^3_{,\alpha 2} \\ u^3_{,\alpha 3} \end{Bmatrix}$$

where displacements are interpolated, as usual, using the shape functions $\boldsymbol{N}_i$.

At this point we need to note that we now have three different reference systems: the Cartesian reference system $\{\boldsymbol{x} \quad \boldsymbol{y} \quad \boldsymbol{z}\}$,the isoparametric system $\{\boldsymbol{\xi} \quad \boldsymbol{\eta} \quad \boldsymbol{\zeta}\}$,and the reference system tangent to the middle surface of the shell $\{\boldsymbol{\alpha}_1 \quad \boldsymbol{\alpha}_2 \quad \boldsymbol{\alpha}_3\}$.

In this case, to evaluate

$$[\boldsymbol{k}_e] = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1}[\boldsymbol{B}]^T[\boldsymbol{D}][\boldsymbol{B}]det[\boldsymbol{J}]d\xi d\eta d\,\zeta, \tag{13}$$

we have to keep in mind that$[\boldsymbol{B}]$is a function of$\{\alpha_1 \quad \alpha_2 \quad \alpha_3\}$,so we need two changes of reference system:

$$\{\alpha_1 \quad \alpha_2 \quad \alpha_3\} \rightarrow \{\xi \quad \eta \quad \zeta\} \tag{14}$$

$$\{\xi \quad \eta \quad \zeta\} \quad \rightarrow \{x \quad y \quad z\}$$

Matrix D is the usual 3D elasticity constitutive matrix.

4 MASS MATRIX OF THE 20-NODES SERENDIPITY ELEMENT. CONSISTENT AND LUMPED MASS MATRICES.

In order to study the transverse frequencies of the structural element concerned by the finite element method, we must develop the mass matrix of the element. When we take the same shape functions for interpolating the geometry of the element in order to discretize the kinetic energy, we have the so-called consistent mass matrix of the finite element.

$$m_e = \int_V \rho N^T N dV = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \rho N^T N det[J] dV \, . \tag{15}$$

If we develop this expression, we find:

$$m_{ij} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \rho abc N_I N_J d\xi d\eta d\zeta = \qquad \rho abc \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \begin{pmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{pmatrix} \begin{pmatrix} N_j & 0 & 0 \\ 0 & N_j & 0 \\ 0 & 0 & N_j \end{pmatrix} d \tag{16}$$

$$= \qquad \rho abc \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \begin{pmatrix} N_i N_j & 0 & 0 \\ 0 & N_i N_j & 0 \\ 0 & 0 & N_i N_j \end{pmatrix} d\xi d\eta d\zeta$$

Another possibility in this regard is to construct the lumped mass matrix. This mass matrix is traditionally less effective than the consistent mass matrix but is more efficient since it is a square diagonal matrix. The mass of the element is concentrated uniformly in all nodes.

$$m_{ij} = \frac{\rho abc}{20} (i = j), \tag{17}$$

where a,b, and c are the finite element dimensions.

In this work, we have opted for a more rational distribution of the masses at the nodes, taking into account the geometry of the element and the distribution of nodes in it.

If L_ξ, L_η and L_ζ are the dimensions of the element according to the isoparametric coordinates of the element, the surface mapping and lengths used to calculate the volume and mass associated with the nodes of the element can be drawn considering the following figures:

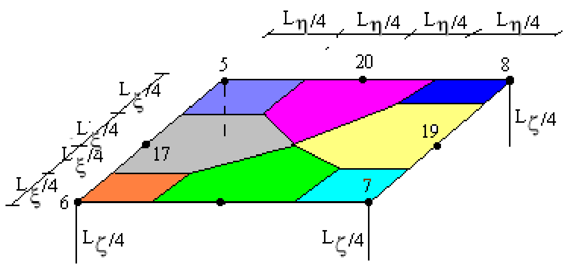

Figure 3: Distribution of areas on the upper and lower surfaces of the 20-nodes element

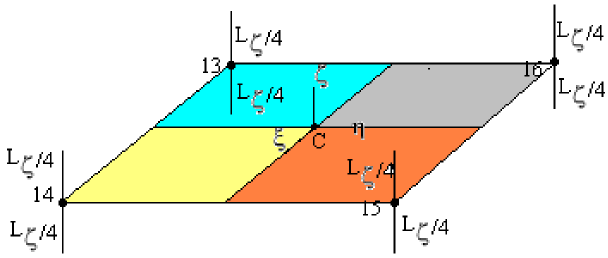

Figure 4: Distribution of areas on the medium surface of the 20-nodes element

So the mass distribution at the nodes is not uniform but takes into account geometry and symmetry considerations of the contribution of each node to the mass matrix of the element.

5 RESULTS

To test the goodness of this work we have previously tested both versions of the element with the hyperbolic paraboloid, whose solutions are known according to the works of Narita and Leissa [16] and Chakravorty [17,18];however those authors used shell elements. The hyperbolic paraboloid test is quite strict both the surface type (doubly curved shell) and the support conditions.

The data we have used for the example are a hyperbolic paraboloid with curved edges, a square planform with 1 · 1 m sides, constant thickness of 0.01 m, radii of curvature that are equal and opposite in value, with R = 2 m, Young's modulus $E=10.92\cdot 10^6 N/m^2$, Poisson's ratio = 0.30, and density= 100 kg/m^3, subjected to a uniformly distributed load of 20 $kN/m^2$and clamping along the four edges. In all cases we have used the lumped mass matrix proposed by the authors.

The frequency associated with the first mode of vibration of the hyperbolic paraboloid according the classical formulation is 17. 53rad/s. The results obtained by the above-named authors are:

– Narita–Leissa: 17.16 rad/s
– Chakravorty: 17.25 rad/s

Displacements associated with the first vibration mode are (see Figure 5):

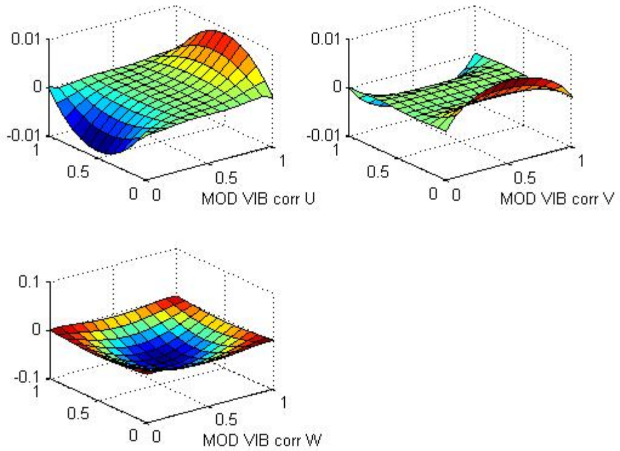

Figure 5: Displacements associated with the first vibration mode of the hyperbolic paraboloid. Classical formulation

If we use the alternative formulation presented in this paper, it is observed that the frequency associated with the first vibration mode is 17.05 rad/s and the displacements associated with the first vibration mode in this case are shown in Figure 6:

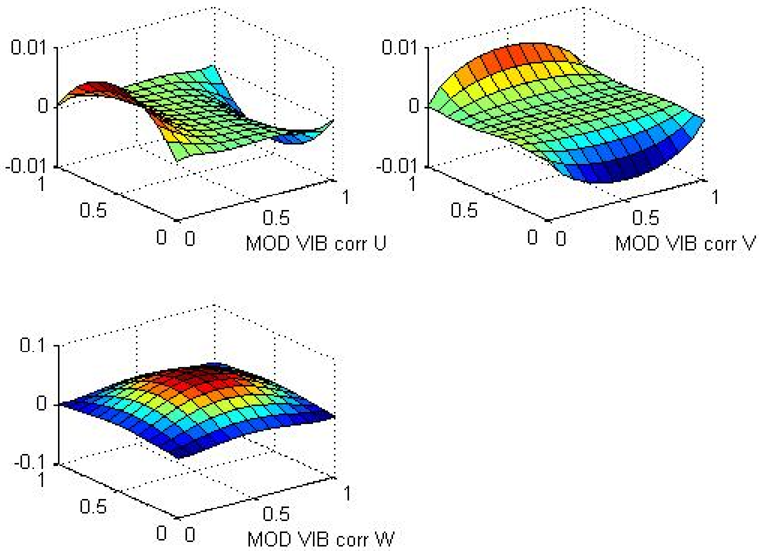

Figure 6: Displacements associated with the first vibration mode of the hyperbolic paraboloid. Formulation in curvilinear coordinates

Given these results, the efficiency of both elements is very high (less than 1% error in both cases). We have also tested the consistent mass matrix. The errors of the solutions obtained from Narita and Leissa or Chakravorty are around 3%.

The next example is a elliptic paraboloid, square planform with 10 · 10 m sides, constant thickness of 0.08 m, Young's modulus E = 21000000 kN/m^2, Poisson's ratio = 0.30, and density = 24 kN/m^3, subjected to a uniformly distributed load of 20 kN/m^2 and mixed support conditions at the edges.

In this case, we set the curvatures to be much greater than in the previous example, and the solution varies according to whether the classic version of the element or the one proposed by the authors is used.

Next we present the displacements u, v, and w associated with the first vibration mode using the classical formulation of the element and the lumped mass matrix proposed by the authors (see Figure 7):

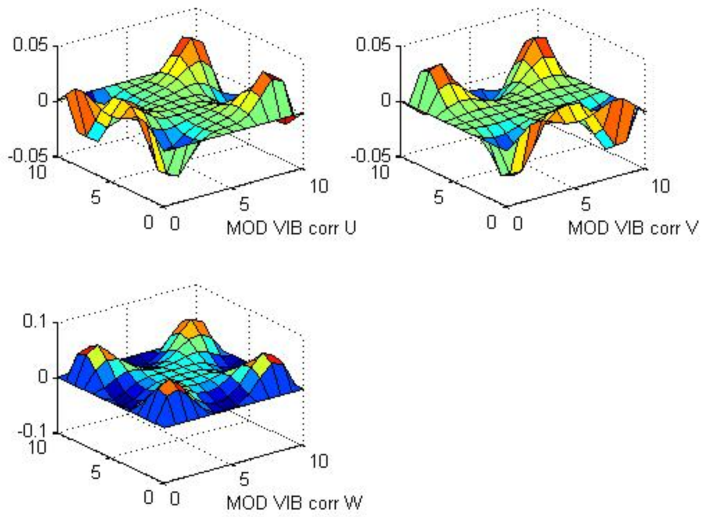

Figure 7: Graphical representation of the first vibration mode. Classical formulation

The first vibration mode is assigned a frequency of 353.8396 rad/s. The values of the first five fundamental frequencies are 353.8396, 364.0470, 364.0471, 373.1142, and 390.0157 rad/s.

If we consider the element formulation adopting the curvilinear system tangent to the middle surface, displacements associated with the first mode of vibration are shown in the following figure (see Figure 8).

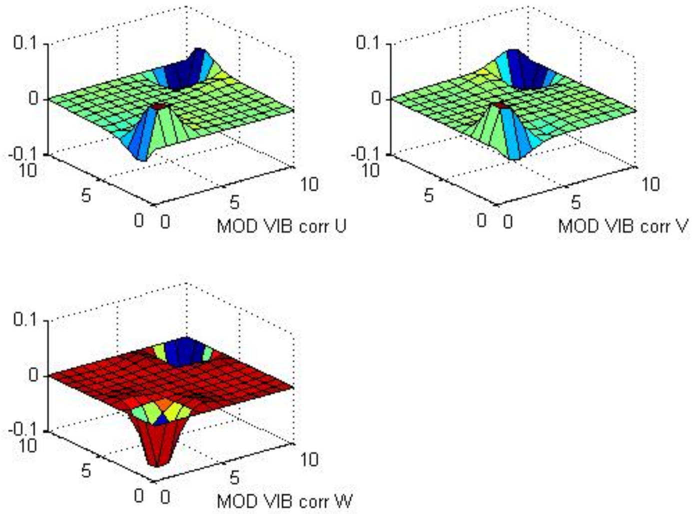

Figure 8: Graphical representation of the first vibration mode. Proposed formulation.

We see that in this case the differences are much greater. The frequencies obtained in this case are: 310.9571, 311.3424, 324.4821, 324.6731,and 380.4881 rad/s.

As we can see, the differences in the first vibration modes are significant. Due to the scarcity of results regarding this kind of shell, we need to keep working on this issue in future work and compare the results obtained with other finite elements.

6 CONCLUSIONS

We have obtained the vibrational frequencies for the elliptic paraboloid and the hyperbolic paraboloid, as examples of doubly curved shells, and we have represented the displacements associated with the first mode of vibration. The results obtained agree with those of other authors [4,5,11].

So we have used both the classical formulation of the 20-nodes element and another considering the strain-displacement relations in a curvilinear system tangent to the middle surface of the shell in order to improve their performance. We have also tested the consistent mass matrix and a variant of the lumped mass matrix taking into account the geometry of the element and the contribution of each node.

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